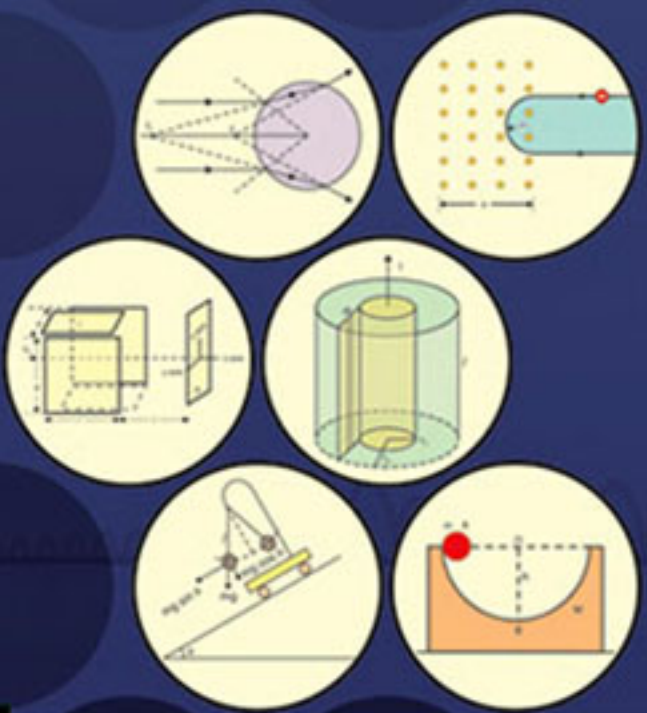
SOLVED PROBLEMS IN PHYSICS



S.L. SRIVASTAVA

SOLVED PROBLEMS IN PHYSICS

VOLUME – 1

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PREFACE

While teaching various courses in physics to the B.Sc. and M.Sc. students over the last thirty-seven years, it has been my experience that a large number of students do not have a clear understanding of the basic concepts of the subject. A slight deviation from what had been taught to them is generally beyond the grasp of average students. It is my firm belief that solving numerical problems can go a long way in aiding comprehension of the underlying basic principles of physics provided that these are suitably designed and properly arranged. It has been my endeavour to fulfil these requirements.

In our country, Intermediate Science (+2 level) is the minimum qualification for appearing in the entrance examinations to different courses in engineering and medical institutions. The minimum qualification for those appearing in civil services examination is graduation and for those appearing in the national eligibility test is a master degree from a recognised University. Students appearing in these examinations will find this book particularly useful. It offers more than two thousand solved problems with complete solutions in a very lucid style and has been written keeping in mind the special needs of aspiring students. It is hoped that the book will also be very useful for those students who seek to make career in the fields of medicine, engineering, biology and earth science. I will consider my efforts amply rewarded if the book is able to instil in the students a sense of confidence and inspiration for these examinations.

S.L. SRIVASTAVA

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1

KINEMATICS IN ONE AND TWO DIMENSIONS

1.1. A man walks 15 km north and then 14.1 km south-east. What is his resultant displacement from the origin?

Solution: Let us represent the displacement along the x axis and y axis, keeping the x axis to the east and y axis to the north as shown in Fig. 1.1. For the first part of the journey the displacement is along the positive side of the y axis and is given $y_1 = 15$ km.

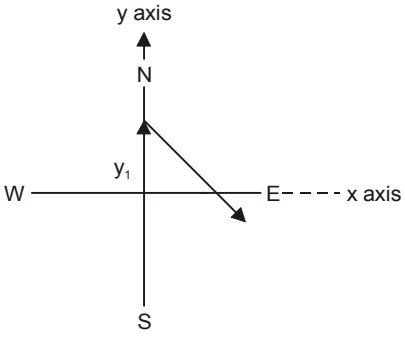


Fig. 1.1

For the second part of the journey the displacement 14.1 km has two components.

The x component of the displacement is

$$14.1 \cos 45 = 14.1 \times \frac{1}{\sqrt{2}} = \frac{14.1}{1.41} = 10 \text{ km}$$

The y component of the displacement is

$$-14.1 \sin 45 = -14.1 \times \frac{1}{\sqrt{2}} = -10 \text{ km}$$

Net displacement along x axis = 10 km

Net displacement along y axis = $15 - 10 = 5$ km

$$\text{Resultant displacement} = \sqrt{100 \quad 25} = 5\sqrt{5} \text{ km}$$

1.2. Two objects are approaching a point O as shown in Fig. 1.2 with equal velocities V. Determine the relative velocity of one with respect to the other.

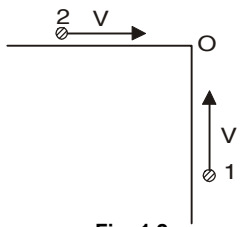


Fig. 1.2

Solution: In Fig. 1.2 the two velocities are with respect to the earth. If we assume that object 2 is at rest, then the earth moves with a

velocity V towards object 2. Now the velocity of object 1 with respect to the earth + velocity of the earth with respect to object 2 will be equal to the velocity of object 1 with respect to object 2, i.e.

$$\boldsymbol{V}_{12} = \boldsymbol{V} - \boldsymbol{V}$$

Thus the velocity of object 1 with respect to object 2 is equal to difference in their velocities with respect to the earth. The speed is worked out as

$$\boldsymbol{V}_{12} = \sqrt{V^2 + V^2} = \sqrt{2}\, V$$

1.3. An object covers distance AB in the following manner. Half of the distance AB, viz. $AB/2 = AC$, is covered with velocity v_0. Of the remaining half, distance CD is covered with velocity v_1 for half of the time and DB with velocity v_2 for the other half of the time. Find the mean velocity of the object.

Solution: Let the total distance AB shown in Fig. 1.3 be equal to x. AC will now be equal to $x/2$.

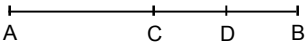


Fig. 1.3

Time taken to cover distance $AC = \dfrac{x}{2v_0}$

Time taken to cover distance $CD = \dfrac{CD}{v_1}$

Time taken to cover the distance $DB = \dfrac{DB}{v_2}$

Now in view of the problem, $\dfrac{CD}{v_1} = \dfrac{DB}{v_2}$ and $CD + DB = \dfrac{x}{2}$

Total distance covered = x

$$\therefore \text{ Mean velocity } = \frac{x}{\dfrac{x}{2v_0} \quad \dfrac{CD}{v_1} \quad \dfrac{DB}{v_2}}$$

$$= \frac{x}{\dfrac{x}{2v_0} \quad \dfrac{2CD}{v_1}} \text{ since } \frac{CD}{v_1} = \frac{DB}{v_2}$$

$$= \frac{x}{\dfrac{x}{2v_0} \quad \dfrac{xv_1}{v_1 \quad v_2} \quad \dfrac{1}{v_1}}$$

$$= \frac{2v_0 \,(v_1 \quad v_2)}{v_1 \quad v_2 \quad 2v_0}$$

1.4. The velocity time graph of a particle is a parabola as shown in the Fig. 1.4. The maximum value of velocity is equal to V. Calculate the distance covered by the particle in time $\frac{T}{2}$.

Solution: The velocity of the particle is given by the equation.

$$v = \frac{4V}{T^2/4}\, t\left(\frac{T}{2} - t\right)$$

or $$\frac{dx}{dt} = \frac{16V}{T^2} \quad \frac{T}{2}\cdot t \quad t^2$$

$$\int_0^S dx = \frac{16V}{T^2}\left[\frac{T}{2}\int_0^{T/2} tdt - \int_0^{T/2} t^2 dt\right]$$

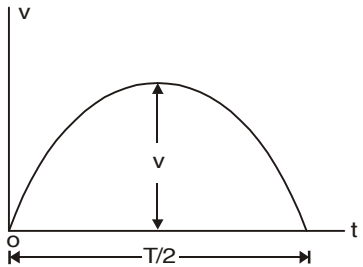


Fig. 1.4

$$S = \frac{16V}{T^2} \quad \frac{T^3}{16} \quad \frac{T^3}{24}$$

$$S = \frac{TV}{3}$$

1.5. A particle of mass m moves linearly under the action of a time varying force. The force time diagram is a parabola as shown in the Fig. 1.5. The maximum value of the force is equal to X. Calculate the displacement of the particle in time T.

Solution: The force F acting on the particle is given by

$$F = \frac{4X}{T^2}\, t\,(T - t)$$

$$\text{acceleration } a = \frac{4X}{mT^2}\, t\,(T - t)$$

$$\text{velocity } v = \frac{4X}{mT^2} \quad T\cdot\frac{t^2}{2} \quad \frac{t^3}{3}$$

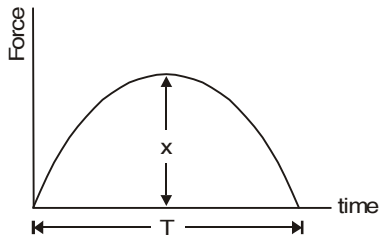


Fig. 1.5

$$\int_0^S dx = \frac{4X}{mT^2}\left[\frac{T}{2}\cdot\int_0^T t^2 dt - \frac{1}{3}\int_0^T t^3 dt\right]$$

$$S = \frac{4X}{mT^2} \quad \frac{T^4}{6} \quad \frac{T^4}{12}$$

$$S = \frac{XT^2}{3m}$$

1.6. Two bodies P & Q connected through light rigid rod of length L are placed as shown in the Fig. 1.6. Body P moves toward O with a constant speed v. Find the velocity of Q.

Solution: Let the position of P and Q from O be X and Y respectively. Let P moves through dx then the displacement of Q is obtained by the relation.

$$L^2 - (X - dx)^2 = (Y + dy)^2$$

$$L^2 - X^2 + 2X dx = Y^2 + 2Y \cdot dy$$

or $$X \cdot dx = Y \cdot dy$$

or $$\frac{X}{Y}\frac{dx}{dt} = \frac{dy}{dt} = v'$$

or $$v' = v \cot 30$$

$$= v \cdot \sqrt{3}$$

$$= 1.732\, v$$

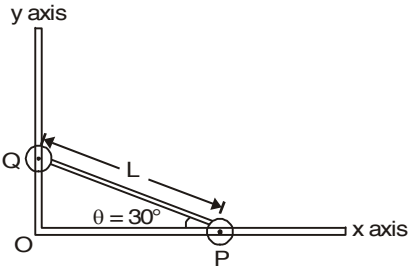


Fig. 1.6

1.7. A point moves in the xy plane according to the equation $x = \alpha t$ and $y = \alpha t\,(1 - \beta t)$ where α and β are positive constant. Find (i) the equation of the trajectory (ii) the velocity and acceleration of the particle at an instant t.

Solution:

(i) $$y = x\left(1 - \frac{\beta x}{\alpha}\right)$$

$$= x - \frac{\beta}{\alpha} \cdot x^2$$

$$= -\frac{\beta}{\alpha}\left[x^2 - \frac{\alpha}{\beta} \cdot x\right]$$

$$= -\frac{\beta}{\alpha}\left[\left(x - \frac{\alpha}{2\beta}\right)^2 - \frac{\alpha^2}{4\beta^2}\right]$$

$$\left(y - \frac{\alpha}{4\beta}\right) = -\frac{\beta}{\alpha}\left[\left(x - \frac{\alpha}{2\beta}\right)^2\right]$$

This is the equation of parabola with concavity downward and vertex at $\left(\frac{\alpha}{2\beta}, \frac{\alpha}{4\beta}\right)$

(ii) $$\frac{dx}{dt} = v_x = \alpha, \quad \frac{dy}{dt} = \alpha - 2\alpha\beta t = v_y$$

$$\therefore v = \sqrt{v_x^2 \quad v_y^2} = \alpha\sqrt{1 + (1 - 2\beta t)^2}$$

$$\frac{d^2x}{dt^2} = a_x = 0, \quad \frac{d^2y}{dt^2} = a_y = -2\alpha\beta$$

$$\therefore a = \sqrt{a_x^2 \quad a_y^2} = 2\alpha\beta$$

1.8. A particle is moving in a plane with velocity given by $u = u_0 i + (a\omega \cos \omega t)\, j$, where i and j are unit vectors along x and y axes respectively. If particle is at the origin at $t = 0$. (a) Calculate the trajectory of the particle. (b) Find the distance from the origin at time $\frac{3\pi}{2\omega}$.

Solution:

(a) $$v_x = \frac{dx}{dt} = u_0$$

$$\therefore x = u_0 dt = u_0 t$$

$$v_y = \frac{dy}{dt} = a\omega \cos \omega t$$

$$\therefore y = \int a\omega \cos \omega t \, dt = a \sin \omega t$$

Path of the particle is given by

$$y = a \sin \frac{\omega \cdot x}{u_0}$$

(b) At $t = \frac{3\pi}{2\omega}$

$$x = u_0 \cdot \frac{3\pi}{2\omega}$$

$$y = a \sin \omega \cdot \frac{3\pi}{2\omega} = -a$$

$\therefore$ Distance of the particle from the origin = $\sqrt{x^2 \quad y^2}$

$$= \left[a^2 + \left(\frac{3\pi u_0}{2\omega} \right)^2 \right]^{\frac{1}{2}}$$

1.9. Starting from the point O along the road OA a person is interested to reach to the point B located in the field at a distance l from A by scooter in the shortest time. The speed of the scooter in the field reduces η times that on the road. Calculate the distance of the point C (see Fig. 1.7) from A at which he will leave the road in order to reach to the point B.

Solution: Let $OC = x$

time t_1 for the part $OC = \frac{x}{v}$

v = velocity of the scooter on the road

time t_2 for the part $CB = \sqrt{OA \quad x^2 \quad l^2} \cdot \frac{\eta}{v}$

$$\text{total time } t = \frac{x + \eta \cdot \sqrt{(OA - x)^2 + l^2}}{v}$$

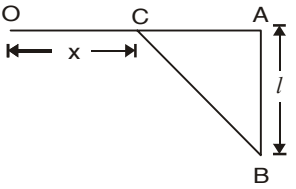


Fig. 1.7

$$\text{Now } \frac{dt}{dx} = 0 = \frac{1}{v}\left[1 + \frac{\eta}{2} \frac{\times 2 (OA - x) \times -1}{\sqrt{(OA - x)^2 + l^2}}\right]$$

$$\text{or } \quad 1 = \frac{\eta (OA - x)}{\sqrt{(OA - x)^2 + l^2}}$$

$$\text{or } \quad AC^2 + l^2 = \eta^2 \cdot AC^2$$

$$\text{or } \quad AC = \frac{l}{\sqrt{\eta^2 - 1}}$$

1.10. A ship goes from A to B at v_1 = 10 km/hr and from B to A at v_2 = 16 km/hr. Find (i) The average velocity of the ship (ii) the velocity of the river current.

Solution:

(i) Let x be the distance between A and B.

Time for the entire path (up and down journey)

$$t = \frac{x}{\frac{10 \quad 5}{18}} + \frac{x}{\frac{16 \quad 5}{18}} = \frac{13 \quad 18x}{400}$$

$$\text{Average velocity} = \frac{2x \quad 400}{13 \quad 18x} = 3.42 \text{ m/sec}$$

(ii) $$10 \times \frac{5}{18} + u = 16 \times \frac{5}{18} - u$$

$$2u = \frac{5}{3}$$

$$u = \frac{5}{6} = 0.83 \text{ m/sec}$$

1.11. A boat moves perpendicular to the bank with velocity of 7.2 km/hr (2 m/sec). The current carries it 150 m down stream. Find (i) the velocity of the current (ii) the time required to cross the river. The river is 0.5 km wide.

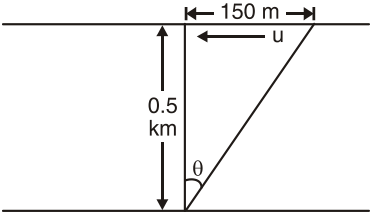


Fig. 1.8

Solution:

$$\tan \theta = \frac{u}{2} = \frac{150}{500}$$

$$\therefore u = 0.6 \text{ m/sec}$$

$$t = \frac{500}{2} = 250 \text{ sec}$$

1.12. A ship is moving towards east with a velocity v_0. Wind blows at an angle θ to the east in the south-east direction with a velocity v. Find (i) the wind velocity relative to the ship (ii) the wind direction in the frame of reference of the ship.

Solution: (i) Let v' be the wind velocity relative to the ship see Fig. 1.9.

$$v' \cos \theta' = v_0 + v \cos \theta$$

$$v' \sin \theta' = v \sin \theta$$

$$v' = \sqrt{(v_0 + v \cos \theta)^2 + v^2 \sin^2 \theta}$$

(ii) $$\tan \theta' = \frac{v \sin \theta}{v_0 + v \cos \theta}$$

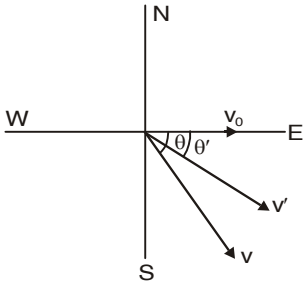


Fig. 1.9

1.13. Two swimmers starting from A wants to reach to the point B by crossing the river in which the current velocity is v_0 by two approaches. One of them crosses the river along the path AB with a velocity v' relative to water while the other swims at right angles to the stream with the same velocity and reaches to the point C (Fig. 1.10b) lying on the other bank. At what velocity u he should run along CB so that both the swimmers reaches B simultaneously.

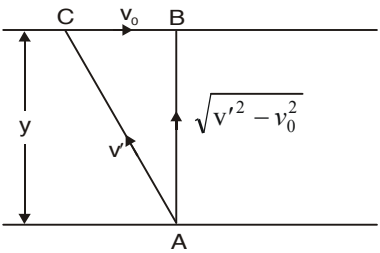


Fig. 1.10a

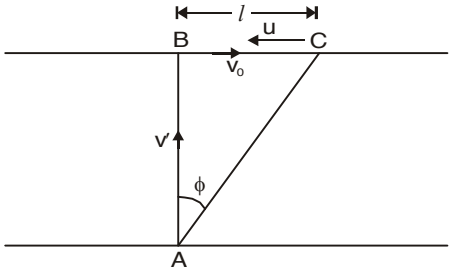


Fig. 1.10b

Solution: In view of Fig. 1.10a one swimmer moves along AC with velocity v'. Due to the current having velocity v_0 it will have a resultant velocity $\sqrt{v^2 \quad v_0^2}$ along AB. Time taken t is given by

$$t = \frac{y}{\sqrt{v^2 \quad v_0^2}} = \frac{y}{v' \sqrt{1 - \frac{v_0^2}{v'^2}}}; \; y = \text{width of the river.}$$

The other swimmer moves along AB with velocity v' and therefore the time t_1 taken for the journey $= \frac{y}{v}$

Due to the current it will reach to point C.

$$\tan\phi = \frac{v_0}{v} = \frac{l}{y}$$

$$\therefore \quad l = \frac{v_0 \cdot y}{v}; \; l = BC.$$

Time taken for the path CB with velocity u is given by

$$t_2 = \frac{l}{u} = \frac{v_0 \cdot y}{v \cdot u}$$

$$t = t_1 + t_2$$

or $$\frac{y}{v\sqrt{1 - \frac{v_0^2}{v^2}}} = \frac{y}{v} + \frac{v_0 \cdot y}{v\,u}$$

or $$\frac{1}{\sqrt{1 - \frac{v_0^2}{v^2}}} = 1 + \frac{v_0}{u}$$

or $$u = \frac{v_0}{\left(1 - \frac{v_0^2}{v^2}\right)^{1/2} - 1}$$

1.14. A boat takes $\frac{5}{4}$ min to cross a 100 m wide river with a speed of $V = 4$ m/sec relative to the water. Assuming the current speed $u = 2$ m/sec find possible up stream or down stream point where the boat will reach on the opposite shore.

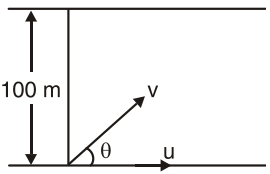


Fig. 1.11a

Solution: $$V\sin\theta = v = \frac{100}{\frac{5}{4}\;60} = \frac{4}{3}$$

$$\therefore \sin\theta = \frac{1}{3}$$

$$\cos\theta = \frac{\sqrt{8}}{3}$$

For up the stream the required velocity

$= V\cos\theta - u$ (see Fig. 1.11a)

$$= 4 \cdot \frac{\sqrt{8}}{3} - 2$$

$= 1.77$ m/sec

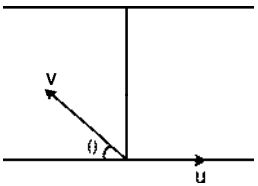


Fig. 1.11b

Distance moved = $1.77 \times 75 = 132.8$ m

For down the stream the required velocity= $V\cos\theta + u$ (see Fig. 1.11b)

$$= 4 \cdot \frac{\sqrt{8}}{3} + 2 = 5.77 \text{ m/sec}$$

Distance moved = 5.77 × 75 = 432.8 m

1.15. A motor boat moving up streams meets a raft moving down stream. After travelling for a period of $t_1 = 30$ min the boat reaches a landing place where it stops for a period of $t_2 = 1$ hr. The boat then travels down stream and meets the raft in $t_3 = 45$ min. Find the current velocity u by assuming the speed of the boat relative to the water $V = 8$ km/hr.

Solution: In view of Fig. 1.12 we have

$u\,(t_1 + t_2 + t_3) + (V - u)\,t_1$
$= (V + u)\,t_3$;
u = current velocity.
$ut_2 = V(t_3 - t_1)$

$$\therefore\ u = \frac{V\ t_3\quad t_1}{t_2}$$

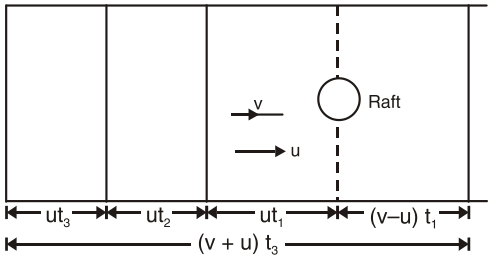


Fig. 1.12

On putting the values $u = \dfrac{8(0.75\quad 0.5)}{1} = 2$ km/hr

1.16. The width of a river is 25 m and in it water is flowing with a velocity of 4 m/min. A boatman is standing on the bank of river. He wants to sail the boat at a point to the other bank which is directly opposite to him. In what time will he cross the river if he can sail the boat at 8 m/min relative to the water.

Solution: The boatman will sail the boat along AC (see Fig. 1.13)

$$\sin\theta = \frac{u}{V} = \frac{4}{8} = \frac{1}{2}$$

$$\theta = 30^o$$

$$\cos\theta = \frac{\sqrt{3}}{2} = \frac{25}{AC}$$

$$\therefore AC = \frac{50}{\sqrt{3}}$$

$$\text{time } t = \frac{AC}{8} = \frac{50}{\sqrt{3}\quad 8} = 3.6 \text{ min.}$$

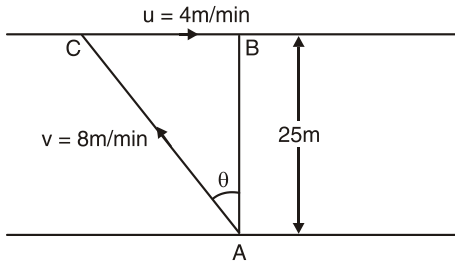


Fig. 1.13

1.17. Two boats A and B move in perpendicular direction to a buoy anchored at some point O on a river. They travel with velocity 1.2 v, where v is the stream velocity, boat A moves along the river, whereas boat B moves

perpendicular to it. After traversing an equal distance from O the two boats return. Find the ratio of the time taken by the two boats.

Solution: Boat A travels along OX and boat B moves along OY (Fig.1.14).

Let $OX = OY = l$. The times of journey of boat A from O to X and from X to O are $\frac{l}{1.2v+v}$ and $\frac{l}{1.2v \quad v}$, respectively.

Total time taken by boat A is given by

$$t_A = \frac{l}{1.2v+v} + \frac{l}{1.2v \quad v}$$

$$= \frac{2 \quad 1.2l}{v \quad 1.2^{\,2} \quad 1}$$

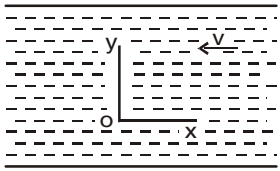


Fig. 1.14

For boat B to move from O to Y its velocity must be at an angle to the direction of the stream velocity in such a way that the resultant velocity is directed towards Y. The magnitude of the resultant velocity is given by

$v \quad 1.2^{\,2} \quad 1^{\,\frac{1}{2}}$. The total time taken by boat B is given by $t_B = \frac{2l}{v \quad 1.2^{\,2} \quad 1^{\,\frac{1}{2}}}$.

$$\therefore \frac{t_A}{t_B} = \frac{1.2}{\sqrt{1.2^{\,2} \quad 1}} = 1.81$$

1.18. In Fig. 1.15 what should be the angle θ at which a swimmer should swim to reach to point B, starting from point A. The speed of the swimmer is v, that of the stream u and the angle ϕ is known.

Solution: A is the point from which the swimmer moves at an angle θ with respect to the width of the river AC. On resolving v along x and y axes we have $v \cos\theta$ as the speed along the y axis and $u + v \sin\theta$ along the x axis.

Now $\tan\phi = \frac{u + v\sin\theta}{v\cos\theta}$

or $v\cos\theta\sin\phi = u\cos\phi + v\sin\theta\cos\phi$

or $v[\sin(\phi - \theta)] = u\cos\phi$

or $\sin(\phi - \theta) = \frac{u}{v}\cos\phi$

or $\theta = \phi - \sin^{-1}\left(\frac{u}{v}\cos\phi\right)$

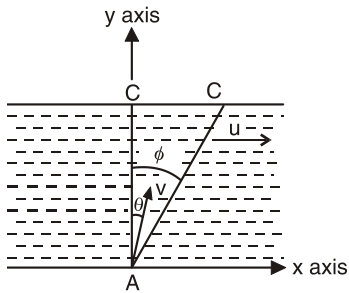


Fig. 1.15

1.19. A man swimming at rate of 5 km/h wants to cross a 120 m wide river in a direction perpendicular to the stream. If the stream speed is 4 km/h then find the direction in which the man swims, and the time he takes to cross the river.

Solution: In Fig. 1.16 *OA* represents the stream speed of 4 km/h and *OB* the swimming speed of 5 km/h. The swimming direction is 90 + θ with respect to the stream.

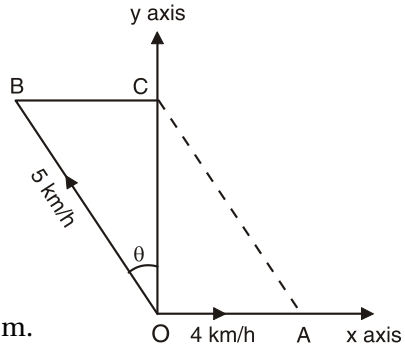


Fig. 1.16

Now $\sin\theta = \frac{4}{5}$

$\cos(90 + \theta) = -\sin\theta = -\frac{4}{5}$

Therefore, the direction in which the man swims is given by $\cos^{-1}\left(-\frac{4}{5}\right)$.

In triangle *BOC*, we have $OC = 120$ m.

$\cos BOC = \cos\theta = \frac{3}{5} = \frac{OC}{OB}$

$\therefore OB = \frac{5}{3} \times OC = \frac{5}{3} \times 120 = 200$ m

Time taken to cross the river $= \frac{200}{5\ \ 1000} \times 3600 = 144$ s

1.20. Two particles are projected in the *xy* plane with equal speeds of $v = 14.1$ m/s from the reference point *O* at angles $\theta_1 = 60°$ and $\theta_2 = 30°$ as shown in Fig. 1.17. What will be their separation after $t = 3$ *s*?

Solution: *x* component of velocity of the particle projected at angle $\theta_1 = v \cos\theta_1$ (on positive side of *x* axis) *x* component of velocity of the particle projected an angle $\theta_2 = v\cos\theta_2$ (on negative side of *x* axis)

Relative velocity along

x axis $= v(\cos\theta_1 + \cos\theta_2)$

y component of velocity of particle projected at an angle $\theta_1 = (v\sin\theta_1 - gt)$ (on positive side of *y* axis)

y component of velocity of particle projected at an angle $\theta_2 = (v\sin\theta_2 - gt)$ (on positive side of *y* axis)

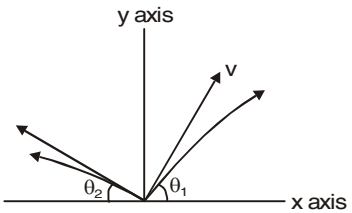


Fig. 1.17

Relative velocity along *y* axis $= v(\sin\theta_1 - \sin\theta_2)$

$$\text{Resultant velocity } V = v\sqrt{(\cos\theta_1 + \cos\theta_2)^2 + (\sin\theta_1 - \sin\theta_2)^2}$$

$$= v\sqrt{2 + 2\cos(\theta_1 + \theta_2)}$$

$$= 2v\cos\frac{(\theta_1 + \theta_2)}{2}$$

Separation between the two particles after time

$$t = 2v\cos\left(\frac{\theta_1 + \theta_2}{2}\right) \times t$$

On substituting the values of θ_1; θ_2; v and t we have,

$$\text{Separation between the two particles} = 2 \times 14.1 \times \cos\left(\frac{30 + 60}{2}\right) \times 3$$

$$= 2 \times 14.1 \times \cos 45 \times 3$$

$$= 2 \times 14.1 \times \frac{1}{1.41}\ \frac{3}{}$$

$$= 60 \text{ m}$$

1.21. A particle initially at rest starts moving from the reference point $x = 0$ along the x axis, with a velocity v that varies as $v = \alpha\sqrt{x}$ where α is constant. (a) How do the velocity and acceleration of the particle vary with time? (b) What is the average velocity of the particle over the first s metres of its path.

Solution: (a) The velocity of the particle is given by

$$v = \alpha\sqrt{x}$$

On differentiating it with respect to t we have

$$\frac{dv}{dt} = \frac{\alpha}{2} \cdot \frac{1}{\sqrt{x}} \cdot \frac{dx}{dt} = \frac{\alpha}{2\sqrt{x}} \cdot v = \frac{\alpha^2}{2};\quad \frac{dx}{dt}\quad v \quad \text{velocity}$$

or $$dv = \frac{\alpha^2}{2}\,dt$$

On integration, we have

$$v = \frac{\alpha^2}{2}t$$

This is the required equation for velocity.

On differentiating the above expression for velocity with respect to t we have acceleration $a = \dfrac{dv}{dt} = \dfrac{\alpha^2}{2}$.

The above equation is independent of time and therefore suggests that the acceleration of the particle remains constant.

(b) Let us start with the equation

$$v = \frac{dx}{dt} = \frac{\alpha^2}{2} t$$

or $dx = \frac{\alpha^2}{2} t\, dt$

On integration, we have

$$x = \frac{\alpha^2 t^2}{4}$$

Let t_1 be the total time taken by the particle to cover a distance of s metres. Then

$$s = \frac{\alpha^2 t_1^2}{4}$$

or $$t_1 = \sqrt{\frac{4s}{\alpha^2}}$$

$$\therefore \text{Average velocity} = \frac{s}{t_1} = \frac{s \cdot \alpha}{2\sqrt{s}} = \frac{\alpha}{2} \sqrt{s}$$

1.22. A particle having an initial velocity v_0 moves along a straight line with deceleration a represented by the equation $a = \alpha\sqrt{v}$, where α is constant. After what time and after covering what distance will the particle be brought to rest.

Solution: Acceleration $a = \frac{dv}{dt} = -\alpha\sqrt{v}$ (given)

Now $\frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v = -\alpha\sqrt{v}$; $\frac{ds}{dt}$ v velocity

or $\sqrt{v}\, dv = -\alpha\, ds$

Integrating the above equation, we have

$$\int_{v_0}^{0} v^{\frac{1}{2}} dv = -\alpha \int_0^s ds$$

or $\frac{2}{3} v_0^{3/2} = \alpha s$ or $s = \frac{2}{3\alpha} v_0^{3/2}$

Again $\frac{dv}{dt} = -\alpha\sqrt{v}$

or $v^{-1/2} dv = -\alpha\, dt$

Integrating the above equation, we have

$$\int_{v_0}^{0} v^{1/2} dv = -\alpha \int_0^t dt$$

or $\frac{2}{\alpha}\sqrt{v_0} = t$

1.23. A small block of mass m slides down an inclined plane forming an angle $\alpha = 30°$ with the horizontal. The coefficient of friction k between the block and the plane is no longer constant but varies with x, the distance travelled by the block on the plane, according to the equation $k = ax$, where a is a constant. Find the distance travelled and the maximum velocity attained by the block before coming to rest.

Solution: The equation of motion of the block along the plane (see Fig. 1.18) is given by

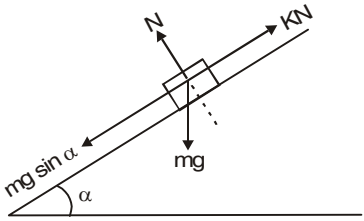


Fig. 1.18

$$mg \sin \alpha - k\, mg \cos \alpha = m \frac{dv}{dt}$$

$$\text{or} \quad g \sin \alpha - kg \cos \alpha = \frac{dv}{dx} \frac{dx}{dt}$$

$$\text{or} \quad g \sin \alpha - kg \cos \alpha = v \frac{dv}{dx}; \quad \frac{dx}{dt} \quad v$$

$$\text{or} \quad (g \sin \alpha - g\, a\, x \cos \alpha)\, dx = v dv$$

On integrating the above equation, we have

$$g \sin \alpha \cdot x - \frac{ax^2}{2} g \cos \alpha = \frac{v^2}{2}$$

This is the equation for the velocity of the block.

On settig $v = 0$ we have

$$x = \frac{2}{a} \tan \alpha \text{ or } x = \frac{2}{a} \times \frac{1}{\sqrt{3}}$$

For obtaining the maximum value of v, we equate $\frac{dv}{dx}$ to zero

i.e., $g \sin \alpha - a\, g\, x \cos \alpha = \frac{v dv}{dx} = 0$

$$\text{or} \quad x = \frac{\tan \alpha}{a}$$

i.e., at a distance $x = \frac{\tan \alpha}{a}$ the velocity of the block will be maximum.

$$\text{Now } \frac{1}{2} v_{\max}^2 = g \sin \alpha \cdot \frac{\tan \alpha}{a} - \frac{a \cdot g}{2} \frac{\tan^2 \alpha}{a^2} \cos \alpha$$

$$= \frac{g \sin \alpha \cdot \tan \alpha}{a} - \frac{g}{2a} \sin \alpha \cdot \tan \alpha$$

$$= \frac{g}{2a} \sin \alpha \cdot \tan \alpha$$

$$\therefore \quad v_{max} = \sqrt{\frac{g}{a} \sin\alpha \cdot \tan\alpha}$$

On putting $\alpha = 30°$, we have

$$v_{max} = \sqrt{\frac{g}{a}} \times \sqrt{\frac{1}{2} \quad \frac{1}{\sqrt{3}}} = \quad \frac{g}{2a\sqrt{3}}^{1/2}$$

1.24. Two particles a and b initially at distances $l_1 = 4$ m and $l_2 = 3$ m from the point O, begin to move towards O with velocities $v_1 = 3$ m/s and $v_2 = 4$ m/s along two mutually perpendicular straight path intersecting at O (see Fig. 1.19). After what time will the distance between the two particles be minimum ? What will be the value of this minimum distance ?

Solution: After time t the distance between the particles is given by $x^2 = (l_1 - v_1 t)^2 + (l_2 - v_2 t)^2$

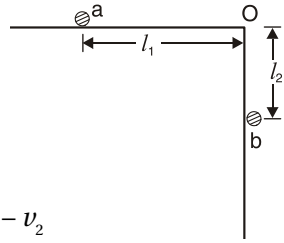


Fig. 1.19

On differentiating this with respect to t and setting the differential to zero we have

$$2x \frac{dx}{dt} = 0$$

$$= 2 (l_1 - v_1 t) \times - v_1 + 2 (l_2 - v_2 t) \times - v_2$$

or $\quad l_1 v_1 - v_1^2 t = - l_2 v_2 + v_2^2 t$

or $\qquad t = \dfrac{l_1 v_1 \quad l_2 v_2}{v_1^2 \quad v_2^2}$

On substituting the values of l_1, l_2, v_1 and v_2, we have

$$t = \frac{4 \quad 3 \quad 3 \quad 4}{16 \quad 9} = \frac{24}{25} \text{ s}$$

In view of the value of t the starting equation takes the form

$$x^2 = l_1^2 + l_2^2 - \frac{2 \quad l_1 v_1 \quad l_2 v_2}{v_1^2 \quad v_2^2}^2 + \frac{l_1 v_1 \quad l_2 v_2}{v_1^2 \quad v_2^2}^2$$

$$= l_1^2 + l_2^2 - \frac{l_1 v_1 \quad l_2 v_2}{v_1^2 \quad v_2^2}^2$$

$$= \frac{l_1 v_2 \quad l_2 v_1}{v_1^2 \quad v_2^2}^2$$

$$\text{or } x_{min} = \frac{|l_1 v_2 \quad l_2 v_1|}{\sqrt{v_1^2 \quad v_2^2}} = \frac{16 \quad 9}{\sqrt{16 \quad 9}} = \frac{7}{5} = 1.4 \text{ m}$$

1.25. The equation of motion of an object moving in the xy plane with constant acceleration a acting along the negative y axis is governed by

the relation $y = \alpha x - \beta x^2$, where α and β are constants. Find the initial velocity of the object.

Solution: $y = \alpha x - \beta x^2$ (given)

On differentiating this with respect to t, we have

$$\frac{dy}{dt} = \alpha \frac{dx}{dt} - 2\beta x \frac{dx}{dt}$$

A further differentiation with respect to t yields

$$\frac{d^2y}{dt^2} = \alpha \frac{d^2x}{dt^2} - 2\beta x \frac{d^2x}{dt^2} - 2\beta \left(\frac{dx}{dt}\right)^2$$

In view of the problem we have $\frac{d^2x}{dt^2} = 0$ and $\frac{d^2y}{dt^2} = -a$

Substituting these values in the above equation, we get

$$-a = -2\beta \left(\frac{dx}{dt}\right)^2 \quad \text{or} \quad \frac{dx}{dt} = \sqrt{\frac{a}{2\beta}}$$

With this value of $\frac{dx}{dt}$, $\frac{dy}{dt}$ at $x = 0$ becomes

$$\frac{dy}{dt} = \alpha \sqrt{\frac{a}{2\beta}}$$

$$\therefore \quad v_0^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{a}{2\beta} + \alpha^2 \frac{a}{2\beta}$$

$$\therefore \quad v_0 = \sqrt{\frac{(1+\alpha^2)a}{2\beta}}$$

1.26. A stone is thrown horizontally with a velocity of 10 m/sec from a height of h above the ground. Find the radius of curvature of its trajectory in 3 sec after the motion begins.

Solution: $x = 10t$, $y = \frac{1}{2} gt^2 = 4.9t^2$

or $y = \frac{4.9x^2}{100}$

$$\frac{dy}{dx} = \frac{4.9x}{50} \text{ and } \frac{d^2y}{dx^2} = \frac{4.9}{50}$$

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{1 \left(\frac{dy}{dx}\right)^{2\,3/2}}$$

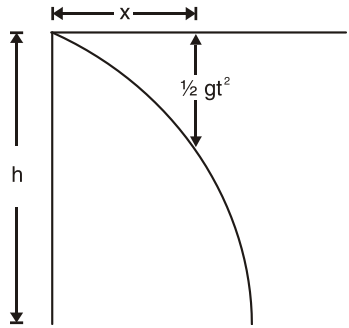


Fig. 1.20

$$\frac{1}{\rho} = \frac{\frac{4.9}{50}}{\left[1+\left(\frac{4.9\times 3}{5}\right)^2\right]^{\frac{3}{2}}}; x = 10 \times 3 = 30 \text{ m}$$

$$\frac{1}{\rho} = \frac{4.9}{50 \quad 9.6436 \quad 3.1}$$

$$\rho = 305 \text{ m}$$

1.27. The equation for the parabola is of the form $x^2 = 2py$ where p is +ve constant. Find the radius of curvature of the parabola at each point.

Solution: $x^2 = 2py$

$$p\frac{dy}{dx} = x \text{ and } p\frac{d^2y}{dx^2} = 1$$

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{1}{p\left[1+\frac{x^2}{p^2}\right]^{\frac{3}{2}}}$$

or $\rho = p\left[1+\frac{x^2}{p^2}\right]^{\frac{3}{2}}$

1.28. A ball is thrown vertically upward from the top of a tower with an initial velocity u reaches the foot in time t_1. When it is thrown vertically downwards with the same initial velocity from the same point it takes time t_2. Calculate the free fall time.

Solution: Let h be the height of the tower then the equation of motion of the ball is given by

$$h = -ut_1 + \frac{1}{2}gt_1^2$$

and

$$h = +ut_2 + \frac{1}{2}gt_2^2$$

$$h(t_1 + t_2) = \frac{1}{2}g t_1 t_2 (t_1 + t_2)$$

or

$$h = \frac{1}{2}g t_1 t_2$$

In case of free fall $h = \frac{1}{2}gt^2$

On comparing the two we have

$$t = \sqrt{t_1 t_2}$$

1.29. Two particles are projected in the earths gravitational field from a point with initial velocities v_1 and v_2 and at angles θ_1 and θ_2 with the horizontal respectively. What will be the path of one particle as seen from the other.

Solution: The separation betwen the two particles along x axis (taken as horizontal) is equal to

$$x = x_2 - x_1 = (v_2 \cos\theta_2 - v_1 \cos\theta_1)\, t$$

Similarly the separation between the particles along y axis is equal to

$$y = \left(v_2 \sin\theta_2\, t - \frac{1}{2} gt^2\right) - \left(v_1 \sin\theta_1\, t - \frac{1}{2} gt^2\right)$$

$$y = (v_2 \sin\theta_2 - v_1 \sin\theta_1)\, t$$

These separations are in fact the coordinates of a point on one trajectory relative to the other

$$\frac{y}{x} = \frac{v_2 \sin\theta_2 - v_1 \sin\theta_1}{v_2 \cos\theta_2 - v_1 \cos\theta_1} = \text{constt.} = m$$

$$\therefore \quad y = mx$$

The path will therefore be a straight line.

1.30. A spherical ball is located at a point A (see Fig. 1.21) at a height h above the earth's surface. As it starts falling freely another similar ball is thrown from the earth's surface from the point O. With what initial velocity and at what angle should the ball be thrown from O in order to collide with the first one at B at an altitude $h_1 = h/2$ after covering a horizontal distance l?

Solution: Time taken by the first ball for the path AB is the same as that of the second ball for the horizontal distance l.

Now $\quad AB = \dfrac{h}{2} = \dfrac{1}{2} gt^2$

or $\quad t = \sqrt{\dfrac{h}{g}}$

Distance $l = v \cos\theta \cdot t$; v is the initial velocity

$$= v \cos\theta \sqrt{\frac{h}{g}}$$

Further $\dfrac{h}{2} = v \sin\theta \cdot t - \dfrac{1}{2} gt^2$

or $\quad \dfrac{h}{2} = v \sin\theta \cdot \sqrt{\dfrac{h}{g}} - \dfrac{h}{2}$

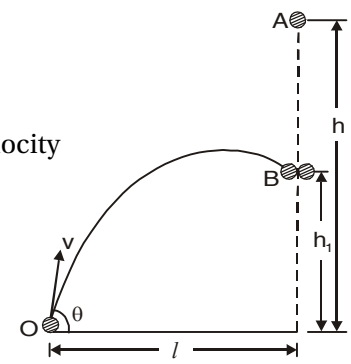


Fig. 1.21

or $h = v \sin\theta \cdot \sqrt{\frac{h}{g}}$

or $v \sin\theta = \sqrt{hg}$

$v^2 (\sin^2\theta + \cos^2\theta) = gh + \frac{l^2 g}{h}$

or $v = \sqrt{\frac{g}{h}\ l^2 \quad h^2}$

$\tan\theta = \frac{\sqrt{gh}}{l\sqrt{g}}\ \sqrt{h} = \frac{h}{l}$

1.31. An aeroplane P initially located at O moves along the path OB inclined at 30° with the horizontal with a constant velocity of 30 m/s. At the same time a bomber Q moving from left to right with a constant horizontal velocity of 126 m/s at a certain height above the ground release a bomb which hits the plane P at B (see Fig. 1.22). Find the height of the bomber Q assuming it was 500 m behind P when it released the bomb.

Solution: The time of journey for the path QB is the same as that of path OB, i.e.,

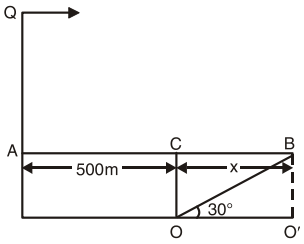


Fig. 1.22

$\frac{500 \quad x}{126} = \frac{x}{30\cos 30} = \frac{x}{15\sqrt{3}}$

or $500 \times 15\sqrt{3} + 15\sqrt{3}\, x = 126\, x$

or $500 \times 15\sqrt{3} = (126 - 26)\, x\ ;\ 15\sqrt{3} = 26$

or $x = 75\sqrt{3}$

Let t be the time taken by the aeroplane in going from O to B. Then we have

$x = 75\sqrt{3} = 30 \cos 30 \times t = 15\sqrt{3}\, t$

$\therefore\ t = 5$ s

Height $AQ = \frac{1}{2}\ gt^2 = \frac{1}{2} \times 9.8 \times 25 = 122.5$ m

Height $O'B = 30 \sin 30 \times t = 30 \times \frac{1}{2} \times 5 = 75$ m

Height of the bomber above the ground = 197.5 m

1.32. A batsman lifts a ball almost from the ground with an aim of providing the maximum horizontal range to it. At the same time a fielder 40 m away from the batsman runs at the speed of 10 m/sec along the line of the hit and catches the ball. Find the velocity of the ball just after it is struck, $g = 10$ m/s^2.

Solution: *First part* : When the fielder is going away from the batsman.

The time taken by the ball to traverse path AD = time taken by the fielder to move distance $BD = t$ (see Fig. 1.23).

Let u be the velocity of the ball at the hit. Then $40 + 10\,t = u \cos 45 \times t$

or $40 + 10\,t = \dfrac{u}{\sqrt{2}}\,t.$

Moreover t will be the time of flight, i.e.,

$$t = \frac{2u \sin 45}{g} = \frac{u\sqrt{2}}{10}$$

On substituting the value of t in the above equation, we have

$$40 + u\sqrt{2} = \frac{u^2}{10}$$

or $u^2 - 10\sqrt{2} \cdot u - 400 = 0$

$$\text{or } u = \frac{10\sqrt{2} \quad \sqrt{200 \quad 1600}}{2}$$

$$= \frac{14.14 \quad \sqrt{1800}}{2}$$

$$= \frac{14.14 \quad 42.43}{2} = 28.28 \text{ m/s}$$

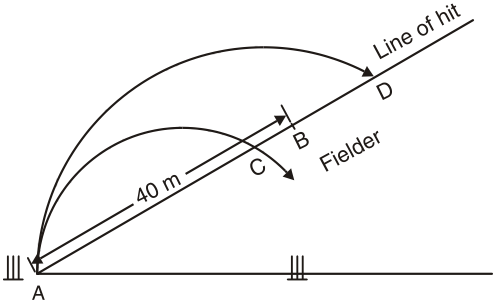


Fig. 1.23

Second part : When the fielder is coming close towards the batsman.

Time taken by the ball to traverse path AC = time taken by the fielder to travel distance $BC = t$

Let v be the velocity of the ball when hit. Then

$40 - 10\,t = v \cos 45\; t = \dfrac{v}{\sqrt{2}}\,t$; here $t = \dfrac{v\sqrt{2}}{10}$ exactly as above.

On substituting the value of t we have

$$v^2 + 10\sqrt{2}v - 400 = 0$$

$$\text{or } v = \frac{10\sqrt{2} \quad \sqrt{200 \quad 1600}}{2}$$

$$\text{or } v = \frac{14.14 \quad 42.43}{2} = 14.14 \text{ m/s}$$

1.33. A ball is thrown from the point O at an angle θ to the horizontal with an initial velocity of v_0. Find (i) range (ii) maximum height (iii) time of flight (iv) equation of trajectory in the coordinates x' and y' (v) $\dfrac{dv}{dt}$ and

$\frac{d}{dt}\langle v\rangle$ at the apex of the trajectory (vi) radius of curvature of the trajectory at point O and O'.

Solution:

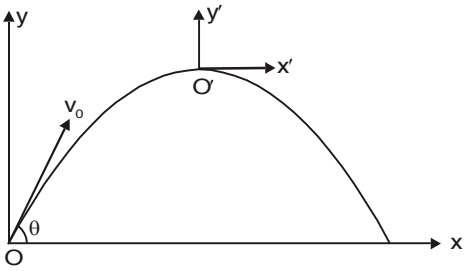


Fig. 1.24

(i) $\frac{v_0^2 \sin 2\theta}{g}$

(ii) $\frac{v_0^2 \sin^2 \theta}{2g}$

(iii) $\frac{2v_0 \sin\theta}{g}$

(iv) $y = x\tan\theta - \frac{1}{2}\, g \cdot \frac{x^2}{v_0^2 \cos^2\theta}$

$$x = \frac{v_0 \cos\theta \cdot v_0 \sin\theta}{g} + x' \qquad \text{and} \qquad y = y' + \frac{v_0^2 \sin^2\theta}{2g}$$

$$\left(y' + \frac{v_0^2 \sin^2\theta}{2g}\right) = \left(v_0 \cos\theta \cdot \frac{v_0}{g} \sin\theta + x'\right) \tan\theta - \frac{g}{2v_0^2 \cos^2\theta} \left[v_0 \cos\theta \cdot \frac{v_0 \sin\theta}{g} + x'\right]^2$$

$$\text{or } y' = -\frac{g}{2v_0^2 \cos^2\theta}\, x'^2$$

(v) $\frac{dv}{dt} = 0,\ \frac{d}{dt}(\langle v\rangle) = 0$

(vi) $y = x\tan\theta - \frac{gx^2}{2v_0^2 \cos^2\theta}$

$$\frac{dy}{dx} = \tan\theta - \frac{gx}{v_0^2 \cos^2\theta}$$

At O, $x = 0$, $y = 0$

$$\therefore \frac{dy}{dx} = \tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{g}{v_0^2 \cos^2\theta}$$

$$\frac{1}{\rho} = \frac{g \big/ v_0^2 \cos^2\theta}{\left[1 + \tan^2\theta\right]^{3/2}}$$

$$\rho \text{ at } O = \frac{v_0^2}{g\cos\theta}$$

$$\frac{dy}{dx} = \frac{-gx'}{v_0^2\cos^2\theta} = 0 \text{ at } O'$$

$$\frac{d^2y}{dx^2} = \frac{-g}{v_0^2\cos^2\theta}$$

$$\therefore \quad \rho' = -\frac{v_0^2\cos^2\theta}{g}$$

1.34. An object is situated on a hill h metres high has an elevation β. In order to hit this object show that the initial velocity of projection should not be less than $\sqrt{gh(1+\text{cosec}\,\beta)}$.

Solution: Let O be the point from which an object is projected with an initial velocity u in order to hit the object positioned at A. In view of Fig. 1.25, we have

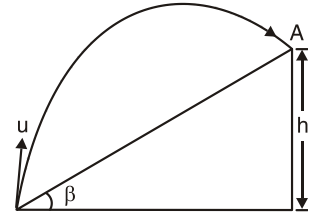


Fig. 1.25

$$OA = h\,\text{cosec}\,\beta$$

Now if u is minimum then OA will be the maximum range, i.e.,

$$h\,\text{cosec}\,\beta = OA = \frac{u^2}{g(1+\sin\beta)}$$

or $$u = \sqrt{gh(1+\text{cosec}\,\beta)}$$

1.35. Two particles A and B starts moving from a high point O at $t = 0$ in the opposite direction with horizontal velocities of 3 m/s and 4 m/s respectively. Due to earth's gravitational field the two particles experiences uniform acceleration $g = 9.8$ m/s^2. How far apart are the particles when their velocity vectors become mutually perpendicular.

Solution: Let us assume that $v_1 = 3$ m/s and $v_2 = 4$ m/s.

In view of the geometry of the Fig. 1.26 we have the following:

$$\theta_1 + \theta_2 = \frac{\pi}{2}$$

$$\sin\theta_1 = \frac{gt}{\sqrt{g^2t^2 \quad v_1^2}}$$

$$\sin\theta_2 = \frac{gt}{\sqrt{g^2t^2 \quad v_2^2}}$$

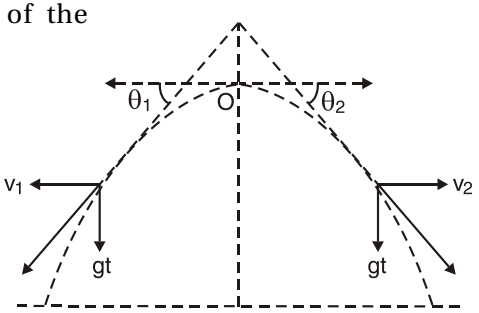


Fig. 1.26

$$\cos\theta_1 = \frac{v_1}{\sqrt{g^2t^2 \quad v_1^2}}$$

$$\text{and } \cos\theta_2 = \frac{v_2}{\sqrt{g^2t^2 \quad v_2^2}}$$

$$\text{Now } \sin(\theta_1 + \theta_2) = \sin 90 = 1 = \frac{gtv_2 \quad gtv_1}{\sqrt{g^2t^2 \quad v_1^2 \quad g^2t^2 \quad v_2^2}}$$

$$\text{or} \quad g^2t^2 \quad v_1^2 \quad g^2t^2 \quad v_2^2 = g^2t^2 \quad v_1^2 \quad v_2^2 + 2g^2t^2v_1v_2$$

$$\text{or} \quad g^4t^4 + g^2t^2 \quad v_1^2 \quad v_2^2 + v_1^2 \, v_2^2 = g^2t^2 \quad v_1^2 \quad v_2^2 + 2g^2t^2v_1v_2$$

$$\text{or} \quad g^4t^4 - 2g^2t^2v_1v_2 + v_1^2 \, v_2^2 = 0$$

$$\text{or} \quad (g^2t^2 - v_1v_2)^2 = 0$$

$$\text{or} \quad g^2t^2 = v_1v_2$$

$$\text{or} \quad t = \frac{\sqrt{v_1v_2}}{g}$$

Separation between the particles = $(v_1 + v_2)\ t = \frac{v_1 \quad v_2}{g}\sqrt{v_1v_2}$ at time t

On substituting the values of v_1, v_2 and g we have

$$\text{Separation between the particles} = \frac{(3 \quad 4)\sqrt{12}}{9.8}$$

$$= \frac{2\sqrt{3}}{1.4}$$

$$= \frac{1.73}{0.7} = 2.47 \text{ m}$$

1.36. Two successive shells fired with velocity $v = 200$ m/s from a point at angles $\theta_1 = 60°$ and $\theta_2 = 30°$ to the horizontal collide in space. Assuming the air drag to be negligible. Calculate the lapse of time between the two firings.

Solution: Let t_1 and t_2 be the time taken by the shells fired at angles θ_1 and θ_2 respectively. In view of Fig. 1.27, we have

$$l = v\cos\theta_1 t_1 = v\cos\theta_2 t_2$$

and

$$h = v\sin\theta_1 t_1 - \frac{1}{2}\, g t_1^2$$

$$= v\sin\theta_2 t_2 - \frac{1}{2}\, g t_2^2$$

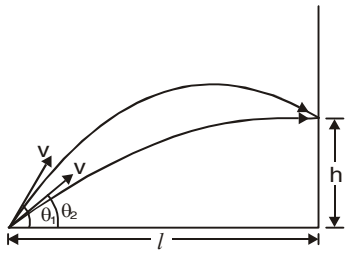


Fig. 1.27

In the light of the second equation, we have

$$v(\sin\theta_1 t_1 - \sin\theta_2 t_2) = \frac{1}{2} g \quad t_1^2 \quad t_2^2$$

$$\text{or} \quad v\left[\sin\theta_1 \cdot \frac{l}{v\cos\theta_1} - \frac{\sin\theta_2 \cdot l}{v\cos\theta_2}\right] = \frac{1}{2} g(t_1 - t_2)\left[\frac{l}{v\cos\theta_1} + \frac{l}{v\cos\theta_2}\right]$$

$$\text{or} \quad \frac{\sin(\theta_1 - \theta_2)}{\cos\theta_1 \cdot \cos\theta_2} = \frac{1}{2} g \frac{(t_1 - t_2)(\cos\theta_1 + \cos\theta_2)}{v\cos\theta_1 \cdot \cos\theta_2}$$

$$\text{or} \quad \Delta t = t_1 - t_2 = \frac{2v\sin(\theta_1 - \theta_2)}{g(\cos\theta_1 + \cos\theta_2)}$$

On substituting the values of the various terms occurring in the above equation we get Δt as follows:

$$\Delta t = \frac{2 \quad 200 \quad \sin(60 \quad 30)}{9.8 \quad (\cos 60 \quad \cos 30)} = \frac{200}{4.9 \quad 1 \quad \sqrt{3}} = \frac{200}{4.9 \quad 2.73} = 14.9 \text{ sec.}$$

1.37. A triangular frame *OAB* with base angles θ and ϕ as shown in the Fig. 1.28 is located in *xy* plane. At what angle α a projectile be projected from *O* so as to graze the vertex *A* and fall at the base *B*.

Solution: For the frame we have.

$$\tan\theta + \tan\phi = \frac{y}{x} + \frac{y}{R \quad x}; \quad R = \text{Range}$$

$$y = AC$$

$$= \frac{yR}{x(R \quad x)}$$

For the projectile we have

$$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}; \; v = \text{velocity of projection.}$$

$$= x\tan\alpha\left[1 - \frac{gx}{2v^2\cos^2\alpha \cdot \tan\alpha}\right]$$

$$= x\tan\alpha \quad 1 \quad \frac{x}{R}$$

$$y = x\tan\alpha \quad \frac{R \quad x}{R}$$

$$\tan\alpha = \frac{yR}{x(R \quad x)} = \tan\theta + \tan\phi$$

$$\therefore \; \alpha = \tan^{-1}[\tan\theta + \tan\phi]$$

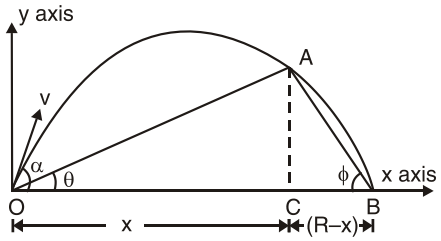


Fig. 1.28

1.38. A particle is projected from the top of a tower of height H at an angle θ with the horizontal. It strikes the ground at A lying at a distance D from the foot of the tower (see Fig. 1.29). Calculate the maximum height attained by the particle.

Solution: Let t be the time of flight.

$t \cdot v \cos\theta = D$;

v = velocity of projection.

$$\therefore\ t = \frac{D}{v\cos\theta}$$

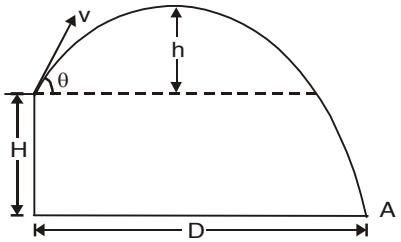


Fig. 1.29

Further

$$-H = v\sin\theta \cdot t - \frac{1}{2} gt^2$$

or $$-H = \frac{v\sin\theta \cdot D}{v\cos\theta} - \frac{1}{2} g \cdot \frac{D^2}{v^2\cos^2\theta}$$

or $$-H = D\tan\theta - \frac{1}{2} g \frac{D^2}{v^2\cos^2\theta}$$

or $$\frac{gD^2}{2v^2\cos^2\theta} = H + D\tan\theta$$

Now $v^2\sin^2\theta = 2gh$; h is the maximum height above the tower

$$\therefore\ h = \frac{v^2\sin^2\theta}{2g} = \frac{\sin^2\theta}{2g} \times \frac{g D^2}{2(H + D\tan\theta)\cdot\cos^2\theta}$$

$$= \frac{D^2\tan^2\theta}{4(H + D\tan\theta)}$$

$$\therefore \text{ Maximum height} = H + \frac{D^2\tan^2\theta}{4(H + D\tan\theta)}$$

1.39. Two walls AB & CD each of height h are at a distance $2h$ from each other (see Fig. 1.30). A particle projected from the point O with a velocity $v = 2\cdot\sqrt{gh}$ just clears the two walls and strike the ground at E. Calculate the time taken by the projectile in passing between the two walls.

Solution: Let us assume that the projectile attains a height h in time t.

$$h = 2\cdot\sqrt{gh}\ \cdot\sin\theta\cdot t - \frac{1}{2} gt^2$$

This is a quadratic equation in t, therefore it will have two roots t_1 and t_2. Now

$$t_1 + t_2 = \frac{4\cdot\sqrt{gh}}{g}\ \sin\theta$$

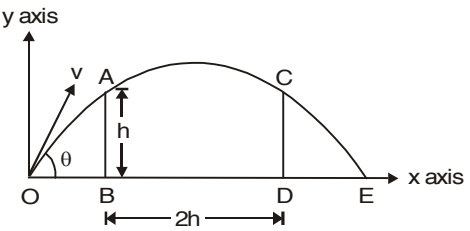


Fig. 1.30

and $t_1 t_2 = 2\,\frac{h}{g}$

Now $(t_2 - t_1)^2 = (t_1 + t_2)^2 - 4\,t_1\,t_2$

$$(t_2 - t_1)^2 = \frac{16h}{g}\,\sin^2\theta - \frac{8h}{g}$$

$$(t_2 - t_1) = \left[\frac{8h}{g}(2\sin^2\theta - 1)\right]^{\frac{1}{2}}$$

Further $2\cdot\sqrt{gh}\cdot\cos\theta \times (t_2 - t_1) = 2h$

or $$t_2 - t_1 = \sqrt{\frac{h}{g}}\cdot\frac{1}{\cos\theta}$$

$$\therefore\ \sqrt{\frac{h}{g}}\cdot\frac{1}{\cos\theta} = \left[\frac{8h}{g}\left(2\sin^2\theta - 1\right)\right]^{\frac{1}{2}}$$

$$\frac{h}{g\cos^2\theta} = \frac{8h}{g}\,(2\sin^2\theta - 1)$$

$$1 = 16\sin^2\theta\cos^2\theta - 8\cos^2\theta$$

$$1 = 16\,(1 - \cos^2\theta)\cos^2\theta - 8\cos^2\theta$$

$$1 = 16\cos^2\theta - 16\cos^4\theta - 8\cos^2\theta$$

$$16\cos^4\theta - 8\cos^2\theta + 1 = 0$$

or $$(4\cos^2\theta - 1)^2 = 0$$

or $$\cos^2\theta = \frac{1}{4}$$

$$\sin^2\theta = \frac{3}{4}$$

With these values

$$t_2 - t_1 = \left[\frac{8h}{g}\left(\frac{2\times 3}{4} - 1\right)\right]^{\frac{1}{2}} = \sqrt{\frac{4h}{g}} = 2\cdot\sqrt{\frac{h}{g}}$$

1.40. A ball is projected in xy plane with a velocity v_0 and at an angle θ with the horizontal. During motion it experiences an air resistance equal to $+\beta v$ where β is a positive constant and v is the instantaneous velocity. Find (i) the time taken to attain the highest point (ii) the maximum height attained by the ball.

Solution:

(i) The equation of motion of the ball along y axis is given by

$$\frac{mdv}{dt} = -(mg + \beta v)$$

$$\text{or} \qquad -\int_{v_0 \sin\theta}^{0} \frac{m dv}{mg + \beta v} = \int_0^t dt$$

$$\frac{m}{\beta} \log\left[1 + \frac{\beta v_0 \sin\theta}{mg}\right] = t$$

(ii) $\dfrac{m dv}{dt} = -(mg + \beta v)$

$$\text{or} \qquad m \cdot \frac{dv}{dh} \cdot \frac{dh}{dt} = -(mg + \beta v)$$

$$\text{or} \qquad -\int_{v_0 \sin\theta}^{0} \frac{m \cdot v dv}{(mg + \beta v)} = \int_0^h dh$$

$$\text{or} \qquad h = \frac{m v_0 \sin\theta}{\beta} - \frac{m^2 g}{\beta^2} \log\left[1 + \frac{\beta v_0 \sin\theta}{mg}\right]$$

1.41. A particle of mass m is projected vertically upward with a velocity v_0 in a medium whose resistance is kv^2. Determine the time at which the particle will come to rest.

Solution: The equation of motion of the particle is given by

$$\frac{m dv}{dt} = -(mg + kv^2)$$

$$\text{or} \qquad -\int_{v_0}^{0} \frac{m dv}{mg \quad kv^2} = \int_0^t dt$$

$$= \frac{m}{k} \int_0^{v_0} \frac{dv}{\frac{mg}{k} \quad v^2} = \int_0^t dt$$

$$t = \frac{m}{k} \sqrt{\frac{k}{mg}} \cdot \tan^{-1} v_o \sqrt{\frac{k}{mg}}$$

$$= \sqrt{\frac{m}{kg}} \cdot \tan^{-1} v_o \sqrt{\frac{k}{mg}}$$

1.42. At what initial speed and at what angle with the horizontal a particle be projected in order to attain a maximum height H and horizontal range R?

Solution: Let the particle be projected at an angle θ with the horizontal with an initial speed v.

$$H = \frac{v^2 \sin^2\theta}{2g} \text{ and } R = \frac{2v^2 \sin\theta \cos\theta}{g}$$

$$R^2 = \frac{4v^4}{g^2}\left[\sin^2\theta\left(1-\sin^2\theta\right)\right]$$

$$= \frac{4v^4}{g^2}\left[\sin^2\theta - \sin^4\theta\right]$$

$$= \frac{4v^4}{g^2}\left[\frac{2Hg}{v^2} - \frac{4H^2g^2}{v^4}\right] = \frac{8Hv^2}{g} - 16H^2$$

$$\therefore \quad v = \sqrt{\frac{g\ R^2\ \ 16H^2}{8H}}$$

$$\sin^2\theta = \frac{2gH}{v^2} = \frac{2gH\cdot 8H}{g\left(R^2+16H^2\right)} = \frac{16H^2}{R^2+16H^2}$$

$$\sin\theta = \frac{4H}{\sqrt{R^2\ \ 16\,H^2}}$$

or $$\theta = \sin^{-1}\ \frac{4H}{\sqrt{R^2\ \ 16\,H^2}}$$

1.43. A particle is projected up an incline plane of inclination ϕ (see Fig. 1.31). At what elevation θ to the horizontal the particle will strike the plane at right angles.

Solution: Time of flight for path *OA* is given by

$$t = \frac{2v\sin(\theta-\phi)}{g\cos\phi}$$

Initial velocity of the particle = $v\cos(\theta-\phi)$

Final velocity at $A = 0 = v\cos(\theta-\phi) - g\sin\phi\cdot t$.

The particle is striking the plane at *A* at 90°.

$$\therefore\ t = \frac{v\cos(\theta-\phi)}{g\sin\phi}$$

$$\therefore\ \frac{2v\sin(\theta-\phi)}{g\cos\phi} = \frac{v\cos(\theta-\phi)}{g\sin\phi}$$

or $2\tan(\theta-\phi) = \cot\phi$

$$2\left[\frac{\tan\theta-\tan\phi}{1+\tan\theta\cdot\tan\phi}\right] = \cot\phi$$

or $2\tan\theta - 2\tan\phi = \cot\phi + \tan\theta$

$\tan\theta = \cot\phi + 2\tan\phi$

or $\theta = \tan^{-1}(\cot\phi + 2\tan\phi]$

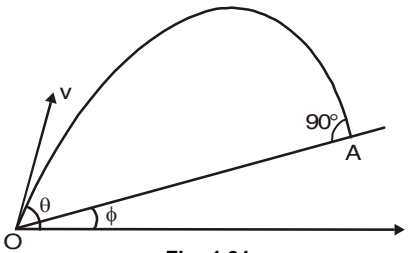


Fig. 1.31

1.44. A projectile is fired from point O with initial velocity v and angle of elevation θ along an incline plane of inclination ϕ. For what value of θ the range on the incline plane is maximum.

Solution: In Fig. 1.32 the coordinates of point A is given by

$$x = v\cos\theta \cdot t = r\cos\phi$$

$$y = v\sin\theta \cdot t - \frac{1}{2} gt^2 = r\sin\phi;\ r = \text{range } OA$$

$$r\sin\phi = v\sin\theta \cdot \frac{r\cos\phi}{v\cos\theta} - \frac{1}{2} g \cdot \frac{r^2\cos^2\phi}{v^2\cos^2\theta}$$

$$\tan\phi = \tan\theta - \frac{gr\cos\phi}{2v^2\cos^2\theta}$$

$$\therefore r = (\tan\theta - \tan\phi) \cdot \frac{2v^2\cos^2\theta}{g\cos\phi}$$

Now $$\frac{dr}{d\theta} = \frac{2v^2}{g\cos\phi} [\sec^2\theta \cdot \cos^2\theta + \tan\theta \times -2\cos\theta\sin\theta - \tan\phi \times -2\cos\theta \cdot \sin\theta] = 0$$

or $1 - 2\sin\theta\cos\theta(\tan\theta - \tan\phi) = 0$

or $$\tan\theta - \tan\phi = \frac{1}{2\sin\theta\cos\theta}$$

or $$\tan\theta - \frac{1}{2\sin\theta\cos\theta} = \tan\phi$$

or $-\cot 2\theta = \tan\phi$

or $$\theta = \frac{\pi}{4} + \frac{\phi}{2}$$

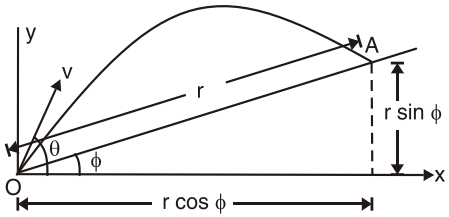


Fig. 1.32

1.45. A bullet of mass M is fired with a velocity of 50 m/sec. at an angle θ with the horizontal. At the highest point of its trajectory it collides head on with a bob of mass 3M suspended by a massless string of length 10/3 metres and gets embedded in the bob. After the collision the string moves through an angle of 120°. Find (i) the angle θ (ii) the vertical and horizontal co-ordinates of the initial position of the bob with respect to the point of firing the bullet (take $g = 10$ m/sec^2). (I.I.T. 1988)

Solution: At the point P (see Fig. 1.33), $u_x = x$ component of velocity $= u\cos\theta$ and $u_y = 0$.

According to conservation of momentum we have

$M \cdot u_x = Mu\cos\theta = (M + 3M)\ v$;

v = final velocity acquired by the bob + bullet

According to conservation of energy

$\frac{1}{2} \cdot (4M)\ v^2 = 4M \cdot gh = 4Mg\ (l + l \sin 30)$

$v^2 = 2gh = 2g \cdot (l + l \sin 30)$

$v^2 = 3gl$

Now $u^2 \cos^2 \theta = 4^2 \cdot 3gl$

$\cos^2 \theta = \frac{16 \quad 3 \quad 10 \quad 10/3}{50^2}$

$\cos \theta = \frac{40}{50} = \frac{4}{5}$

$\theta = \cos^{-1} \frac{4}{5} = 37°$

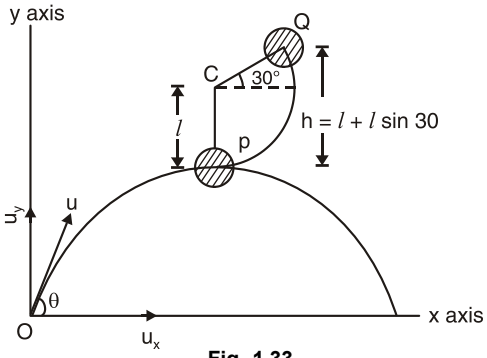


Fig. 1.33

(ii) Time of ascent $t = \frac{u \sin \theta}{g} = \frac{50 \times 0.6}{10} = 3$ sec

$x = u \cos \theta \cdot t$

$= 50 \times \frac{4}{5} \times 3 = 120$ m

$y = u \sin \theta \cdot t - \frac{1}{2} gt^2$

$= 50 \times 0.6 \times 3 - \frac{1}{2} \times 10 \times 9$

$= 90 - 45$

$= 45$ m

1.46. A particle projected from the origin ($x = y = 0$) moves in a path lying in the xy plane with a velocity $V = 2i + 4xj$, where i and j are the unit vectors along x and y axes. Find the equation of motion of the particle.

Solution: $V = 2i + 4xj$

or $iV_x + jV_y = 2i + 4xj$

or $V_x = 2$ and $V_y = 4x$

We know that $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$; $\frac{dy}{dt} = 4x$ and $\frac{dx}{dt} = 2$.

On substituting these values, we have

$4x = \frac{dy}{dx} \cdot 2$

or $2xdx = dy$

On integrating the above, we get

$x^2 = y$

This is the required equation for the trajectory of the particle.

1.47. A lift starts going up with a uniform acceleration $a = 1.4$ m/s^2. After $t = 2.0$ s of its journey a bolt from its ceiling starts falling freely. Assume the floor-to-ceiling distance of the lift to be equal to 2.8 m, find the displacement and the distance travelled by the bolt.

Solution: As the lift is going up with an acceleration a, the acceleration of the bolt relative to the lift $= g + a$. Let the time of free fall of the bolt be t.

Then, $2.8 = \frac{1}{2}(g + a)t^2$

or

$$t = \sqrt{\frac{2 \quad 2.8}{9.8 \quad 1.4}} = \sqrt{\frac{5.6}{11.2}} = \sqrt{0.5} = 0.707 \text{ s}$$

The velocity of the lift after 2.0 s $= 1.4 \times 2 = 2.8$ m/s

Now the distance covered by the lift (Fig. 1.34)

$$S_2 = 2.8t + \frac{1}{2}at^2$$

$$= 2.8 \times 0.707 + \frac{1}{2} \times 1.4 \times 0.5$$

$$= 1.98 + 0.35 = 2.33 \text{ m}$$

Displacement of the bolt $S = 2.8 - 2.33 = 0.47$ m

On considering the motion of the bolt we have

$$0 = (2.8)^2 - 2 \times 9.8 \times S_1$$

or $S_1 = \frac{(2.8)^2}{2 \quad 9.8} = 0.4$ m

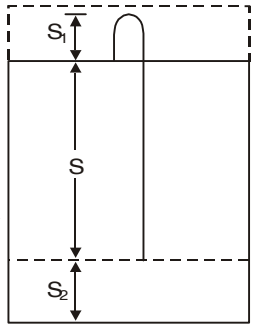


Fig. 1.34

Distance covered by the bolt $= 2S_1 + S = 2 \times 0.4 + 0.47 = 1.27$ m

1.48. A particle moves along a straight line with an acceleration $a = 6\sqrt[3]{s}$. At $t = 2$ sec its displacement is 27 m and its velocity is 27 m/sec. Find the velocity and acceleration at $t = 4$ sec.

Solution:
$$\frac{dv}{dt} = 6s^{\frac{1}{3}}$$

$$v\,dv = 6s^{\frac{1}{3}}\,ds$$

On integration

$$v^2 = 9s^{\frac{4}{3}} + C$$

$$27^2 = 9 \cdot (27)^{\frac{4}{3}} + C$$

$$\therefore C = 0$$

Now
$$v = 3s^{\frac{2}{3}}$$

$$\frac{ds}{dt} = 3\,s^{\frac{2}{3}}$$

$$s^{-\frac{2}{3}}\,ds = 3\,dt$$

On integration we have

$$s^{\frac{1}{3}} = t + C_1$$

At $t = 2$ sec $s = 27$ m

$$\therefore\ C_1 = 1$$

$$\therefore\ s^{\frac{1}{3}} = t + 1$$

At $t = 4$ sec

$$s = (4 + 1)^3 = 125 \text{ m}$$

$$v = 3 \times (125)^{\frac{2}{3}} = 75 \text{ m/sec}$$

$$a = 6 \times (125)^{\frac{1}{3}} = 30 \text{ m/sec}^2$$

1.49. A particle moves along a verticle line with an acceleration $a = 2 \cdot \sqrt{v}$. At $t = 2$ sec its displacement $s = \frac{64}{3}$ m and its velocity $v = 16$ m/sec. Find displacement, velocity and acceleration at $t = 3$ sec.

Solution: $\frac{dv}{dt} = 2\,v^{\frac{1}{2}}$

or $v^{-\frac{1}{2}}\,dv = 2\,dt$

On integration

$$\sqrt{v} = t + C$$

At $t = 2$ sec $v = 16$ m/sec

$$\therefore\ C = 2$$

$$\therefore\ \sqrt{v} = t + 2$$

Now $v = \frac{ds}{dt} = t^2 + 4t + 4$

$$ds = t^2 \cdot dt + 4t \cdot dt + 4 \cdot dt$$

On integration

$$s = \frac{t^3}{3} + 2t^2 + 4t + C_1$$

$$\frac{64}{3} = \frac{8}{3} + 8 + 8 + C_1$$

$$C_1 = \frac{8}{3}$$

Therefore $s = \frac{t^3}{3} + 2t^2 + 4t + \frac{8}{3}$

At $t = 3$ sec

$$s = 27 + 18 + 12 + \frac{8}{3}$$
$$= 41.7 \text{ m}$$
$$v = (3+2)^2 = 25 \text{ m/sec}$$
$$a = 2 \cdot \sqrt{25} = 10 \text{ m/sec}^2$$

1.50. A particle initially at rest completed $N = 1.5$ revolutions around a circle of radius $R = 5$ m at a constant tangential acceleration during time $\tau = 10$ sec. Calculate (i) the mean value of speed $\langle v \rangle$ (ii) the magnitude of the mean velocity $|\langle V \rangle|$ (iii) the magnitude of the mean acceleration $|\langle a \rangle|$.

Solution:

(i) $\langle v \rangle = \frac{\text{Total distance travelled}}{\text{Total time}} = \frac{1.5 \times 2\pi R}{\tau} = \frac{3\pi R}{\tau} = 4.7 \text{ m/sec}$

(ii) $|\langle V \rangle| = \frac{\text{Displacement}}{\text{Time}} = \frac{2R}{\tau} = 1 \text{ m/sec}$

(iii) $|\langle a \rangle| = \frac{\text{Change in velocity}}{\text{Total time}} = \frac{2V}{\tau} = \frac{2}{\tau} \cdot \frac{2\pi RN}{\tau}$

$$= \frac{6\pi R}{\tau^2} = 0.942 \text{ m/sec}^2.$$

1.51. A disc D and a fix vertical cylinder C of radius $R = 0.25$ m connected through a horizontal thread CD of length $l = 2$ m lies in a horizontal plane. On imparting an initial velocity of $v = 2$ m/s to the disc it starts moving on the path shown in the Fig. 1.35a. What time will the disc take to strike the cylinder ?

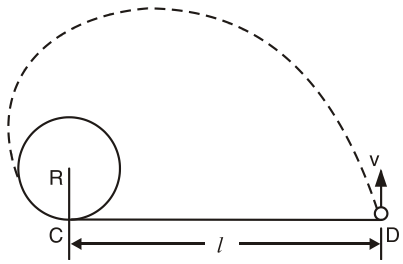


Fig. 1.35a

Solution: As the thread moves it describes an angle. Let us assume θ to be the total angle described by the thread till the disc strikes the cylinder. Consider a section of the path AB described by the disc (see Fig. 1.35b). The length of the path

$AB = (l - R\phi)\, d\phi$

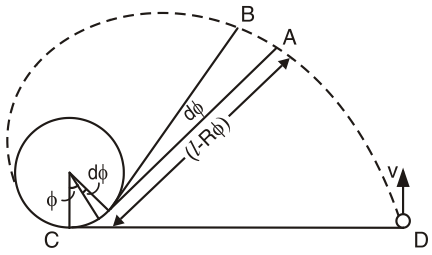


Fig. 1.35b

Total distance covered by the disc $= \int_0^{\theta} (l - R\phi)\, d\phi$

$= l\theta - \frac{R\theta^2}{2}$

Further, $l = R\theta$

$\therefore\ \theta = \frac{l}{R}$

On substituting the value of θ the total distance covered by the disc

$= \frac{l^2}{R} - \frac{l^2}{2R} = \frac{l^2}{2R}$

Time taken by the disc $t = \frac{\text{Length covered}}{\text{Velocity } v} = \frac{l^2}{2Rv}$

On substituting the values of various terms we have

$t = \frac{l^2}{2Rv} = \frac{4}{2 \quad 0.25 \quad 2} = 4\,\text{s}$

1.52. A ball, initially at rest, starts falling down a smooth plane inclined at angle $\alpha = 30°$ with the horizontal. After falling through the distance $h = 4$ m, the ball rebounds elastically off the plane (Fig. 1.36a). At what distance from the point of impact will the ball rebound for the second time ?

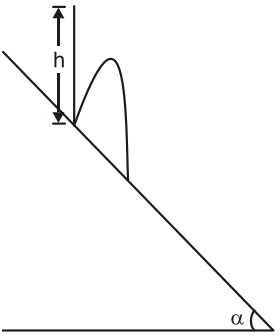


Fig. 1.36a

Solution: The ball strikes the inclined plane with a velocity $v = \sqrt{2gh}$ where h is the distance travelled by the ball in air. The initial velocity of the ball at the first impact is v. In view of the Fig. 1.36b, v makes an angle α with the y-axis. Velocity v will have two components, $v \sin\alpha = v_x$ along the plane and $v \cos\alpha = v_y$ perpendicular to the plane.

Let l be the distance travelled along the plane between the first and second impact.

Then, according to the laws of motion, $l = v \sin\alpha \cdot t + \frac{1}{2} \cdot g \sin\alpha \cdot t^2$; $g \sin\alpha$ is the component of acceleration due to gravity along the plane. Here t is the time of flight.

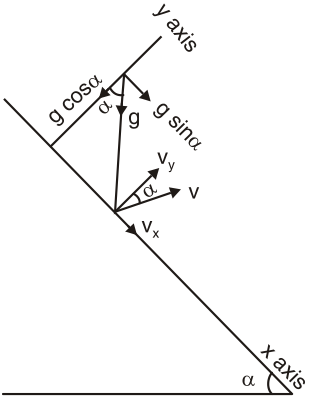


Fig. 1.36b

Further, $v \cos\alpha \cdot t - \frac{1}{2} g \cos\alpha \cdot t^2 = 0$

$$\therefore t = \frac{2v}{g}$$

With this value of t, $l = \frac{2v^2 \cdot \sin\alpha}{g} + \frac{1}{2} g \cdot \sin\alpha \cdot \frac{4v^2}{g^2}$

$$= \frac{4v^2 \sin\alpha}{g} = \frac{4}{g} \sin\alpha \times 2gh$$

$$= 8h \sin\alpha$$

On substituting the value of $\alpha = 30°$ and $h = 4$ m, we have

$$l = 8 \times 4 \times \frac{1}{2} = 16 \text{ m}$$

1.53. A particle is projected from a point P on the earth with a velocity v_0 at an angle of 45° with the horizontal at any instant $t = 0$. Find out the magnitude and direction of the angular momentum of the particle about the point P at the instant $t = \frac{v_0}{g}$. (IIT, 1984)

Solution: The particle will describe a parabolic path in the earth's gravitational field (see Fig. 1.37). After a lapse of time $t = \frac{v_0}{g}$ the particle will be at some point $Q(x, y)$. The two components of velocities at Q are given by

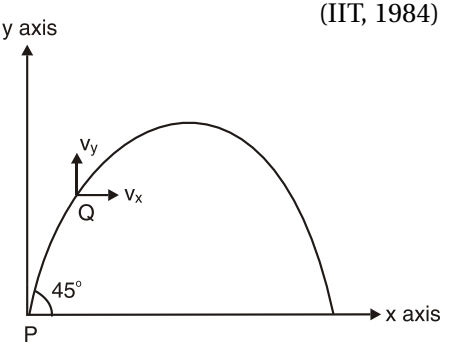


Fig. 1.37

$$v_x = v_0 \cos 45 = \frac{v_0}{\sqrt{2}}$$

and $v_y = v_0 \sin 45 - gt = v_0 \sin 45 - v_0$; since $\frac{v_0}{g} = t$

$$= \frac{v_0}{\sqrt{2}} \quad 1 \quad \sqrt{2}$$

The position of point Q is given by

$$x = v_0 \cos 45 \times \frac{v_0}{g} = \frac{v_0^2}{g\sqrt{2}} \text{ and}$$

$$y = v_0 \sin 45 \times \frac{v_0}{g} - \frac{1}{2} g \quad \frac{v_0}{g}^2$$

$$= \frac{v_0^2}{2g} \quad \sqrt{2} \quad 1$$

The angular momentum is defined as

$$L = m\mathbf{r} \times \mathbf{v}$$

$$= m\left[(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) \times (\mathbf{i}v_x + \mathbf{j}v_y + \mathbf{k}v_z)\right]$$

$$= m\left[\mathbf{i}(y \cdot v_z - v_y \cdot z) + \mathbf{j}(v_x \cdot z - x \cdot v_z) + \mathbf{k}(v_y \cdot x - v_x \cdot y)\right]$$

Since the particle is moving in the xy plane the only component of L that will exist is $m\,(x \cdot v_y - y \cdot v_x)$ along the z axis.

$$\therefore L_z = m\,(x \cdot v_y - y \cdot v_x) = m\left[\frac{v_0^2}{g\sqrt{2}}\frac{v_0}{\sqrt{2}}\left(1-\sqrt{2}\right) - \frac{v_0^2}{2g}\left(\sqrt{2}-1\right)\cdot\frac{v_0}{\sqrt{2}}\right]$$

$$= \frac{mv_0^3}{2g\sqrt{2}} \quad \sqrt{2} \quad 2 \quad \sqrt{2} \quad 1 \quad = \frac{mv_0^3}{2g\sqrt{2}}$$

The negative sign indicates that the angular momentum will be along the negative side of z axis.

1.54. A body projected with an initial velocity $v = 50$ m/s from the point O at an angle θ to the horizontal is hurled at a horizontal disance $l = 125$ m from the edge of a platform and hits the ground at a distance x from the vertical edge (see Fig. 1.38). At what value of θ, will x be minimum, given $g = 10$ m/s^2.

Solution: In view of Fig. 1.38, x will be minimum only when l corresponds to the maximum horizontal range, i.e.,

$$l = \frac{v \cdot \cos\theta \cdot 2v \sin\theta}{g};\ \frac{2v \sin\theta}{g} = \text{time of flight}$$

$$l = \frac{v^2}{g} \sin 2\theta$$

Now l will be maximum with larger value of θ

$$\sin 2\theta = \frac{lg}{v^2}$$

$$\therefore \theta = \frac{1}{2} \sin^{-1} \frac{lg}{v^2}$$

$$= \frac{1}{2} \sin^{-1} \frac{125 \quad 10}{50 \quad 50}$$

$$= \frac{1}{2} \sin^{-1}(0.5) = 15° \text{ or } 75°$$

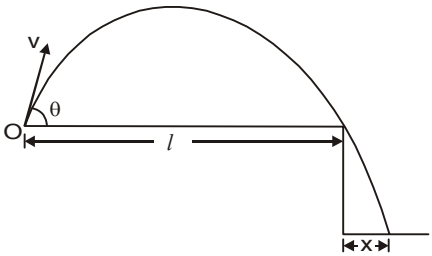


Fig. 1.38

The maximum value of θ is thus 75°.

1.55. Two identical spherical balls are simultaneously thrown towards each other from points P and Q horizontally separated by 8 m, and situated at heights 4 and 8 m above the ground. One ball is thrown from P

horizontally with a speed of 8 m/s while the other is thrown downward with an initial speed of v at an angle of 45° to the horizontal (Fig. 1.39). The two balls collide in space. Calculate (i) the initial speed of ball thrown from the point Q, (ii) coordinates of the point of collision and (iii) velocities of the two balls before collision. $g = 10$ m/s^2.

Solution: The paths of the two balls are shown in Fig. 1.39. The equation of motion of ball projected from P is given by

$$8t = x \text{ and } 4 - y = \frac{1}{2} gt^2;\ t = \text{time taken}$$

Similarly the equation of motion of the ball projected from Q is given by

$$v \cos 45 \cdot t = (8 - x) \text{ and}$$

$$8 - y = v \sin 45 \cdot t + \frac{1}{2} gt^2$$

$$\text{Now } 8 - 4 + \frac{1}{2} gt^2 = \frac{vt}{\sqrt{2}} + \frac{1}{2} gt^2$$

$$\text{or } 4\sqrt{2} = vt$$

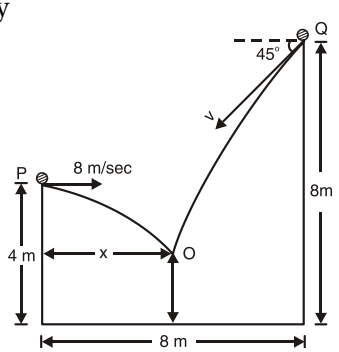


Fig. 1.39

On substituting the value of vt in the equation of motion of the ball projected from Q along the x axis, we have

$$\frac{4\sqrt{2}}{\sqrt{2}} = 8 - x$$

$$\therefore x = 4 \text{ m}$$

Now $8t = x$ is the equation of motion of the ball thrown from P along the x axis. On substituting the value of x in this equation we have

$$t = \frac{x}{8} = 0.5 \text{ s}$$

Further $vt = 4\sqrt{2}$

$$\therefore v = \frac{4\sqrt{2}}{0.5} = 8 \times 1.41 = 11.28 \text{ m/s}$$

Now $4 - y$ is the coordinate of the point of collision O with respect to P.

$$\therefore 4 - y = \frac{1}{2} \times 10 \times 0.25 = 5 \times 0.25 = 1.25 \text{ m (downward)}$$

Thus the coordinate of point of collision O is (4, 2.75)

v_P = velocity of ball projected from P at the point O (point of collision)

$$= \sqrt{64 \quad 20 \quad 1.25} = \sqrt{89} = 9.434 \text{ m/s}$$

v_Q = velocity of ball projected from Q at the point O (point of collision)

$= \sqrt{11.28\cos 45^{\circ\,2} \quad 11.28\sin 45^{\circ\,2} \quad 2 \quad 10 \quad 5.25}$

$= \sqrt{63.62 \quad 63.62 \quad 105}$

$= \sqrt{232.24}$

$= 15.24$ m/s

1.56. Three tortoises are located at the vertices of an equilateral triangle of side $a = 3$ m. They start moving simultaneously with constant velocity $v = 1$ m/s. The first heads for the second, the second for the third and the third for the first, *i.e.*, all the time they move in cyclic order. After what time will they meet ?

Solution: Let three tortoises be located at the corners of the triangle ABC (Fig. 1.40). A is moving towards B, B towards C and C towards A. In triangle ABC, $OA = OB = OC = \frac{2}{3}\sqrt{3} \cdot \frac{a}{2} = \frac{a}{\sqrt{3}}$

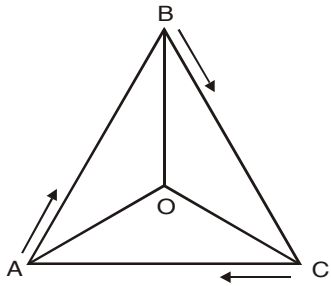


Fig. 1.40

The velocity of the tortoise at A can be resolved along AO and perpendicular to AO. Similarly the velocities of the tortoise at B and C can be resolved respectively along BO and perpendicular to BO, and CO and perpendicular to CO. The velocity of the tortoise along AO = velocity along BO = velocity along $CO = v\cos 30 = \frac{v\sqrt{3}}{2}$.

With this velocity, the three tortoise will meet at O.

Time taken by a tortoise $= t = \frac{\text{distance}}{\text{velocity}} = \frac{AO \text{ or } BO \text{ or } CO}{v\sqrt{3}/2}$

$= \frac{a}{\sqrt{3}} \times \frac{2}{v\sqrt{3}} = \frac{2a}{3v}$

On substituting the values of a and v we have

$t = \frac{2}{3} \times \frac{3}{1} = 2$s

2

PARTICLE DYNAMICS

2.1. Two blocks of mass 3 m and m are placed in contact on a frictionless horizontal table. A force F is applied to the block of mass 3 m as shown in the Fig. 2.1. Find the force of contact. What will be the force of contact if the force is applied on the block of mass m ?

Solution: Let a be the acceleration of the system as a whole. Then we have

$$F = (3m + m)\, a$$

or $$a = \frac{F}{4m}$$

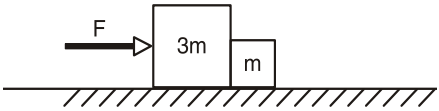


Fig. 2.1

Force of contact F_1 in this condition = Force exerted on block of mass m

or $$F_1 = m \times \frac{F}{4m} = \frac{F}{4} N$$

If the force is applied on the block of mass m, then the force of contact F_2 = foce exerted on block of mass $3m$.

or $$F_2 = 3m \times \frac{F}{4m} = \frac{3F}{4} N$$

2.2. A sphere of mass $m = \sqrt{3}$ g fitted on smooth rod of lenth $2l = 1$ m can slide freely along its length. Initially the rod containing the sphere at its middle is at rest. Now it executes a translational motion with a constant acceleration $a = 2$ m/s^2 making an angle $\alpha = 60°$ with it (see Fig. 2.2). Find: (i) the reaction force exerted by the rod on the sphere and (ii) the time at which the sphere will leave the rod.

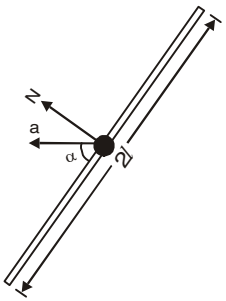


Fig. 2.2

Solution: (i) The force acting on the sphere is ma. The component of ma perpendicular to the length of the rod is the reaction force and is equal to $ma \sin \alpha$, i.e.,

$$N = m\, a \sin \alpha$$

$$= \sqrt{3} \times 10^{-3} \times 2 \times \frac{\sqrt{3}}{2} = 3 \times 10^{-3} \text{ N}$$

(ii) The component force $ma \cos \alpha$ will provide motion to the sphere along the length of the rod. The acceleration along the length of the rod = $a \cos \alpha$. Now, according to the laws of motion,

$$l = 0 + \frac{1}{2} a \cos \alpha \, t^2$$

where t is the time taken by the sphere to traverse a length l.

$$\therefore \; t = \sqrt{\frac{2l}{a \cos}}$$

On substituting the values we have

$$t = \sqrt{\frac{2 \quad 0.5}{2 \quad 0.5}} = 1 \text{ s}$$

2.3. Find the magnitude of the force F required to keep the mass m in equilibrium on an incline plane (see Fig. 2.3). The coefficient of friction is μ.

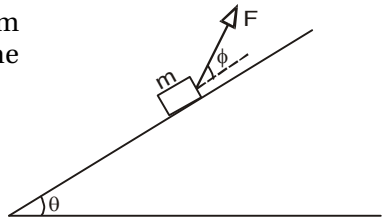


Fig. 2.3

Solution:

$$F \cos \phi + \mu \, mg \cos \theta = mg \sin \theta$$

$$\text{or } F = \frac{mg \left[\sin \theta - \mu \cos \theta \right]}{\cos \phi}$$

2.4. Two masses m_1 and m_2 connected through a massless and inextensible strings are placed on a smooth cylindrical surface as shown in Fig. 2.4a & b. Find the angle of equilibrium.

Solution: In Fig. 2.4a take moment about O.

$$m_2 gr \sin \theta = m_1 g \cdot r \cos \theta; \; r = \text{radius of cylinder}$$

$$\tan \theta = \frac{m_1}{m_2}$$

$$\text{or} \qquad \theta = \tan^{-1} \frac{m_1}{m_2}$$

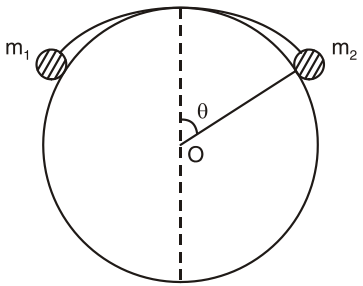


Fig. 2.4a

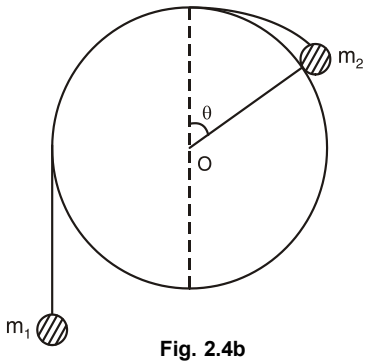


Fig. 2.4b

In Fig. 2.4b take moment about O.

$m_2 g.r \sin\theta = m_1 gr$

$\therefore \quad \sin\theta = \dfrac{m_1}{m_2}$

or $\quad \theta = \sin^{-1} \dfrac{m_1}{m_2}$

2.5. A particle moves along the path $r = 3\theta$. If the angle $\theta = t^2$ rad find the velocity of the particle when $\theta = 45°$.

Solution: $\theta = t^2, \ \dot{\theta} = 2t$

$r = 3t^2, \ \dot{r} = 6t$

$$v = \sqrt{\left(r\dot{\theta}\right)^2 + \dot{r}^2} = \sqrt{\left(3\theta \cdot 2\sqrt{\theta}\right)^2 + \left(6\sqrt{\theta}\right)^2}$$

$$= \sqrt{36\theta^3 + 36\theta} = \sqrt{36 \ \frac{3.14^3}{64} \ \frac{36 \ 3.14}{4}}$$

$$= \sqrt{17.4 \quad 28.26} = 6.75 \text{ m/sec}$$

2.6. Two incline planes marked as 1 and 2 coincides with the cords of a circle of radius R (see Fig. 2.5). An object slides down along each plane without any initial velocity. Show that the time of sliding along both the planes is same.

Solution: In view of the Fig. 2.5 length of plane 1 = $2R \sin\alpha$ and length of plane 2 = $2R \sin\beta$.

Equation of motion of the object along the plane is given by

$$2R \sin\alpha = \frac{1}{2} \ g \sin\alpha \cdot t_1^2$$

and $$2R \sin\beta = \frac{1}{2} \ g \sin\beta \cdot t_2^2$$

From the two equation we have $t_1 = t_2$.

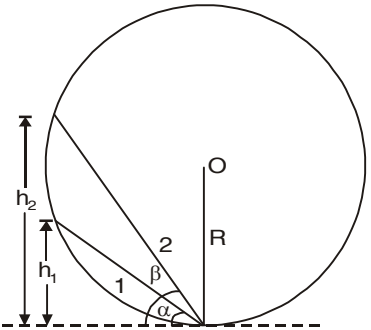


Fig. 2.5

2.7. A particle moves along a straight path and is acted upon by a resistive force which is proportional to the cube of the instantaneous speed. The initial speed is v_0 and after a lapse of time τ the speed is $\dfrac{v_0}{2}$. Find the speed after the time 5τ.

Solution: $\quad \dfrac{mdv}{dt} = -kv^3$

$$\int_{v_0}^{\frac{v_o}{2}} m\frac{dv}{v^3} = \int_0^{\tau} -kdt$$

$$\frac{m}{2}\left|\frac{1}{v^2}\right|_{v_0}^{\frac{v_o}{2}} = k\tau$$

$$\frac{m}{2}\cdot\frac{3}{v_0^2} = k\tau$$

Further $$\int_{v_0}^{V}\frac{mdv}{v^3} = \int_0^{5\tau} -kdt$$

$$\frac{m}{2}\left[\frac{1}{V^2}-\frac{1}{v_0^2}\right] = 5\,k\tau$$

or $$\frac{m}{2}\left[\frac{1}{V^2}-\frac{1}{v_0^2}\right] = 5\times\frac{m}{2}\times\frac{3}{v_0^2}$$

$$\frac{1}{V^2} = \frac{16}{v_0^2}$$

or $$V = \frac{v_0}{4}$$

2.8. A ball of mass m starts moving from rest from the point O along the track OCB (see Fig. 2.6). Find the reaction R on the ball at C if the curve OCB is defined by equation $y = h\sin\left(\frac{\pi x}{l}\right)$ where $h = \frac{l}{3}$.

Solution: $y = h\sin\left(\frac{\pi x}{l}\right)$

$$\frac{dy}{dx} = \frac{h\pi}{l}\cdot\cos\left(\frac{\pi x}{l}\right)$$

with $x = \frac{l}{2}$ i.e. at C, $\frac{dy}{dx} = O$

Now, $$\frac{d^2y}{dx^2} = -\frac{h\cdot\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right)$$

at C, $$\frac{d^2y}{dx^2} = -\frac{h\pi^2}{l^2} = -\frac{\pi^2}{3l}$$

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{1\quad\left(\frac{dy}{dx}\right)^{2\ \frac{3}{2}}} = -\frac{\pi^2}{3l}$$ here ρ is negative.

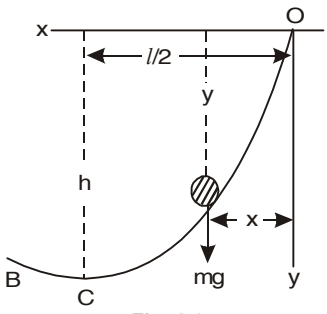


Fig. 2.6

Equation of motion is $R - mg = \dfrac{mv^2}{\rho}$

or $R = mg + \dfrac{2mgh}{\rho}$; $v^2 = 2gh$

$= mg + 2mg \cdot \dfrac{l}{3} \times \dfrac{\pi^2}{3l}$

$= mg\left[1 + \dfrac{2\pi^2}{9}\right]$

2.9. A block of mass m slides down a frictionless incline of angle α, mass M and length L which is on a horizonal frictionless plane. The block starts from rest from the top of the incline. How long the block will take to reach to the bottom ?

Solution: Let s be the position of the block of mass m at any time t and χ be the position of the centre of mass.

The acceleration along the plane is given by

$$\frac{d^2s}{dt^2} - \frac{d^2\chi}{dt^2}\cos\alpha = g\sin\alpha$$

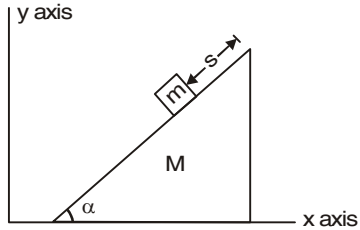


Fig. 2.7

No net force is acting along x axis therefore the net momentum along the horizontal direction = 0

$$(M + m)\frac{d\chi}{dt} - m\cos\alpha \cdot \frac{ds}{dt} = 0$$

On differentiation $\dfrac{d^2\chi}{dt^2} = \dfrac{m\cos\alpha}{(M+m)}\dfrac{d^2s}{dt^2}$

Now $\dfrac{d^2s}{dt^2} - \dfrac{m\cos^2\alpha}{M+m} \cdot \dfrac{d^2s}{dt^2} = g\sin\alpha$

$\therefore \dfrac{d^2s}{dt^2} = \dfrac{(M+m)g\sin\alpha}{M + m\sin^2\alpha}$

On integration we have

$s = \mathrm{L} = \dfrac{1}{2}\dfrac{(M+m)g\sin\alpha}{M + m\sin^2\alpha} \cdot t^2$

$\therefore t = \sqrt{\dfrac{2L\left(M + m\sin^2\alpha\right)}{(M+m)g\sin\alpha}}$

2.10. A thin uniform chain is held vertically above a table with its lower end just touching it. The mass of the chain is m and its length is l. The

chain is released. Calculate the force that the chain exerts on the table t seconds after it is released.

Solution: Let the upper end moves through x.

velocity $v = \sqrt{2gx}$

The length of the chain lying on the table in time $t = \frac{1}{2} gt^2$.

Now there are two forces which will act (i) the weight of the portion of the chain lying on the table (ii) the force due to impact.

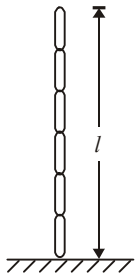


Fig. 2.8

Momentum imparted to the table = $\frac{m}{l} \cdot dx \quad v$

Force due to impact = $\frac{m}{l} \cdot dx \cdot \frac{v}{dt} = \frac{mv^2}{l} = 2mg \frac{x}{l}$

Weight of the chain of length $x = \frac{m}{l} \cdot x \cdot g$

Net force = $\frac{3mgx}{l} = \frac{3}{2} \cdot \frac{m}{l} \cdot g^2t^2$

2.11. Two blocks A of mass 8 kg and B of 4 kg are attached through a massless string are placed as shown in Fig. 2.9. Block B falls through 50 cm from rest. Determine its speed if (i) there is no friction (ii) the coefficient of friction between block A and the horizontal plane is 0.25.

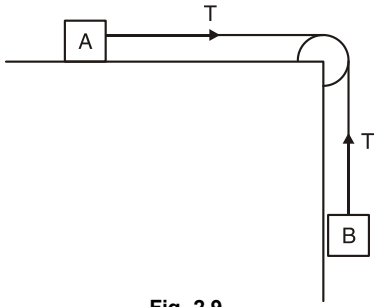


Fig. 2.9

Solution:

(i) $T = 8a$

$4g - T = 4a$

On adding the two $a = \frac{g}{3}$

Now $v^2 = 0 + 2 \times \frac{g}{3} \times 0.5 = \frac{g}{3}$

$v = 1.81$ m/sec

(ii) $T - 0.25 \times 8g = 8a$

$4g - T = 4a$

on adding $2g = 12a$

$a = \frac{g}{6}$

$$v^2 = 0 + 2 \times \frac{g}{6} \times 0.5 = \frac{g}{6}$$

$$v = 1.28 \text{ m/sec}$$

2.12. Two particle each of mass m are connected by a light string of length $2L$ as in Fig. 2.10a. A continuous force F is applied at the mid point of the string ($x = 0$) at right angles to the initial position of the string. Show that the acceleration of m is directed at right angles to F is given by

$$a = \frac{F}{2m} \frac{x}{\sqrt{L^2 \quad x^2}}$$

where x is the perpendicular distance of one of the particles from the line of action of F.

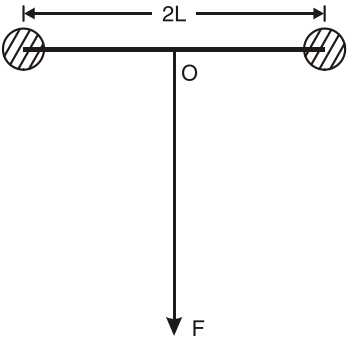


Fig. 2.10a

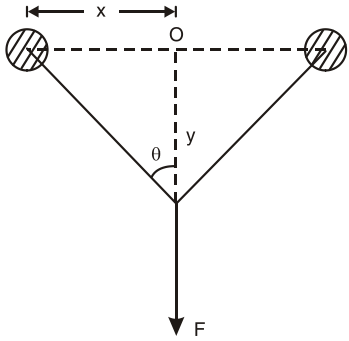


Fig. 2.10b

Solution: Let the mid point O be the displaced downwards through y. Along this direction we have

$2T \cos \theta - F = 0$; T = tension in the string.

or $T = \dfrac{F}{2 \cos \theta}$

Further in the horizontal direction we have

$T \sin \theta = ma$

or $a = \dfrac{T \sin \theta}{m}$

$$= \frac{F}{2m} \cdot \tan \theta$$

$$= \frac{F}{2m} \cdot \frac{x}{\sqrt{L^2 \quad x^2}}$$

2.13. Two block P & Q of masses 6 kg and 4 kg respectively are connected through a rope of mass 2 kg. The two masses are placed on a frictionless

plateform as in Fig. 2.11. The system of the blocks is pulled in the forward direction by a force of 60 N applied on the block *P*. Find the acceleration of the system and tension in the rope at *A*, *B* and *C*.

Solution: The common acceleration of the system is given by

$$a = \frac{F}{\text{Total mass}} = \frac{60}{12} = 5 \text{ m/sec}^2$$

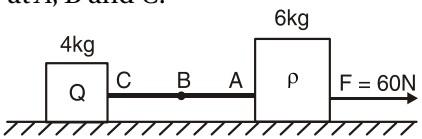


Fig. 2.11

Tension at *A* is the force required to pull the block *Q* along with the rope. The total mass = 6 kg.

$T_A = 6 \times 5 = 30$ N.

Tension at *B* is the force requird to pull the block *Q* along with the portion *BC* at the rope. Total mass = 5 kg.

$T_B = 5 \times 5 = 25$ N

Tension at $C = T_C = 4 \times 5 = 20$ N

2.14. What is the tension in a rod of length *L* and mass *M* at a distance *y* from F_1 when the rod is acted on by two unequal forces F_1 and F_2 ($< F_1$) as shown in Fig. 2.12. (IIT 1978, 1993)

Solution: The acceleration of the rod is given by

$$a = \frac{F_1 \quad F_2}{M}$$

Let *T* be the tension at *O*. For the motion of part *OP* we have

$$F_1 - T = \frac{M}{L} \cdot y \cdot a$$

$$\therefore T = F_1 - \frac{M}{L} \cdot y \cdot \frac{F_1 \quad F_2}{M}$$

$$T = F_1 \quad 1 \quad \frac{y}{L} \quad + \frac{F_2 y}{L}$$

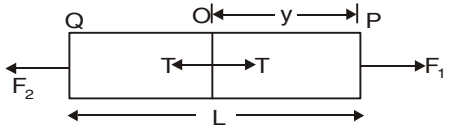


Fig. 2.12

2.15. Block *A* of mass *m* and block *B* of mass 2*m* are placed on a fixed triangular wedge by means of a massless in extensible string and a frictionless pulley as shown in Fig. 2.13. The wedge is inclined at 45° to the horizontal on both sides. The coefficient of friction between block *A* and the wedge is $\frac{2}{3}$ and that between block *B* and the wedge is $\frac{1}{3}$. If the system of *A* and *B* is released from rest, find

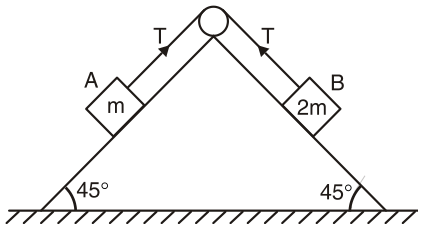


Fig. 2.13

(i) The acceleration of A.

(ii) Tension in the string.

(iii) The magnitude and direction of the force of friction acting on A.

(IIT 1997)

Solution: Equation of motion of the two blocks is given by

$$2ma = 2mg \sin 45 - T - 2mg \cos 45 \times \frac{1}{3}$$

$$ma = T - mg \sin 45 - mg \cos 45 \times \frac{2}{3}$$

On adding we have

$$3ma = \frac{mg}{\sqrt{2}} - \frac{4mg}{3\sqrt{2}}$$

This equation suggests that a is negative, which is not possible. Here the force of friction is large and therefore the masses will not move.

$\therefore$ Acceleration of A = acceleration of $B = 0$

(ii) $2mg \sin 45 = T + \frac{2mg}{3} \cdot \cos 45$

$$T = \frac{4mg}{3\sqrt{2}}$$

(iii) For block A, $T = \frac{4mg}{3\sqrt{2}}$ up the plane and $\frac{mg}{\sqrt{2}}$ down the plane

$$\therefore f = \frac{4mg}{3\sqrt{2}} - \frac{mg}{\sqrt{2}} = \frac{mg}{3\sqrt{2}} \text{ down the plane.}$$

2.16. A particle of mass m is moving in a circular path of constant radius r such that its centripetal acceleration a_c is varying with time t_c as $a_c = k^2rt^2$ where k is a constant. What is the power delivered to the particle by the force acting on it. (IIT 1994)

Solution: $a_c = \frac{v^2}{r} = k^2rt^2$

Kinetic enrgy $= \frac{1}{2} mv^2 = \frac{1}{2} mk^2r^2t^2$

Power $P = \frac{d}{dt} \left(\frac{1}{2} mk^2r^2t^2 \right) = mk^2r^2t$

2.17. A smooth semi-circular wire-track of radius R is fixed in a vertical plane (Fig. 2.14a). One end of a massless spring of natural length $\frac{3R}{4}$ is attached to the lowest point O of the wire-track. A small ring of mass m, which can slide on the track, is attached to the other end of the spring. The ring is held stationary at point P such that the spring makes an angle of 60° with the vertical. The spring constant $K = \frac{mg}{R}$. Consider the instant

when the ring is released and (i) draw the free body diagram of the ring (ii) determine the tangential acceleration of the ring and the normal reaction.

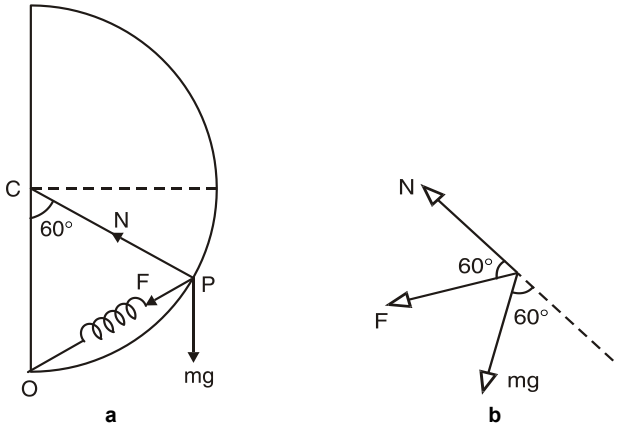


Fig. 2.14

Solution: $OP = 2R \cos 60 = R$

Stretch in the spring $= R - \dfrac{3R}{4} = \dfrac{R}{4}$

Spring force $F = \dfrac{mg}{R} \times \dfrac{R}{4} = \dfrac{mg}{4}$

Tangential force $= F \sin 60 + mg \sin 60 = \dfrac{5g\sqrt{3}}{8}$

Radial force $= N + F \cos 60 - mg \cos 60$

Since the radial force is zero

$\therefore N = mg \cos 60 - F \cos 60$

$$= mg \quad \frac{mg}{4} \quad \times \frac{1}{2}$$

$$= \frac{3}{8} mg$$

2.18. Two small solid spheres of masses m_1 and m_2 ($m_1 > m_2$) are attached through a thread and thrown over a pulley of mass M. Find (i) the common acceleration a (ii) the tensions T_1 & T_2 of the threads.

Solution:

(i) The equation of motion of the spheres is given by

$m_1 g - T_1 = m_1 a$

$T_2 - m_2 g = m_2 a$

and $(T_1 - T_2)\, r = \dfrac{Mr^2}{2}\, \alpha$

$$\text{or } (T_1 - T_2) = \frac{Ma}{2}$$

$$(m_1 - m_2)\, g - (T_1 - T_2) = (m_1 + m_2)\, a$$

$$\text{or } a = \frac{m_1 \quad m_2 \quad g}{m_1 \quad m_2 \quad \frac{M}{2}}$$

(ii) $$T_1 = m_1 g - \frac{m_1 \quad m_1 \quad m_2 \quad g}{m_1 \quad m_2 \quad \frac{M}{2}}$$

$$= \frac{m_1 g \quad 2m_2 \quad \frac{M}{2}}{m_1 \quad m_2 \quad \frac{M}{2}}$$

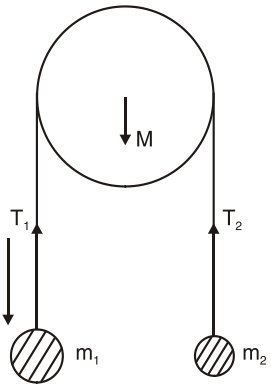


Fig. 2.15

$$T_2 = m_2 g + \frac{m_2 \quad m_1 \quad m_2 \quad g}{m_1 \quad m_2 \quad \frac{M}{2}} = \frac{m_2 g \quad 2m_1 \quad \frac{M}{2}}{m_1 \quad m_2 \quad \frac{M}{2}}$$

2.19. A ball of mass m attached by a string AB of length l is placed on a sphere of radius r as shown in the Fig. 2.16. The point of suspension A is at a distance d from the surface of the sphere. Find (i) the tension in the string (ii) the reaction force which the sphere acts on the ball.

Solution: In view of the various forces we have

$$T\cos\theta + N\cos\phi = mg$$

and $$T\sin\theta = N\sin\phi$$

$$N \cdot \frac{x}{r} = T \cdot \frac{x}{l}$$

$$\text{or } N = \frac{Tr}{l}$$

On substituting this value we have

$$T\cos\theta + T \cdot \frac{r}{l}\cos\phi = mg$$

$$T \cdot \left[\frac{d+p}{l} + \frac{r}{l} \cdot \frac{(r-p)}{r}\right] = mg$$

$$T \quad \frac{d \quad r}{l} \quad = mg$$

$$\therefore\ T = mg \cdot \frac{l}{d \quad r}$$

$$\text{and } N = mg \cdot \frac{r}{d \quad r}$$

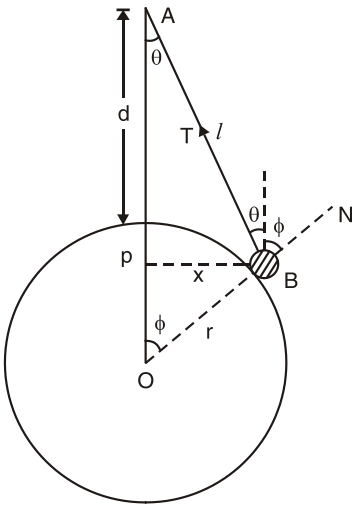


Fig. 2.16

2.20. A particle of mass m is projected vertically upward with a velocity v_0 in a medium whose resistance is kv. Determine (i) the time for the particle to come to rest (ii) the maximum height attained.

Solution: (i) Equation of motion of the particle

$$\frac{mdv}{dt} = -(kv + mg)$$

$$\int_0^{v_0} \frac{dv}{g \quad v\frac{k}{m}} = \int_0^{t_1} dt$$

$$t_1 = \frac{m}{k} \log \quad 1 \quad \frac{v_0 k}{mg}$$

(ii) $$\frac{mdv}{dt} = \frac{mdv}{dh} \cdot v = -(mg + kv)$$

or $$\frac{mvdv}{mg \quad kv} = -dh$$

Put $mg + kv = \chi$

$kdv = d\chi$

$$\int_{mg + kv_o}^{mg} m \frac{(\chi - mg)}{k^2} d\chi = \int_0^h dh$$

or $$h = \frac{m}{k^2} \quad kv_0 \quad mg \log \frac{mg \quad kv_0}{mg}$$

2.21. An auto vehicle of mass m moves along a horizontal track. At a certain instant the engine is turned off. The resistive force that acts on it is $\alpha + \beta v^2$ where α and β are positive constants and v is instantaneous speed. Find (i) the time taken by the vehicle to come to rest (ii) the distance travelled by it.

Solution:

(i) $$\frac{mdv}{dt} = -(\alpha + \beta v^2)$$

$$\int_{v_0}^{0} \frac{mdv}{\beta\left[\frac{\alpha}{\beta} + v^2\right]} = \int_0^t dt$$

$$t = \frac{m}{\beta} \tan^{-1}\left(v_o \cdot \sqrt{\frac{\beta}{\alpha}}\right)$$

(ii) $\frac{mdv}{dt} = \frac{mdv}{d\chi} \cdot v = -(\alpha + \beta v^2)$

$$\int_{v_0}^{0} \frac{mvdv}{\alpha + \beta v^2} = \int_{0}^{S} -d\chi$$

$$S = \frac{m}{2\beta} \log\left(1 + \beta \frac{v_0^2}{\alpha}\right)$$

2.22. A mass m moves along a straight path under the action of a force F. During motion it experiences a resistive force equal to kv^2 where k is constant and v is the instantaneous speed. Find the distance travelled when its speed changes from v_1 to v_2.

Solution:

$$\frac{mdv}{dt} = \frac{mdv}{dx} \cdot v = F - kv^2$$

$$\int_{v_1}^{v_2} \frac{mvdv}{F \quad kv^2} = \int_{O}^{S} dx$$

$$S = \frac{m}{2k} \log \frac{F \quad kv_1^2}{F \quad kv_2^2}$$

2.23. A point P moves on a circular path in the counter clockwise direction. The length of the arc sweeped out is defined by relation $s = t^3 + 3$. The radius of the circle is $r = 4$ m. Determine the component velocities and accelerations.

Solution: Length of the arc = 4 m

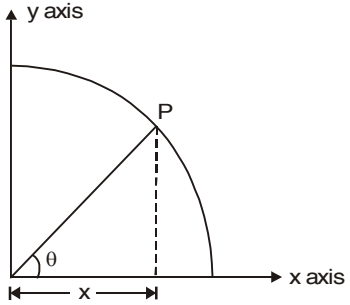


Fig. 2.17

$$\theta = \frac{S}{r} = 1 \text{ rad} = 57.3°$$

$$\omega = \frac{d\theta}{dt} = \frac{3t^2}{4} = 0.75 \text{ rad/sec}$$

Now $x = r\cos\theta$ and $y = r\sin\theta$

$$v_x = \frac{dx}{dt} = -r\omega \sin\theta$$

$$= -4 \times 0.75 \sin 57.3°$$

$$= -2.52 \text{ m/sec}$$

$$v_y = \frac{dy}{dt} = r\omega\cos\theta = 4 \times 0.75 \cos 57.3° = 1.62 \text{ m/sec}$$

$$a_x = \frac{d^2x}{dt^2} = -r\omega^2\cos\theta - r\sin\theta \cdot \alpha$$

$$= -4 \times 0.75^2 \times \cos 57.3° - 4 \sin 57.3° \times 1.5; \alpha = \frac{6t}{4} = 1.5$$

$$= -1.21 - 5.049 = -6.26 \text{ m/sec}^2$$

$$a_y = \frac{d^2y}{dt^2} = -r\omega \sin\theta + r\cos\theta \cdot \alpha$$
$$= -4 \times 0.75^2 \sin 57.3° + 4 \times \cos 7.3° \times 1.5$$
$$= -1.89 + 3.24 = 1.35 \text{ m/sec}^2$$

2.24. A body of mass m, thrown at angle α to the horizontal plane with an initial velocity v_0 moves under the action of the force of gravity and the force of resistance $R = -rv$. Determine the height h and the distance s to the base of the maximum height along the horizontal plane.

Solution: For maximum height

$$\frac{mdv}{dt} = -(mg + rv)$$

or $$\frac{mdv}{dx} \cdot v = -(mg + rv)$$

or $$-\frac{mvdv}{mg \quad rv} = dx$$

Put $mg + rv = p$

$$-\frac{m}{r^2} \quad \frac{p \quad mg}{p} \quad dp = dx$$

On integration we have

$$-\frac{m}{r^2} [p - mg \log p] = x$$

or $$-\frac{m}{r^2} \left| \left[mg + rv - mg \log (mg + rv) \right] \right|_{v_o \sin\alpha}^{0} = |x|_0^h$$

or $$-\frac{m}{r^2} [mg - mg \log mg - mg - rv_0 \sin\alpha + mg \log (mg + rv_0 \sin\alpha)] = h$$

or $$h = \frac{m}{r} v_0 \sin\alpha - \frac{m^2 g}{r^2} \log \quad 1 \quad \frac{rv_o \sin}{mg}$$

For horizontal distance x

$$\frac{mdv}{dt} = -rv$$

or $$\frac{mdv}{v} = \quad rdt$$

$$\log v = -\frac{rt}{m} + c$$

At $t = 0$, $v = v_0 \cos\alpha$, $\therefore\ c = \log v_0 \cos\alpha$

Therefore $$\log v = -\frac{rt}{m} + \log v_0 \cos\alpha$$

or $$v = v_0 \cos\alpha\, e^{-rt/m}$$

On integration we have

$$x = -\frac{m}{r} v_0 \cos\alpha \cdot e^{-rt/m} + c$$

At $t = 0$, $x = 0$; $\therefore c = \frac{m}{r} v_0 \cos\alpha$

$$\therefore x = \frac{m}{r} \cdot v_0 \cos\alpha (1 - e^{-rt/m})$$

For vertical distance y

$$\frac{mdv}{dt} = -(mg + rv)$$

$$-\frac{mdv}{mg \quad rv} = dt$$

On integration we have

$$\frac{m}{r} \log (mg + rv) = -t + c_0$$

or $\log (mg + rv) = -\frac{rt}{m} + c$

At $t = 0$; $v = v_0 \sin\alpha$ $\therefore c = \log (mg + r v_0 \sin\alpha)$

$$\therefore \log \left(\frac{mg + rv}{mg + rv_0 \sin\alpha} \right) = -\frac{rt}{m}$$

or $v = \frac{dy}{dt} = \left(\frac{mg}{r} + v_0 \sin\alpha \right) e^{-rt/m} - \frac{mg}{r}$

On further integration

$$y = -\frac{m}{r} \left(\frac{mg}{r} + v_0 \sin\alpha \right) e^{-rt/m} - \frac{mgt}{r} + c_1$$

At $t = 0$, $y = 0$; $\therefore c_1 = \frac{m}{r} \left(\frac{mg}{r} + v_0 \sin\alpha \right)$

or $y = \frac{m}{r} \left(v_o \sin\alpha + \frac{mg}{r} \right) \quad 1 \quad e^{rt/m} \quad - \frac{mgt}{r}$

For horizontal distance s.

Time of flight $t = t_1 + t_2$ where t_1 = time for going up and t_2 = time for coming down

$$\frac{mdv}{dt} = -(mg + rv)$$

$$-\frac{mdv}{mg + rv} = dt$$

On integration

$$-\frac{m}{r}\log(mg+rv)=t+c_1$$

At $t=0$, $v=v_0\sin\alpha$ $\therefore$ $c_1=-\frac{m}{r}\log(mg+rv_0\sin\alpha)$

or $t=\frac{m}{r}\log\left(\frac{mg+r\,v_0\sin\alpha}{mg+rv}\right)$

At $t=t_1$, $v=0$;

$$\therefore\ t_1=\frac{m}{r}\log\left(1+\frac{r v_o\sin\alpha}{mg}\right)$$

For downward journey

$$\frac{mdv}{dt}=mg-rv$$

$$\frac{mdv}{mg\quad rv}=dt$$

On integration we have

$$-\frac{m}{r}\cdot\log(mg-rv)=t+c$$

At $t=0$, $v=0$; $\therefore$ $c=-\frac{m}{r}\cdot\log mg$

or $t=\frac{m}{r}\log\ \frac{mg}{mg\quad rv}$

At $t=t_2$, $v=v_0\sin\alpha$; $\therefore$ $t_2=\frac{m}{r}\log\frac{mg}{mg-rv_o\sin\alpha}$

$$\therefore\ \text{Total time } t=t_1+t_2=\frac{m}{r}\log\left[\frac{mg+rv_0\sin\alpha}{mg-rv_o\sin\alpha}\right]$$

Further $s=\frac{x}{2}$

$$=\frac{mv_0\cos\alpha}{2r}\left[1-\frac{mg-rv_0\sin\alpha}{mg+rv_0\sin\alpha}\right];\ 1-\frac{mg-rv_0\sin\alpha}{mg+rv_0\sin\alpha}=1-e^{-rt/m}$$

$$=\frac{v_0^2\sin2\alpha}{2g\left[1+\frac{rv_0}{mg}\sin\alpha\right]}$$

2.25. A mass M with no initial velocity falls from a height H onto a helical spring. The mass forces the spring to contract by h. What is the time of contraction if we neglect the action of forces of resistance and consider the mass of the spring to be negligible?

Solution:

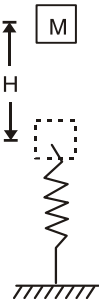


Fig. 2.18

$$\frac{mdv}{dt} = mg - Kx;$$

$$mg(H+h) = \frac{1}{2} Kh^2$$

$$\therefore K = \frac{2mg(H \quad h)}{h^2}$$

$$\frac{dv}{dt} = g - \frac{2g(H \quad h)}{h^2} x$$

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot v = g - \frac{2g(H \quad h)x}{h^2}$$

or $$\int v dv = \int g \cdot dx - \int \frac{2g(H+h)}{h^2} x dx$$

$$v^2 = 2gx - \frac{2g(H+h)x^2}{h^2} + c$$

At $x = 0$, $v = v_0 \therefore c = v_0^2$

Further $v_0 = \sqrt{2gH}$

$$\therefore v = \frac{dx}{dt} = \sqrt{2gh \quad 2gx \quad \frac{2g(H \quad h)x^2}{h^2}}$$

$$\frac{h}{\sqrt{2g(H \quad h)}} \cdot \frac{dx}{\sqrt{\frac{Hh^2}{H \quad h} \quad \frac{h^2 x}{H \quad h} \quad x^2}} = dt$$

$$= \frac{h}{\sqrt{2g(H \quad h)}} \frac{dx}{\frac{Hh^2}{H \quad h} \quad \frac{h^4}{4(H \quad h)^2} \quad \frac{h^2}{2(H \quad h)} \quad x^{2}^{\frac{1}{2}}} = dt$$

On itegration

$$t = -\frac{h}{\sqrt{2g(H \quad h)}} \sin^{1} \frac{\frac{h^2}{2(H \quad h)} \quad x}{\frac{h}{2(H \quad h)} \quad h^2 \quad 4(H \quad h)H^{\frac{1}{2}}} \Bigg|_0^h$$

$$= \frac{h}{\sqrt{2g\,(H \quad h)}} \left[\frac{\pi}{2} + \sin^{-1} \frac{h}{h+2H} \right]$$

$$= \frac{h}{\sqrt{2g\,(H \quad h)}} \left[\frac{\pi}{2} + \tan^{-1} \frac{h}{2\sqrt{H\,(H+h)}} \right]$$

2.26. A mass m falls in air without any initial velocity. Assuming the force of air resistance to be proportional to the second power of velocity, $R = rv^2$, find the velocity and position of the mass as function of time.

Solution: The equation of motion is given by

$$\frac{mdv}{dt} = mg - rv^2$$

or $$\frac{dv}{1 - \frac{r}{mg} \cdot v^2} = gdt$$

put $$\sqrt{\frac{r}{mg}} \cdot v = p$$

$$\sqrt{\frac{mg}{r}} \cdot \frac{dp}{1-p^2} = gdt$$

or $$\frac{dp}{1 \quad p^2} = \sqrt{\frac{rg}{m}} \cdot dt$$

$$\frac{1}{2} \left[\frac{dp}{1-p} + \frac{dp}{1+p} \right] = \sqrt{\frac{rg}{m}}\ dt$$

On integration we have

or $$\frac{1}{2} \log \frac{1 \quad p}{1 \quad p} = \sqrt{\frac{rg}{m}} \cdot t + C$$

or $$\frac{1}{2} \log \frac{1 + \sqrt{\frac{r}{mg}} \cdot v}{1 - \sqrt{\frac{r}{mg}} \cdot v} = \sqrt{\frac{rg}{m}} \cdot t + C$$

At $t = 0$, $v = 0$; $\therefore\ C = 0$

$$\tan h^{-1} \sqrt{\frac{r}{mg}} \cdot v = \sqrt{\frac{rg}{m}} \cdot t$$

or $$v = \sqrt{\frac{mg}{r}}\ \tan h \sqrt{\frac{rg}{m}} \cdot t$$

$$v = \frac{dx}{dt} = \sqrt{\frac{mg}{r}} \cdot \tan h \sqrt{\frac{rg}{m}} \cdot t$$

$$dx = \sqrt{\frac{mg}{r}} \cdot \tan h \sqrt{\frac{rg}{m}} \cdot t \cdot dt$$

$$x = \frac{\sqrt{\frac{mg}{r}}}{\sqrt{\frac{rg}{m}}} \cdot \log \cos h \sqrt{\frac{rg}{m}} \cdot t + C_o$$

$$x = \frac{m}{r} \log \cos h \sqrt{\frac{rg}{m}} \cdot t + C_o$$

at $t = 0$, $x = 0$; $\therefore C_o = 0$

or $x = \frac{m}{r} \cdot \log \cos h \sqrt{\frac{rg}{m}} \cdot t$

2.27. A particle of mass m moves in the $x\,y$ plane under the action of a force F of constant magnitude rotating clockwise in the plane at a constant angular velocity ω. At the initial moment the force is directed along the x axis and the particles velocity is v_0. Find the momentum of the particle as a function of time.

Solution:

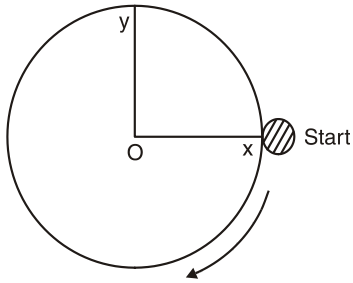


Fig. 2.19

$$\frac{mdv_x}{dt} = F \cos \omega t \cdot i$$

$$mv_x = i \frac{F}{\omega} \cdot \sin \omega t + A$$

At $t = 0$, $v_x = v_0$; $\therefore A = mv_0$

$$\therefore mv_x = mv_0 + i \frac{F}{\omega} \cdot \sin \omega t$$

$$\frac{mdv_y}{dt} = -F \sin \omega t \,.\, j$$

$$mv_y = \frac{F}{\omega} \cos \omega t \cdot j + B$$

At $t = 0$, $v_y = 0$; $\therefore B = -\frac{F}{}$

$$mv_y = -\frac{F}{\omega} (1 - \cos \omega t)\, j$$

$$\Delta p = mv_0 + i \frac{F}{\omega} \sin \omega t - j \frac{F}{\omega} (1 - \cos \omega t)$$

2.28. The speed of a particle is determined in terms of distance s that it has travelled via equation $v = v_0 - bs$ where b is constant. Find the value of s and v as a function of time.

Solution: $v = \dfrac{ds}{dt} = v_0 - bs$

or $\dfrac{ds}{v_o \quad bs} = \quad dt$

$\log (v_0 - bs) = - bt + C$

At $t = 0$, $s = 0$; $\therefore\ C = \log v_0$

or $\log \ 1 \quad \dfrac{bs}{v_0} = - bt$

or $s = \dfrac{v_o}{b} (1 - e^{-bt})$

$\dfrac{ds}{dt} = v = v_0 e^{-bt}$

2.29. A jet of water with a sectional area $A = 6$ cm^2 strikes a wall at an angle of $\theta = 60°$ to the normal and rebounds elastically from the wall without change in velocity. Find the force acting on the wall if the velocity of water in the jet is $v = 12$ m/sec.

Solution: Mass of water flowing per sec $= A\rho v$

Momentum of water flowing per sec $= p = A\rho v^2$

Incident normal momentum per sec $= A\rho v^2 \cdot \cos 60$

Reflected normal momentum per sec $= - A\rho v^2 \cos 60$

Force $= 2\, A\rho v^2 \cos 60 = A\rho v^2$

$= 6 \times 10^{-4} \times 1000 \times 144$

$= 86.4$ N

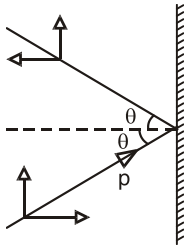


Fig. 2.20

2.30. The potential energy of a particle has the form $U = a \quad \dfrac{x}{y} \quad \dfrac{y}{z}$ where a is constant. Find (i) the force acting on the particle, (ii) the work done on the particle is going from (1, 1, 1) to (2, 2, 4).

Solution:

(i) $F_x = - \dfrac{dU}{dx} = - \dfrac{a}{y}$, $F_y = - \dfrac{dU}{dy} = \dfrac{a}{y^2} + \dfrac{a}{z}$

and $F_z = - \dfrac{dU}{dz} = - \dfrac{ay}{z^2}$

$F = - \dfrac{a}{y} i + \quad \dfrac{a}{y^2} \quad \dfrac{a}{z} \quad j - \dfrac{ay}{z^2} k$

U at 1, 1, 1 = 0

U at 2, 2, 4 = a 1 $\frac{1}{2}$ = $\frac{a}{2}$

Work done = $-\frac{a}{2}$

2.31. A block of mass m = 400 kg placed on a 30° incline plane is acted upon by a force of F = 4800 N at an angle of 30° with the incline see Fig. 2.21. The coefficient of friction between the block and plane is $\mu = 0.2$. Find the acceleration of the block.

Solution: The equation of motion of the block is given by

$400\,a = 4800 \cos 30 - 400\,g \sin 30 - \mu N$

where $N = 4800 \sin 30 + 400\,g \cos 30$

$= 2400 + 3394.72$

$= 5794.72$

$400\,a = 4156.8 - 1960 - 1158.9$

$a = \frac{1037.9}{400}$

$= 2.6\ m/\text{sec}^2$

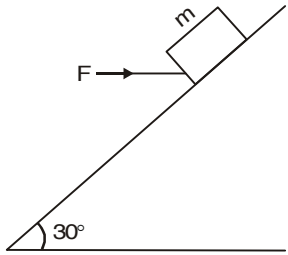


Fig. 2.21

2.32. A particle of mass 2 kg has a velocity of $v_1 = (3i - 12tj)$ m/sec and another particle of mass 3 kg has velocity $v_2 = 2i$ m/sec. At an instant t = 0.5 sec find the velocity and acceleration of the centre of mass.

Solution:

x component of the velocity of $cm = \frac{2 \quad 3 \quad 3 \quad 2}{5} = 2.4$

y component of the velocity of $cm = \frac{-6 \times 2 + 3 \times 0}{5} = -2.4$

$\therefore\ v_{cm} = (2.4\,i - 2.4\,j)$ m/sec

x component of the acceleration of $cm = 0$

y component of the acceleration of $cm = \frac{2 \times -12 + 0}{5}$

$= -4.8\ \text{m/sec}^2$

$a_{cm} = -4.8\,j$

2.33. Two blocks of masses 2 kg & 3 kg connected through a spring are placed on a frictionless horizontal table (see Fig. 2.22). An initial velocity of 10 m/sec is imparted to the 2 kg block in the direction of 3 kg block. Determine (i) the velocity of the centre of mass (ii) the velocities of the two blocks after the kick in the reference of centre of mass.

Solution:

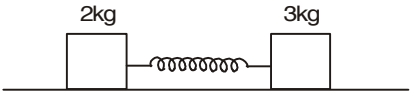


Fig. 2.22

(i) $v_{cm} = \dfrac{2 \quad 10}{5} = 4$ m/sec

(ii) v_1 = velocity of 2 kg mass

$= 10 - 4 = 6$ m/sec

v_2 = velocity of 3 kg mass $= 0 - 4 = -4$ m/sec

2.34. Two blocks of masses m_1 & m_2 connected by a spring of force constant K are placed on a smooth horizontal surface. Initially the blocks are at rest. A constant horizontal force F is applied on the block of mass m_2 as shown in Fig. 2.23. Neglecting the mass of the spring find the extension produced in it.

Solution: Acceleration of centre of mass is given by

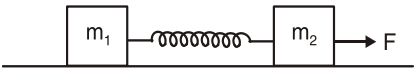


Fig. 2.23

$$a_{cm} = \frac{F}{m_1 \quad m_2}$$

A force $F_1 = m_1 a$ acts towards left.

$$F_1 = \frac{m_1 F}{m_1 \quad m_2}$$

Similarly a force $F_2 = m_2 a = \dfrac{m_2 F}{m_1 \quad m_2}$ acts on block of mass m_2.

Net force acting on $m_2 = F = F - F_2 = F - \dfrac{m_2 F}{m_1 \quad m_2} = \dfrac{m_1 F}{m_1 \quad m_2}$

Amount of work done by these forces will be equal to

$$F_1 x_1 + F' \cdot x_2 = \frac{m_1 F \quad x_1 \quad x_2}{m_1 \quad m_2}$$

This will be equal to increase in potential energy hence

$$\frac{m_1 F \quad x_1 \quad x_2}{m_1 \quad m_2} = \frac{1}{2} \text{ K } (x_1 + x_2)^2$$

$$\text{or } x_1 + x_2 = \frac{2 m_1 F}{K \quad m_1 \quad m_2}$$

2.35. A prism of height h and mass M holding a small block of mass m is kept on a smooth horizontal surface as shown in the Fig. 2.24. As the mass slides the prism starts moving. Find the distance covered by the prism when the block reaches at the bottom.

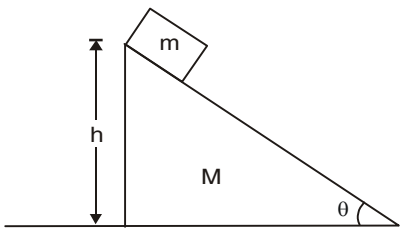


Fig. 2.24

Solution:

$$md_1 - Md_2 = 0$$

d_1 = displacement of block of mass m

d_2 = displacement of prism.

$$d_1 + d_2 = h\cot\theta$$

$$d_2\left(1 + \frac{M}{m}\right) = h\cot\theta$$

$$d_2 = \frac{mh\cot}{M \quad m}$$

2.36. A pulley fixed to a support carries a massless thread with spheres of masses m_1 and m_2 attached at its ends as shown in the Fig. 2.25. Neglecting the mass of the pulley find the acceleration of the centre of mass.

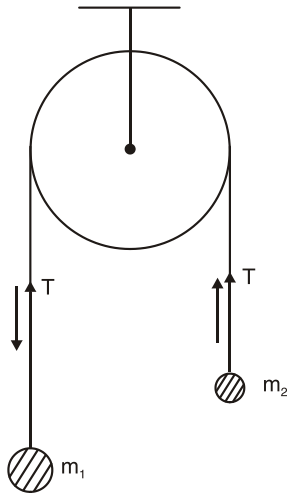


Fig. 2.25

Solution: The position of the centre of mass is given by

$y_{cm} = \frac{m_1 y_1 \quad m_2 y_2}{m_1 \quad m_2}$; y_1 and y_2 are the positions of the two masses.

Velocity of centre of mass is given by

$$v_{cm} = \frac{\frac{m_1 dy_1}{dt} - \frac{m_2 dy_2}{dt}}{m_1 + m_2}$$

Acceleration of the centre of mass is given by

$$a_{cm} = \frac{m_1 \quad m_2}{m_1 \quad m_2} \quad a;\ a = \frac{d^2 y_1}{dt^2} = \frac{d^2 y_2}{dt^2}$$

The equation of motion of the two spheres is given by

$m_1 a = m_1 g - T$

$m_2 a = \mathrm{T} - m_2 g$

$$\therefore\ a = \frac{m_1 \quad m_2}{m_1 \quad m_2} \quad g$$

On substituting the value of a the acceleration of centre of mass is given by

$$a_{cm} = \frac{m_1 \quad m_2}{m_1 \quad m_2}^2 \cdot g$$

2.37. Two spheres of masses m_1 and m_2 connected by a massless spring forms a system as shown in the Fig. 2.26. They fall under earth's gravitational field with initial velocities v_1 and v_2 respectively. Calculate the total momentum and radius vector of the centre of inertia.

Solution: The position of the centre of mass is given by

$y_{cm} = \frac{m_1 y_1 \quad m_2 y_2}{m_1 \quad m_2}$; y_1 and y_2 are the positions of the two masses.

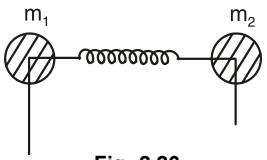


Fig. 2.26

$(m_1 + m_2) \frac{dy_{cm}}{dt}$ = Total momentum

$= m_1 \frac{dy_1}{dt} + m_2 \frac{dy_2}{dt}$; $\frac{dy_1}{dt}$ and $\frac{dy_2}{dt}$ are the velocities of the two masses after time t.

$= m_1 (v_1 + gt) + m_2 (v_2 + gt)$

$= (m_1 v_1 + m_2 v_2) + (m_1 + m_2)\, gt$

$= p_o + (m_1 + m_2)\, gt$; p_o = total initial momentum.

$\frac{dy_{cm}}{dt} = \frac{m_1 v_1 \quad m_2 v_2}{m_1 \quad m_2} + gt$

$= v_0 + gt$; $v_0 = \frac{m_1 v_1 \quad m_2 v_2}{m_1 \quad m_2}$

On integration we have

$y_{cm} = v_o t + \frac{1}{2}\, gt^2$

2.38. A closed system consists of two particles of masses m_1 and m_2 moving at right angles to each other with velocities v_1 and v_2 respectively. Calculate (i) the momentum of the two particles (ii) the total kinetic energy in the frame of reference of centre of inertia.

Solution: (i) Let us assume that particles of masses m_1 and m_2 are moving along x and y axis respectively.

x component of $v_{cm} = \frac{m_1 v_1}{m_1 + m_2}$

y component of $v_{cm} = \frac{m_2 v_2}{m_1 \quad m_2}$

x component of velocity of particle of mass m_1 in the reference frame of centre of mass $= v_1 - \frac{m_1 v_1}{m_1 + m_2} = \frac{v_1 m_2}{m_1 + m_2}$

Momentum of particle of mass m_1 along x axis $= \frac{m_1 m_2 v_1}{m_1 \quad m_2}$

Similarly the momentum of particle of mass m_1 along y axis = $\frac{m_1 m_2 v_2}{m_1 + m_2}$

Net momentum = $\frac{m_1 m_2}{m_1 + m_2} \sqrt{v_1^2 \quad v_2^2}$

y component of velocity of particle of mass m_2 in the reference frame of centre of mass = $v_2 - \frac{m_2 v_2}{m_1 \quad m_2} = \frac{m_1 v_2}{m_1 \quad m_2}$

Momentum of particle of mass m_2 along y axis = $\frac{m_1 m_2 v_2}{m_1 \quad m_2}$

Similarly the momentum of particle of mass m_2 along x axis = $\frac{m_1 m_2 v_1}{m_1 \quad m_2}$

Net momentum = $\frac{m_1 m_2}{m_1 \quad m_2} \sqrt{v_1^2 \quad v_2^2}$

(ii) Total kinetic energy of the two particles is equal to

$$E = \frac{1}{2} m_1 \left(\frac{m_2 v_1}{m_1 \quad m_2} \right)^2 + \frac{1}{2} m_1 \left(\frac{m_2 v_2}{m_1 \quad m_2} \right)^2 + \frac{1}{2} m_2 \left(\frac{m_1 v_1}{m_1 \quad m_2} \right)^2 + \frac{1}{2} m_2 \left(\frac{m_1 v_2}{m_1 \quad m_2} \right)^2$$

$$= \frac{1}{2} \frac{m_1 m_2}{m_1 \quad m_2^{\,2}} \quad m_2 v_1^2 \quad m_2 v_2^2 \quad m_1 v_1^2 \quad m_1 v_2^2$$

$$= \frac{1}{2} \frac{m_1 m_2}{m_1 \quad m_2} \quad v_1^2 \quad v_2^2$$

2.39. A 6 kg goat standing on a boat is at a distance of 12 m from the shore. The goat moved through a distance of 5 m on the boat towards the shore and then halts. Assuming the mass of boat equal to 24 kg determine the position of the goat from the shore. Neglect the friction between the boat and water.

Solution: Let d_1 be the displacement of the goat and d_2 be the displacement of the boat, then

$d_1 + d_2 = 5$

Further

$6 \cdot d_1 = 24\ (5 - d_1)$

$30\ d_1 = 24 \times 5$

$\therefore\ d_1 = \frac{24 \times 5}{30} = 4$ m

Initially the goat was 12 m away from the shore. Its position will now be 12 – 4 = 8 m from the shore.

2.40. Three particles of masses m_1, m_2 and m_3 are moving along x, y and z axes with velocities, v_1, v_2 and v_3 respectively. Find the total kinetic energy of the system about the centre of mass.

Solution: The x, y and z components of the velocity of centre of mass is given by

$$V_x = \frac{m_1 v_1}{m_1 + m_2 + m_3},\ V_y = \frac{m_2 v_2}{m_1 + m_2 + m_3} \text{ and } V_z = \frac{m_3 v_3}{m_1 + m_2 + m_3}$$

The component velocities of particle of mass m_1 with respect to centre of mass is given by

$$v_1 - \frac{m_1 v_1}{m_1 \quad m_2 \quad m_3},\ \frac{m_2 v_2}{m_1 \quad m_2 \quad m_3},\ \frac{m_3 v_3}{m_1 \quad m_2 \quad m_3}$$

Similarly the component velocities of particle of mass m_2 and m_3 with respect to centre of mass is given

$$\frac{m_1 v_1}{m_1 + m_2 + m_3},\ v_2 - \frac{m_2 v_2}{(m_1 + m_2 + m_3)},\ \frac{m_3 v_3}{m_1 + m_2 + m_3}$$

$$\text{and } \frac{m_1 v_1}{m_1 \quad m_2 \quad m_3},\ \frac{m_2 v_2}{m_1 \quad m_2 \quad m_3},\ v_3 - \frac{m_3 v_3}{m_1 \quad m_2 \quad m_3}$$

$$\text{Total kinetic energy } = \frac{1}{2(m_1 + m_2 + m_3)^2} \quad m_1 \quad m_2 \quad m_3 \quad {}^2 v_1^2$$

$$+ m_1 m_2^2 v_2^2 + m_1 m_3^2 v_3^2 \quad m_2 m_1^2 v_1^2$$

$$m_2 \quad m_1 \quad m_3 \quad {}^2 v_2^2 \quad m_2 \quad m_3^2 v_3^2 \quad m_3 \quad m_1^2 v_1^2$$

$$m_3 \quad m_2^2 v_2^2 \quad m_3 \quad m_1 \quad m_2 \quad {}^2 v_3^2$$

$$= \frac{1}{2(m_1 + m_2 + m_3)^2} \quad m_1 m_2^2 v_1^2 \quad m_1 m_3^2 v_1^2 \quad 2 m_1 m_2 m_3 v_1^2 \quad m_1 m_2^2 v_2^2$$

$$m_1 m_3^2 v_3^2 + m_2 m_1^2 v_1^2 + m_2 m_1^2 v_2^2 + m_2 m_3^2 v_2^2 + 2 m_1 m_2 m_3 v_2^2 + m_2 \cdot m_3^2 v_3^2$$

$$+ m_3 m_1^2 v_1^2 + m_3 m_2^2 v_2^2 + m_3 m_1^2 v_3^2 + m_3 m_2^2 v_3^2 + 2 m_1 m_2 m_3 v_3^2 \Big]$$

$$= \frac{1}{2(m_1 + m_2 + m_3)^2} \Big\{ \Big[m_1 m_2 (m_1 + m_2 + m_3) + m_1 m_3 (m_1 + m_2 + m_3) \Big] v_1^2$$

$$+ v_2^2 \quad m_1 m_2 \quad m_1 \quad m_2 \quad m_3 \quad m_2 m_3 \quad m_1 \quad m_2 \quad m_3$$

$$+ \Big[m_1 m_3 (m_1 + m_2 + m_3) + m_3 m_2 (m_1 + m_2 + m_3) \Big] v_3^2 \Big\}$$

$$= \frac{1}{2 \quad m_1 \quad m_2 \quad m_3}$$

$$v_1^2 \quad m_1 m_2 \quad m_1 m_3 \quad v_2^2 \quad m_1 m_2 \quad m_2 m_3 \quad v_3^2 \quad m_1 m_3 \quad m_2 m_3$$

$$= \frac{1}{2\ m_1\ \ m_2\ \ m_3}$$

$$m_1m_2\ \ v_1^2\ \ v_2^2 \quad m_1m_3\ \ v_1^2\ \ v_3^2 \quad m_2m_3\ \ v_3^2\ \ v_2^2$$

$$= \frac{1}{2\ m_1\ \ m_2\ \ m_3} \left[m_1m_2v_{12}^2 + m_1m_3v_{13}^2 + m_2m_3v_{23}^2 \right]$$

where v_{12}, v_{13} and v_{23} are the relative velocities.

2.41. A block of mass M with a semicircular track of radius R rest on a horizontal frictionless surface. A uniform cylinder of radius r and mass m is released from rest at the top point A (see Fig. 2.27). The cylinder slips in the semi-circular frictionless track (a) How far the block moved when the cylinder reaches the bottom (point B) of the track ? (b) How fast is the block moving when the cylinder reaches the bottom of the track.

(IIT 1983)

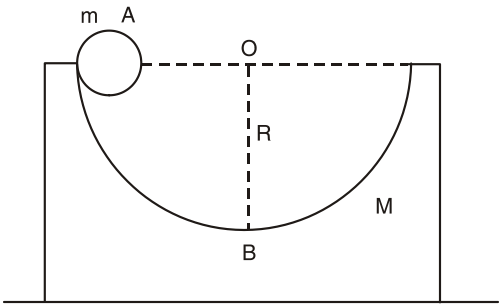


Fig. 2.27

Solution: (a) The system is initially at rest hence $V_{cm} = 0$

Let v_1 and v_2 be the velocities of the cylinder and block at the point B relative to the ground.

$$\frac{mv_1 + Mv_2}{m+M} = 0$$

$$\therefore \quad mv_1 + Mv_2 = 0$$

$$\text{or} \quad \frac{m\ \ r_1}{dt} + \frac{M\ \ r_2}{dt} = 0$$

$$\text{or} \quad m\Delta r_1 + M\Delta r_2 = 0$$

$$\text{or} \quad md_1 = Md_2$$

where $\Delta r_1 = d_1$ and $\Delta r_2 = d_2$, here d_1 and d_2 are opposite to each other.

When the cylinder moves from A to B it has a horizontal displacement $(R - r)$ towards right relative to the block. Let d_2 be the displacement of the block to the left relative to the ground. The displacement of the cylinder relative to the ground is given by

$$d_1 = (R - r) - d_2$$

Now $m\,[(R-r) - d_2] = M \cdot d_2$

$$\therefore \quad d_2 = \frac{m\ R\ \ r}{(M\ \ m)}$$

(b) According to conservation of energy we have

$$mg\,(R-r) = \frac{1}{2}\,mv_1^2 + \frac{1}{2}\,Mv_2^2$$

$$\text{or} \quad mg\,(R-r) = \frac{1}{2}\,m \cdot \left(-\frac{M}{m}\right)^2 v_2^2 + \frac{1}{2}\,Mv_2^2$$

$$mg\,(R-r) = \frac{M}{2}\ \ \frac{M}{m}\ \ 1\ \ v_2^2$$

$$\therefore \qquad v_2^2 = \frac{2m^2 g\,(R\ \ r)}{M\,(M\ \ m)}$$

$$\text{or} \qquad v_2 = m \cdot \sqrt{\frac{2g\,(R-r)}{M\,(M+m)}}$$

2.42. Two uniform thin rods A and B of lenth 0.6 m each and of mass 0.01 kg and 0.02 kg respectively are joined, end to end. The combination is pivoted at the higher end P as shown in Fig. 2.28a such that it can freely rotate about the point P in a vertical plane. A small object of mass 0.05 kg, moving horizontally hits the lower end of the combination and sticks to it. What should be the velocity of the object so that the system could be raised to the horizontal position. (IIT 1994)

Solution: According to conservation of angular momentum we have

$mv \cdot 2l = I\omega$

v = velocity of the object

where I is the moment of inertia of the system about P.

$I = I_A + I_B + I_C$

I_A = moment of inertia of bar A about P and is equal to

$$I_A = \frac{1}{3}\,M_A \cdot l^2$$

$$= \frac{1}{3} \times 0.01 \times (0.6)^2 = 1.2 \times 10^{-3}\ \text{kg.m}^2$$

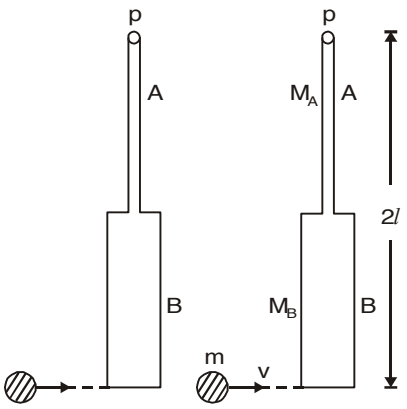


Fig. 2.28a Fig. 2.28b

I_B = moment of inertia of bar B about P and is equal to

$$I_B = \frac{1}{12} M_B \cdot l^2 + M_B \left(\frac{3l}{2}\right)^2$$

$$= \frac{28}{12} M_B l^2$$

$$= \frac{28}{12} \times 0.02 \times (0.6)^2$$

$$= 1.68 \times 10^{-2} \text{ kg.m}^2$$

I_C = moment of inertia of small object about P is equal to

$$I_C = m \cdot (2l)^2$$

$$= 0.05 \times (2 \times 0.6)^2$$

$$= 7.2 \times 10^{-2} \text{ kg.m}^2$$

$$I = 9 \times 10^{-2} \text{ kg.m}^2$$

$$\omega = \frac{m.v.2l}{I}$$

$$= \frac{0.05 \quad v \quad 2 \quad 0.6}{9 \quad 10^{-2}}$$

$$= \frac{2v}{3}$$

Further $\frac{1}{2} I\omega^2 = (m + M_A + M_B)\, Y \cdot g$, Y = centre of mass

$$Y = \frac{M_A \cdot \frac{l}{2} \quad M_B \cdot \frac{3l}{2} \quad m \cdot 2l}{M_A \quad M_B \quad m}$$

$$= \frac{0.01 \quad 0.3 \quad 0.02 \quad 0.9 \quad 0.05 \quad 1.2}{0.01 \quad 0.02 \quad 0.05}$$

$$= \frac{81}{80}$$

On putting the value of Y we have

$$\frac{1}{2} \times 9 \times 10^{-2} \times \left(\frac{2v}{3}\right)^2 = \frac{81}{80} (0.05 + 0.01 + 0.02) \times 9.8$$

On solving $v = 6.3$ m/sec

2.43. A rocket rising vertically up attains an altitude of 1500 m and a velocity of 300 m/sec. It explodes into three fragments of equal mass. One fragment moves vertically up with a speed of 400 m/sec. The second fragment moves with a speed of 200 m/sec towards right and right angles to the initial direction of motion. Find (i) the velocity of the third fragment

(ii) the position of the centre of mass with respect to the ground 4 sec after the explosion.

Solution: Let us assume that the rocket is ascending along y axis.

Its initial momentum = $M \cdot 300\, j$; j = unit vector along y axis.

Final momentum = $\frac{M}{3}\ 200i\ \ \frac{M}{3}\ 400j\ \ \frac{M}{3}v$

According to conservation of momentum we have

$$M \cdot 300\, j = \frac{M}{3}\ 200i\ \ \frac{M}{3}\ 400j\ \ \frac{M}{3}v$$

or $v = (-200\, i + 500\, j)$

(ii) The forces due to explosion are internal. They will not affect the motion of the centre of mass. The x component of the centre of mass will not be affected. The y component of centre of mass is given by

$$y = 1500 + 300 \times 4 - \frac{1}{2} \times 9.8 \times 4^2$$
$$= (2700 - 78.4)\ \text{m}$$
$$= 2.62\ \text{km}$$

2.44. A force $F = \alpha t\,(\tau - t)$ is applied to a stationary body of mass m for a time τ. Calculate (i) the momentum of the body when the force was discontinued, (ii) the distance travelled by the body under the action of force.

Solution:

(i) $\frac{m \cdot dv}{dt} = \alpha t\,(\tau - t)$

$m \cdot dv = \alpha t\,(\tau - t)\, dt$

On integration we have

$$mv = \frac{\alpha t^2 \tau}{2} - \frac{\alpha t^3}{3}$$

On putting $\tau = t$ we have

$$mv = \frac{\alpha \tau^3}{2} - \frac{\alpha t^3}{3}$$
$$= \frac{\alpha \tau^3}{6}$$

(ii) $\frac{m \cdot dx}{dt} = \frac{\alpha \tau}{2} \cdot t^2 - \frac{\alpha t^3}{3}$

$$m \cdot dx = \frac{\alpha \tau}{2} \cdot t^2 \cdot dt - \frac{\alpha}{3} \cdot t^3 \cdot dt$$

On integration

$$x = \frac{\alpha\tau \cdot t^3}{6m} - \frac{\alpha}{12m} \cdot t^4$$

On putting $\tau = t$, we have

$$x = \frac{\alpha \cdot \tau^4}{6m} - \frac{\alpha \cdot \tau^4}{12m}$$

$$= \frac{\alpha \tau^4}{12m}$$

2.45. A particle of mass m initially at rest moves under the action of a periodic force $F = F_0 \sin \omega t$ where F_0 and ω are positive constants. Find the distance covered by the particle in time t.

Solution:

$$F = \frac{m \cdot d^2x}{dt^2} = F_0 \sin \omega t$$

$$\frac{d^2x}{dt^2} = \frac{F_0}{m} \sin \omega t$$

On integration, we have

$$\frac{dx}{dt} = -\frac{F_0}{m\omega} \cos \omega t + A;\ A = \text{constant of integration}$$

At $t = 0$, $\frac{dx}{dt} = 0$

$$\therefore \quad A = \frac{F_0}{m\omega}$$

$$\therefore \quad \frac{dx}{dt} = \frac{F_0}{m\omega} (1 - \cos \omega t)$$

or $$\int_0^X dx = \frac{F_0}{m\omega} \left[\int_0^t dt - \int_0^t \cos \omega t \, dt \right]$$

$$X = \frac{F_0}{m\omega} \left[t - \frac{\sin \omega t}{\omega} \right]$$

$$= \frac{F_0}{m\omega^2} [\omega t - \sin \omega t]$$

2.46. A particle of mass m initially at rest starts moving by an application of a force $F = F_0 \cos \omega t$. Where F_0 and ω are positive constants. Determine (i) the time after which it will stop for the first time (ii) the distance moved by the particle in this time (iii) the maximum velocity attained by the particle over this distance.

Solution:

(i) $F = ma = F_0 \cos \omega t$

$$\therefore \frac{dv}{dt} = \frac{F_0}{m} \cdot \cos \omega t$$

$$dv = \frac{F_0}{m} \cos \omega t \cdot dt$$

On integration, we have

$$v = \frac{F_0}{m\omega} \cdot \sin \omega t + A$$

at $t = 0$, $v = 0$; $\therefore A = 0$

hence $v = \frac{F_0}{m\omega} \cdot \sin \omega t$

Velocity v will be zero when $t = \frac{\pi}{\omega}$

(ii) Time for the motion $= \frac{\pi}{\omega}$

Now $\frac{dx}{dt} = \frac{F_0}{m\omega} \cdot \sin \omega t$

$$\int_0^X dx = \int_0^{\pi/\omega} \frac{F_0}{m\omega} \cdot \sin \omega t \cdot dt$$

$$X = \left| \frac{-F_0}{m\omega^2} \cdot \cos \omega t \right|_0^{\pi/\omega}$$

$$= \frac{2F_0}{m\omega^2}$$

(iii) $v = \frac{F_0}{m\omega} \cdot \sin \omega t$

$$\frac{dv}{dt} = 0 = \frac{F_0}{m} \cdot \cos \omega t$$

This is possible with $\omega t = \frac{\pi}{2}$ or $t = \frac{\pi}{2\omega}$

$$\therefore v_{max} = \frac{F_0}{m\omega} \cdot \sin \cdot \omega \frac{\pi}{2\omega}$$

$$= \frac{F_0}{m\omega}$$

2.47. A motor boat of mass m moving on a lake with velocity v_0 experiences a resistive force of water $F = -rv$. At an instant $t = 0$ the engine fails. Determine (i) the time for which the boat will move after the failure of the engine (ii) the velocity of boat when it has covered a distance s.

Solution:

(i) $F = \dfrac{m \cdot dv}{dt} = -rv$

$$\int_{v_0}^{v} \frac{dv}{v} = \int_0^t -\frac{r\,dt}{m}$$

$$\log \frac{v}{v_0} = -\frac{rt}{m}$$

or $v = v_0\, e^{-rt/m}$

From the above equation $v = 0$ when $t = \infty$

(ii) $m \cdot \dfrac{dv}{dt} = -rv$

$$m \cdot \frac{dv}{dx} \cdot \frac{dx}{dt} = -rv$$

$$-dv = -\frac{r\,dx}{m}$$

$$\int_{v_0}^{v} dv = -\frac{r}{m} \int_0^s dx$$

$$v - v_0 = -\frac{rs}{m}$$

or $v = v_0 - \dfrac{rs}{m}$

2.48. A particle of mass m is projected vertically up in the earths gravitational field with an initial velocity v_0. During its motion it experiences a draging force $F = kv^2$ due to air. Determine its velocity when it comes down.

Solution: In going up the net force acting on the particle = $(mg + kv^2)$

Work done = $m \cdot \dfrac{dv}{dt} \cdot dx = -(mg + kv^2)\, dx$

or $-\displaystyle\int_{v_0}^{0} \frac{mv\,dv}{mg \quad kv^2} = \int_0^h dx$; h = height upto which it will reach.

or $-\dfrac{m}{k} \log \sqrt{\left(\dfrac{mg}{mg + kv_0^2}\right)} = h$

When the body comes down then the equation of motion is given by

$$m \cdot \frac{dv}{dt} = (mg - kv^2)$$

or $m \cdot \dfrac{dv}{dt} \cdot dx = (mg - kv^2)\, dx$

$$\int_0^{v} \frac{m v dv}{mg - kv^2} = \int_h^0 dx$$

or $$-\frac{m}{k} \log \sqrt{\frac{mg - kv'^2}{mg}} = h$$

hence $$-\frac{m}{k} \log \sqrt{\frac{mg - kv'^2}{mg}} = -\frac{m}{k} \log \sqrt{\frac{mg}{mg + kv_0^2}}$$

or $$1 - \frac{kv'^2}{mg} = \frac{mg}{mg + kv_0^2}$$

$$1 - \frac{mg}{mg + kv_0^2} = \frac{kv'^2}{mg}$$

$$kv'^2 = \frac{mg + kv_0^2}{mg + kv_0^2}$$

$$v' = \sqrt{\frac{v_0^2}{1 + kv_0^2 / mg}}$$

$$= \frac{v_0}{\sqrt{1 + kv_0^2 / mg}}$$

2.49. A grain of sand of mass m initially at rest moves in a bucket of water in the vertical downward direction. During its motion it experiences a resistive force equal to kv where k is a positive constant. Determine (i) the velocity of the grain at an insant t (ii) the force acting on the grain at that instant.

Solution:

(i) The equation of motion is given by

$$\frac{mdv}{dt} = mg - kv$$

or $$\frac{mdv}{mg + kv} = dt$$

or $$-\frac{m}{k} \log (mg - kv) = t + c; \; c = \text{constt. of integration}$$

At $t = 0$, $v = 0$

$$\therefore \; c = -\frac{m}{k} \log mg$$

$$\text{or} \quad -\frac{m}{k}\log\frac{mg - kv}{mg} = t$$

$$\text{or} \quad 1 - \frac{kv}{mg} = e^{-kt/m}$$

$$\text{or} \quad v = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

(ii) $$m \cdot \frac{dv}{dt} = m \cdot \frac{mg}{k} \times \frac{k}{m} \cdot e^{-kt/m}$$

$$= mg\,e^{-kt/m}$$

2.50. Discuss the motion of a rocket with total mass m, moving with a velocity v under the action of external forces.

Solution: Let v be the velocity of the rocket in the upward direction. It momentum = mv. In time Δt a mass Δm of gases is ejected from the rocket. Let $-v_r$ be the downward velocity of the gas ejected relative to the rocket (see Fig. 2.29). The velocity v' of the gas relative to the earth is equal to.

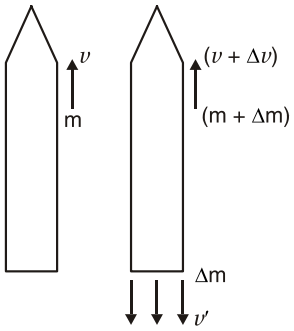


Fig. 2.29

$$v' = -v_r + v = v - v_r$$

$$\Delta m v' = -\Delta m\,(v_r - v)$$

At the end of the time Δt, the mass of the rocket and the unburnt fuel decreases to $(m - \Delta m)$ and the velocity becomes $v + \Delta v$.

$$\text{Change of momentum} = [(m - \Delta m)\,(v + \Delta v) - \Delta m\,(v_r - v)] - mv$$

$$= mv + m\Delta v - \Delta m v + \Delta m \Delta v - \Delta m v_r + \Delta m v - mv$$

$$= m\Delta v - \Delta m v_r$$

This must be equal to $F_{ext} \cdot \Delta t$, hence

$$F_{ext} \cdot \Delta t = m\Delta v - \Delta m v_r$$

$$\text{or} \quad F_{ext} = \frac{m\,v}{t} - \frac{m}{t} \cdot v_r$$

$$\text{or} \quad ma = F_{ext} + v_r \cdot \frac{m}{t}$$

$$\text{or} \quad \frac{mdv}{dt} = -v_r \cdot \frac{dm}{dt} + F_{ext};\ dm = \text{change in mass which is negative.}$$

(i) Assending rocket with $F_{ext} = mg$

Equation of motion is given by

$$\frac{mdv}{dt} = -v_r \frac{dm}{dt} - mg$$

$$\int_{v_0}^{v} dv = \int_{m}^{m_0} -v_r \cdot \frac{dm}{m} - \int_{0}^{t} g\,dt$$; m_0 is the mass of the rocket only.

v_0 = initial velocity.

$$v - v_0 = v_r \cdot \log_e \frac{m}{m_0} - gt$$

or $$v = v_0 + v_r \cdot \log_e \frac{m}{m_0} - gt$$

(ii) Ascending rocket subjected to thrust only i.e. $F_{ext} = 0$

Equation of motion is given by

$$\frac{mdv}{dt} = -v_r \cdot \frac{dm}{dt}$$

or $$\int_{v_0}^{v} dv = -v_r \int_{m}^{m_0} \frac{dm}{m}$$

$$v - v_0 = -v_r \log_e \cdot \frac{m_0}{m}$$

or $$v = v_0 + v_r \log_e \frac{m}{m_0}$$

2.51. A rocket of mass m_0 initially at rest moves by ejecting gas with a velocity u relative to the rocket. The external force is absent. Find its velocity when its mass becomes m.

Solution:

$$\frac{mdv}{dt} = -u \frac{dm}{dt} + F_{\text{external}}$$

$$m \cdot \frac{dv}{dt} = -u \frac{dm}{dt} \; ; F_{\text{external}} = 0$$

or $$\int_{0}^{v} dv = -u \int_{m_0}^{m} \frac{dm}{m}$$

$$v = u \log_e \frac{m_0}{m}$$

2.52. A rocket of mass m_0 moves with a constant acceleration a in the absence of external force by ejecting gas at a velocity u relative to the rocket. In this process the mass of the rocket will continuously reduce. Find the mass of the rocket at an instant t.

Solution:

$$ma = -u\frac{dm}{dt} + F_{\text{ext}}$$

or $$\int_0^t \frac{a\,dt}{u} = \int_{m_0}^{m} \frac{dm}{m}\ ;\ F_{\text{ext}} = 0$$

or $$-\frac{at}{u} = \log_e \frac{m}{m_0}$$

or $$m = m_0 e^{at/u}$$

2.53. A rocket of mass 3000 kg is set to fire in the vertical up direction. The gas is ejected at a speed of 500 m/sec. Calculate the mass of the gas ejected per sec in order (i) to overcome its weight (ii) to provide an initial acceleration of 9.8 m/sec^2 in the upward direction.

Solution:

(i) $$ma = -u\frac{dm}{dt} + mg = 0$$

$$\therefore \frac{dm}{dt} = \frac{mg}{u}$$

On putting the values, we have

$$\frac{dm}{dt} = \frac{3000 \quad 9.8}{500} = 58.8 \text{ kg/sec.}$$

(ii) $$m(g + a) = u \cdot \frac{dm}{dt}$$

or $$\frac{dm}{dt} = \frac{m(g \quad a)}{u}$$

On putting the values, we have

$$\frac{dm}{dt} = \frac{3000\,(9.8 \quad 9.8)}{500} = 12 \times 9.8 = 117.6 \text{ kg/sec}$$

2.54. A rocket of mass m_0 moves in the absence of external force by ejecting gas with exhaust velocity of 1.6 km/sec. For what ratio of the mass of the fuel to the mass of rocket it will attain a velocity of 11.2 km/sec.

Solution:

$$v = v_0 + v_r \log_e \frac{m_0}{m}$$

$$\frac{v}{v_r} = \log_e \frac{m_0}{m}\ ;\ v_0 = 0$$

$$\frac{11.2}{1.6} = 7 = \log_e \cdot \frac{m_0}{m}$$

or $\frac{m_0}{m} = e^7$

or $\frac{\text{mass of fuel}}{\text{mass of rocket}} = \frac{m_0 \quad m}{m} = e^7 - 1 = 1096 - 1 = 1095$

2.55. A cart containing m_0 mass of water moves under the action of a froce F along a horizontal plane. At the bottom of the cart there is a hole through which water comes out at a constant velocity of μ kg/sec. Assuming the initial velocity of the cart to be zero. Determine the velocity and the acceleration of the cart as a function of time by neglecting the frictional forces.

Solution:

$$\frac{F}{m} = \frac{dv}{dt} = -\frac{dv}{dm} \cdot \frac{dm}{dt} = -\frac{dv}{dm} \cdot \mu$$

$$F \cdot \int_{m_0 \quad \mu t}^{m_0} \frac{dm}{m} = -\mu \int_{v}^{0} dv$$

$$\mu V = \mathrm{F} \cdot \log_e \frac{m_0}{m_0 - \mu t}$$

$$\therefore V = \frac{F}{\mu} \cdot \log_e \frac{m_0}{m_0 - \mu t}$$

$$\frac{dV}{dt} = \text{acceleration} = \frac{F}{\mu} \times \frac{m_0 - \mu t}{m_0} \times \frac{m_0 \mu}{(m_0 - \mu t)^2}$$

$$= \frac{F}{m_0 - \mu t}$$

2.56. A flat car of mass m_0 initially at rest moves by applying a constant force F on a horizontal plane surface (see Fig. 2.30). From a stationary hopper sand spills on the car at a constant rate of μkg/sec. Neglecting the force of friction determine the velocity and the acceleration of the car in this process of adding mass every second.

Solution:

$F = \frac{d}{dt}(mv)$

or $Ft = mv + A$

at $t = 0$, $v = 0$

$\therefore A = 0$

hence $F \cdot t = mv$

Now $F \cdot t = (m_0 + \mu t)v'$

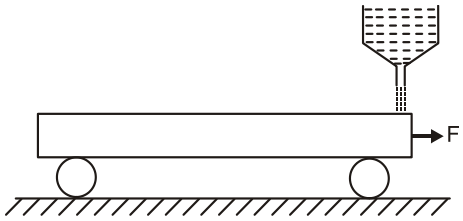


Fig. 2.30

$$\therefore\ v' = \frac{F \cdot t}{m_0 + \mu t}$$

$$\text{Acceleration} = \frac{dv}{dt} = \frac{F}{m_0}\ \frac{d}{dt}\left[t \times \left(1 + \frac{\mu t}{m_0}\right)^{-1} \right]$$

$$= \frac{F}{m_0 \left(1 + \frac{\mu t}{m_0}\right)^2}$$

2.57. A sphere initially at rest rolls down a smooth hill consisting of curved and straight paths from the point *O* situated at height *h* above the ground (see Fig. 2.31). At what value of the height *AB* of the hill will sphere cover maximum horizontal distance *x* along the straight path *BC*. What is this distance equal to ?

Solution: Let $AB = h_1$. In view of conservation of energy the velocity of the sphere at *A* is given by $v = \sqrt{2g\ \ h\ \ \ h_1}$

$$\therefore\ x = v.t = \sqrt{2g\ \ h\ \ \ h_1}\ \cdot \sqrt{\frac{2h_1}{g}} = 2\sqrt{h\ \ \ h_1\ \ h_1}\ ;$$

t is the time taken for the free fall distance h_1.

In the above equation x is a function of h_1. Differentiate x with respect to h_1 and equate it to zero.

$$\frac{dx}{dh_1} = \sqrt{\frac{h\ \ h_1}{h_1}}\ \ \sqrt{\frac{h_1}{h\ \ h_1}} = 0$$

or $h - 2h_1 = 0$

or $h_1 = \frac{h}{2}$

For this value of h_1

$$x = 2\sqrt{\frac{h}{2}\ \ \frac{h}{2}} = h$$

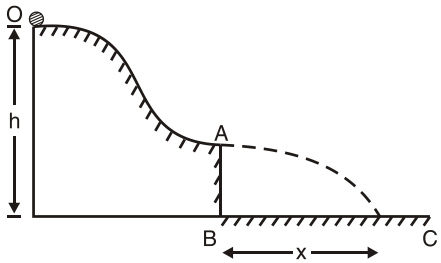


Fig. 2.31

2.58. A block of mass $m = 1$ kg is placed on a plane inclined at an angle $\theta = 60°$ with the horizontal pulled by a string attached to it at an angle ϕ with the plane (see Fig. 2.32a). The coefficient of friction k between the block and the plane is equal to 0.14. Find the angle ϕ at which the tension T in the string will be minimum. What is it equal to?

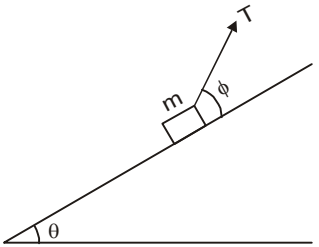


Fig. 2.32a

Solution: The various forces acting along and perpendicular to the plane are shown in Fig. 2.32b. The normal force $N = mg\cos\theta - T\sin\phi$. Under equilibrium the forces along the plane are represented by the equation.

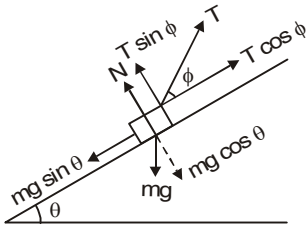


Fig. 2.32b

$$mg\sin\theta + f_s = T\cos\phi$$

where f_s = frictional force

$$= k\,(mg\cos\theta - T\sin\phi)$$

On substituting the value of f_s in the above equation, we have

$$mg\sin\theta + k\cdot mg\cos\theta - kT\sin\phi = T\cos\phi$$

$$T = \frac{mg\sin\theta + k\,mg\cos\theta}{(k\sin\phi + \cos\phi)}$$

In the above equation T is a function of ϕ. Differentiate T with respect to ϕ and equate it to zero.

$$\frac{dT}{d\phi} = \frac{-(mg\sin\theta + k\,mg\cos\theta)(k\cos\phi - \sin\phi)}{(k\sin\phi + \cos\phi)^2}$$

$\frac{dT}{d\phi}$ will be zero only when $k\cos\phi - \sin\phi = 0$

or $\tan\phi = k = 0.14$

$\therefore\ \phi = 8°$

$$\text{Now } T_{\min} = \frac{mg\sin\theta + k\,mg\cos\theta}{\left(k^2/\sqrt{1+k^2}\right) + \left(1/\sqrt{1+k^2}\right)}$$

$$= \frac{mg\sin\theta + k\,mg\cos\theta}{\sqrt{1+k^2}}$$

On substituting the values of m, g, θ and k, we have

$$T_{\min} = \frac{9.8 \quad \frac{\sqrt{3}}{2} \quad 0.14 \quad 9.8 \quad \frac{1}{2}}{\sqrt{1 \quad (0.14)^2}}$$

$$= \frac{4.9 \quad 1.87}{\sqrt{1.02}}$$

$$= 9.072 \text{ N}$$

2.59. A body in the form of a small cubical block of mass m can move freely without friction on a horizontal plane. The body is initially at rest and starts moving when a force $F = \alpha t$, α being a positive constant applied at an angle $\theta = 60°$ with the horizontal at an instant $t = 0$ (see Fig. 2.33a). Find (i) the time at which the body will detach from the plane and (ii) the velocity acquired and the distance travelled by the body at the time of its breaking off the plane.

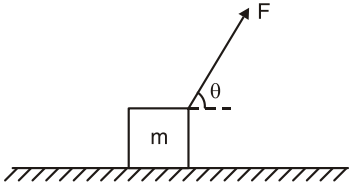


Fig. 2.33a

Solution: With reference to Fig. 2.33b the normal reaction force = $mg - \alpha t \sin\theta$. When the body breaks off the plane the normal reaction force is equal to zero. Thus

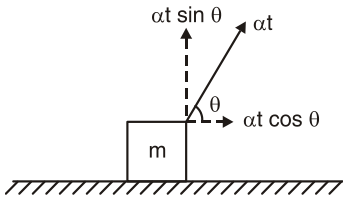


Fig. 2.33b

$$mg = \alpha t \sin\theta$$

$$\text{or } t = \frac{mg}{\alpha \sin\theta} = \frac{2mg}{\alpha\sqrt{3}}\ s$$

In other words the body will break off at a time $t = \dfrac{2mg}{\alpha\sqrt{3}}\ s$

From Newton's second law of motion, we have

$$\frac{d}{dt}(mv) = \alpha t \cos\theta$$

$$\text{or } mv = \alpha \cos\theta \int t\,dt = \alpha \cos\theta \cdot \frac{t^2}{2}$$

$$\text{or } v = \frac{m^2g^2}{\alpha^2 \sin^2\theta} \cdot \frac{\alpha \cos\theta}{2m} = \frac{mg^2 \cos\theta}{2\alpha \sin^2\theta}$$

$$= \frac{mg^2}{2\alpha} \times \frac{2}{3} = \frac{mg^2}{3\alpha}\ \text{m/s}$$

$$\text{Further } v = \frac{ds}{dt} = \frac{\alpha}{2m} \cos\theta \cdot t^2$$

$$\therefore s = \frac{\alpha \cos\theta}{2m} \cdot \int t^2 dt = \frac{\alpha \cos\theta \cdot t^3}{6m} = \frac{\alpha \cos\theta}{6m} \frac{m^3g^3}{\alpha^3 \sin^3\theta}$$

$$= \frac{m^2g^3}{6\alpha^2} \frac{\cos\theta}{\sin^3\theta}$$

$$= \frac{m^2g^3}{6\alpha^2} \frac{4}{3\sqrt{3}} = \frac{2m^2g^3}{9\alpha^2\sqrt{3}}\ \text{m}$$

2.60. A bar of mass $m = 600$ g lies on a smooth horizontal plane. The bar is initially at rest but starts moving along the plane when a force

$F = 1.96$ N (constant in magnitude) is applied to it. The direction of the force, making an angle β with the horizontal, is no longer constant but varies directly with x, the distance covered by the bar according to the equation $\beta = ax$ where a is a positive constant. Find the velocity of the bar.

Solution: In Fig. 2.34, the driving force is $F\cos\beta = 1.96\cos\cdot(ax)$. According to Newton's second law of motion,

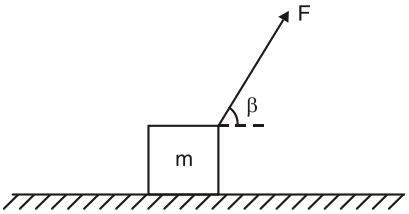


Fig. 2.34

$$\text{Force} = m\frac{dv}{dt} = 1.96\cos(ax)$$

$$\text{or} \quad \frac{dv}{dt} = \frac{1.96}{m}\cos(ax)$$

$$\text{or} \quad \frac{dv}{dx}\frac{dx}{dt} = \frac{1.96}{m}\cos(ax)$$

$$\text{or} \quad v\,dv = \frac{1.96}{m}\cos(ax)\,dx;\ \frac{dx}{dt} = v$$

On integrating the above equation we have

$$\frac{v^2}{2} = \frac{1.96}{ma}\sin(ax) \text{ or } v = \sqrt{\frac{2\times1.96}{600\times10^{-3}\times a}\sin\beta} = \sqrt{\frac{98}{15a}\sin\beta}$$

2.61. A body of mass $m = 1$ kg is slowly pulled up by a force F (constant in magnitude) along the surface of a hill as shown in Fig. 2.35. The coefficient of friction k between the body and the plane is 0.1. Find the amount of work done by this force if the height of the hill $h = 4.5$ m and the length of its base $l = 5$ m.

Solution: A force F is applied on the body of mass m. Under equilibrium we have

$$F = mg\sin\theta + k\,mg\cos\theta$$

Let the body moves through the dx along the hill. The amount of work done by the force is given by

$$F\cdot dx = (mg\sin\theta + k\,mg\cos\theta)\,dx$$

$$dx = mg\frac{dh}{dx}\cdot dx + k\,mg\frac{dl}{dx}\cdot dx;$$

$$\cos\theta = \frac{dl}{dx} \text{ and } \sin\theta = \frac{dh}{dx}$$

Here dl and dh represent the displacement along x and y axis respectively.

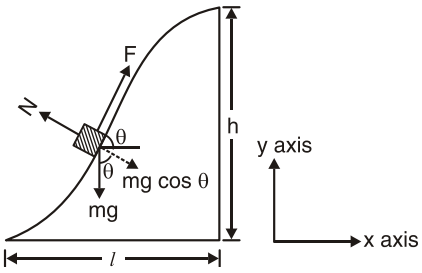


Fig. 2.35

$$\text{Net work done} = \int F\,dx = \int_0^h mg\,dh + k\,mg\int_0^l dl = mg\,h + k\,mg\,l$$
$$= mg\,(h + kl)$$

On substituting the values of h, k, l, m and g, we have

Work done $= 1 \times 9.8\,(4.5 + 0.1 \times 5)$

$= 9.8 \times 5$

$= 49$ J

2.62. A body of mass $M = 9.8$ kg with a small disc of mass $m = 200$ g placed on its horizontal surface ab, rests on a smooth horizontal plane (see Fig. 2.36). The disc can freely move along the smooth groove abc of the mass M. To what height (relative to its initial position) will the disc rise after separating from the body M when an initial velocity $v = 5$ m/s is given to it in the horizontal direction ?

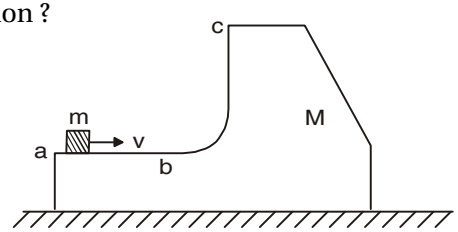


Fig. 2.36

Solution: According to the law of conservation of momentum, we have $mv = (M + m)\,v'$; (v' is the velocity with which the two bodies in contact are moving)

$$\therefore\ v' = \frac{mv}{M\ \ m}$$

Loss in the kinetic energy of the disc

$$= \frac{1}{2}\,mv^2 - \frac{1}{2}\,(M + m)\,v'^2$$

$$= \frac{1}{2}\,mv^2 - \frac{1}{2}\,(M + m)\,\frac{m^2v^2}{(M\ \ m)^2}$$

$$= \frac{1}{2}\,mv^2\ \ 1\ \ \frac{m}{M\ \ m}\ \ = \frac{1}{2}\,\frac{mMv^2}{M\ \ m}$$

With this energy the disc will move up to a height h after separating from the body M, i.e.,

$$\frac{1}{2}\,\frac{mMv^2}{M\ \ m} = mgh$$

$$\text{or}\quad h = \frac{v^2}{2g}\ \ \frac{M}{M\ \ m}$$

On substituting the values, we have

$$h = \frac{5\ \ 5}{2\ \ 9.8} \times \frac{9.8}{10} = 1.25 \text{ m}$$

2.63. Body A is placed on a wedge forming an angle $\alpha = 30°$ with the horizontal (Fig. 2.37a). What is the acceleration that should be imported to the wedge in the horizontal direction for body A to fall freely ?

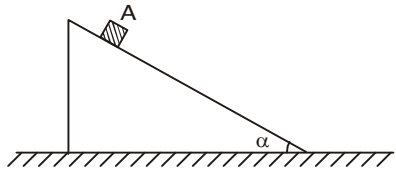


Fig. 2.37a

Solution: Let an acceleration a be imparted to the wedge from right to the left. The distance travelled by the wedge in time $t = AB = (1/2)\ at^2$. During the same time t the body falls freely through a distance

$AA' = (1/2)\ gt^2$ (see Fig. 2.37b).

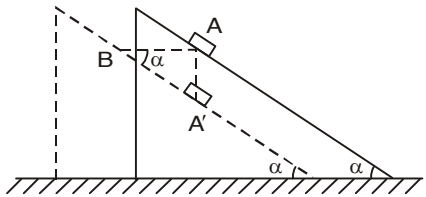


Fig. 2.37b

$$\text{Now } \cot\alpha = \frac{AB}{AA} = \frac{a}{g}$$

$$\therefore a = g\cot\alpha = 9.8 \times \cot 30$$

$$= 9.8 \times \sqrt{3}$$

$$= 16.97 \text{ m/s}^2$$

2.64. A body starting from the top of a right-angled prism slides down along its hypotenuse AB (see Fig. 2.38a). The length of the base of that prism is l. The coefficient of friction between the body and the surface AB of the prism is $k = 0.14$. At what value of angle α will the body take the least time for sliding down the surface ?

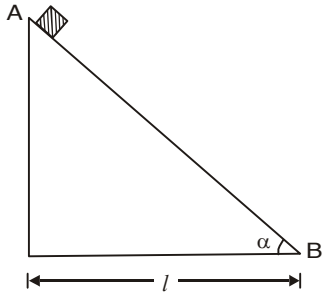


Fig. 2.38a

Solution: Various forces acting along and perpendicular to the surface AB are shown in Fig. 2.38b.

Normal reaction force $= N = mg\cos\alpha$.

Frictional force $= f_s = k\ mg\cos\alpha$.

Net downward force along the surface $= mg\sin\alpha - k\ mg\cos\alpha$.

Net downward acceleration along the surface $= g\sin\alpha - kg\cos\alpha$.

Time taken to cover the surface AB is given by

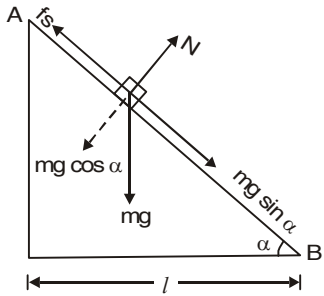


Fig. 2.38b

$$l\sec\alpha = \frac{1}{2}\ (g\sin\alpha - kg\cos\alpha)\ t^2$$

$$\text{or } t = \sqrt{\frac{2l}{g\ (\sin \quad \cos \quad k\cos^2\ )}}$$

Here t is a function of α. Find out the value of $\frac{dt}{d}$ and equate it to zero.

$$\frac{dt}{d} = \sqrt{\frac{2l}{g}} \frac{d}{dt} (\sin\alpha \cos\alpha - k\cos^2\alpha)^{-1/2}$$

$$= -\frac{1}{2}\sqrt{\frac{2l}{g}} \frac{(\cos 2\alpha + k \sin 2\alpha)}{(\sin\alpha\cos\alpha - k\cos^2\alpha)^{3/2}}$$

On equating it to zero we have $\cos 2\alpha + k \sin 2\alpha = 0$

or $\tan 2\alpha = -\frac{1}{k} = -\frac{1}{0.14}$

$\therefore \quad a = 49°$

2.65. A right-angled triangular block *A* has a body *B* of mass *m* placed on it as shown in Fig. 2.39a. Block *A* is given a horizontal acceleration *a* directed towards the left. What is the maximum value of this acceleration so that *B* is stationary relative to *A*? The coefficient of friction μ is such that $\mu < \cot\alpha$.

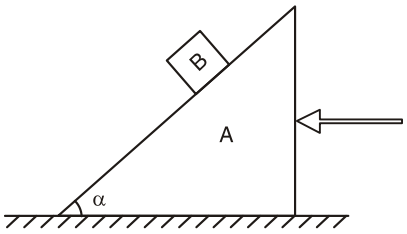


Fig. 2.39a

Solution: The various forces along and perpendicular to the inclined plane are shown in Fig. 2.39b. Normal reaction force = $mg\cos\alpha - ma\sin\alpha$.

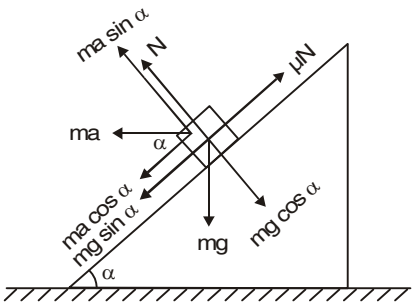


Fig. 2.39b

The net force along the inclined plane should be equal to zero since there is no motion, i.e., $mg\sin\alpha + ma\cos\alpha = \mu N = \mu\,(mg\cos\alpha - ma\sin\alpha)$

or $ma\,(\cos\alpha + \mu\sin\alpha) = mg\,(\mu\cos\alpha - \sin\alpha)$

or $a = \frac{g(\mu\cos\alpha - \sin\alpha)}{\cos\alpha + \mu\sin\alpha} = \frac{g(\mu\cot\alpha - 1)}{(\cot\alpha + \mu)}$

2.66. Prism 1 of mass m_1 and with angle α (Fig. 2.40a) rests on a horizontal surface.

Bar 2 of mass m_2 is placed on the prism. Assuming the friction to be negligible, find the acceleration of the prism.

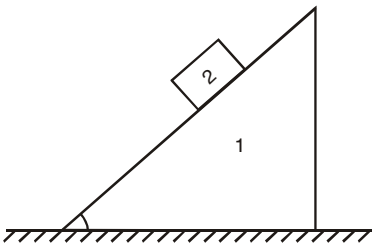


Fig. 2.40a

Solution: The various forces are shown in Fig. 2.40b. During the descent of the mass m_2 a downward force $m_2 g\sin\alpha$ acts along the plane.

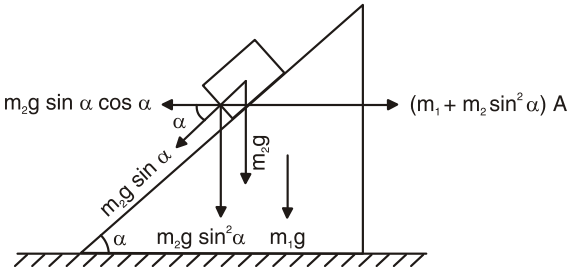


Fig. 2.40b

On resolving the force $m_2 g \sin \alpha$ along two mutually perpendicular axes we have a force $m_2 g \sin \alpha \cos \alpha$ parallel to the base of the prism and $m_2 g \sin^2 \alpha$ acting perpendicular to the base of the prism. Thus the total downward force = $(m_2 g \sin^2 \alpha + m_1 g)$.

Mass corresponding to this force = $(m_2 \sin^2 \alpha + m_1)$.

According to Newton's third law of motion we have,

$m_2 g \sin \alpha \cos \alpha = (m_2 \sin^2 \alpha + m_1) A$

where A is the acceleration of the prism.

$$\therefore A = \frac{m_2 g \sin \alpha \cos \alpha}{m_2 \sin^2 \alpha + m_1}$$

$$= \frac{g \sin \alpha \cos \alpha}{\sin^2 \alpha + (m_1 / m_2)}$$

2.67. Two blocks of masses M_1 and M_2 are kept touching each other on an inclined plane of inclination α with the horizontal (see Fig. 2.41a). Show that (i) the force of interaction between the blocks is $\frac{(\mu_1 - \mu_2) M_1 M_2 g \cos \alpha}{M_1 + M_2}$ and (ii) the minimum value of α at which the block just starts sliding is $\alpha = \tan^{-1} \frac{\mu_1 M_1 + \mu_2 M_2}{M_1 + M_2}$

Where μ_1 and μ_2 are the coefficients of friction between the block M_1 and the inclined plane and between the block M_2 and the inclined plane respectively.

Solution: (i) The various forces along and perpendicular to the plane are represented in Fig. 2.41b.

Let F be the forces of interaction between the block.

The equation of motion of block of mass M_1 is given by

$M_1 g \sin \alpha + F - \mu_1 M_1 g \cos \alpha = M_1 a$

Similarly the equation of motion of block of mass M_2 is given by

$M_2 g \sin \alpha - F - \mu_2 M_2 g \cos \alpha = M_2 a$

On dividing the two equations, we have

$$\frac{M_1 g \sin\alpha + F - \mu_1 M_1 g \cos\alpha}{M_2 g \sin\alpha - F - \mu_2 M_2 g \cos\alpha} = \frac{M_1}{M_2}$$

or $M_1 M_2 g \sin\alpha + M_2 F - \mu_1 M_1 M_2 g \cos\alpha = M_1 M_2 g \sin\alpha - M_1 F - \mu_2 M_1 M_2 g \cos\alpha$

or $F(M_1 + M_2) = (\mu_1 - \mu_2) M_1 M_2 g \cos\alpha$

or $F = \dfrac{(\mu_1 - \mu_2)}{M_1 + M_2} M_1 M_2 g \cos\alpha$

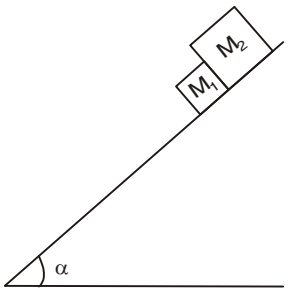


Fig. 2.41a

(ii) On putting $\alpha = 0$, the two equations of motion take the form $M_1 g \sin\alpha + F - \mu_1 M_1 g \cos\alpha = 0$ and $M_2 g \sin\alpha - F - \mu_2 M_2 g \cos\alpha = 0$

On adding them, we have

$$(M_1 + M_2) g \sin\alpha = (\mu_1 M_1 + \mu_2 M_2) g \cos\alpha$$

or $\tan\alpha = \dfrac{\mu_1 M_1 + \mu_2 M_2}{M_1 + M_2}$

or $\alpha = \tan^{-1}\left(\dfrac{\mu_1 M_1 + \mu_2 M_2}{M_1 + M_2}\right)$

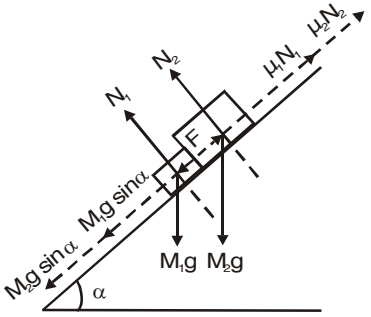


Fig. 2.41b

2.68. In the arrangement shown in Fig. 2.42a, the mass of the bar m and of the wedge M, as well as the wedge angle α, are known. The mass of pulley and the thread are negligible. Find the acceleration of the wedge M.

Solution: As the bar of mass m slides down the inclined plane through a distance x it acquires a velocity v relative to the plane. In the same time the wedge moves through a distance x and gains the speed v (see Fig. 2.42b). The resultant velocity of the bar is given by

$$V = \sqrt{v^2 + v^2 + 2v^2 \cos(180 - \alpha)} = \sqrt{2v^2 (1 - \cos\alpha)}$$

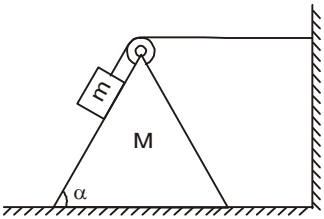


Fig. 2.42a

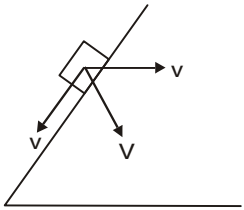


Fig. 2.42b

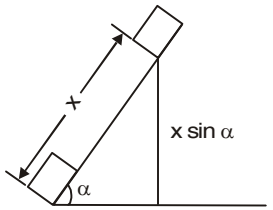


Fig. 2.42c

The loss in potential energy of the bar due to its fall through a height $x \sin \alpha$ is equal to $mg\,x \sin \alpha$. In view of the principle of conservation of energy we have $mg\,x \sin \alpha = \frac{1}{2} Mv^2 + \frac{1}{2} m\,[2v^2 (1 - \cos \alpha)]$

$$\text{or} \quad v^2 = \frac{2mg\,x \sin \alpha}{M + 2m(1 - \cos \alpha)}$$

Let a be the acceleration of the wedge. Then $v^2 = 2ax$

$$\text{or} \quad 2ax = \frac{2mg\,x \sin \alpha}{M + 2m(1 - \cos \alpha)}$$

$$\text{or} \quad a = \frac{mg \sin \alpha}{M + 2m(1 - \cos \alpha)}$$

2.69. A block of weight $W = 4$ kgf lies on a plane of length $l = 50$ cm inclined at an angle $\alpha = 30°$ with the horizontal (see Fig. 2.43a). With what force should the block be pressed against the surface in order to keep it in equilibrium. The coefficient of friction k between the block and the plane is 0.5.

Solution: The block will be in equilibrium when the net force along the plane is zero, i.e.,

$kN = W \sin \alpha$

Further in the direction perpendicular to the plane, we have $F + W \cos \alpha = N$ or $F = N - W \cos \alpha$

$$= \frac{W}{k} \sin \alpha - W \cos \alpha$$

On substituting the values of various terms, we have

$$F = \frac{4}{0.5} \times \frac{1}{2} - 4 \times \frac{\sqrt{3}}{2}$$

$$= 4 - 2\sqrt{3}$$

$$= 4 - 2 \times 1.73 = 0.54 \text{ kgf}$$

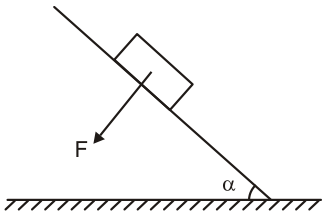


Fig. 2.43a

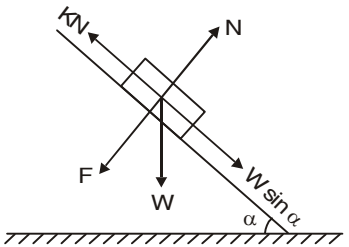


Fig. 2.43b

2.70. A wedge of mass $m_1 = 4$ kg and wedge angle $\alpha = 30°$ lies on a smooth horizontal plane. A rod of mass $m_2 = 1$ kg fixed between the guides and capable of sliding along that y-axis rests on it (Fig. 2.44a). Find the acceleration of both the bodies and the reaction of the wedge.

Solution: In Fig. 2.44b the forces acting on the rod are shown: (i) $m_2 g$ = weight of the rod, (ii) F = the force of reaction of the guide, and (iii) N = the force of reaction of the wedge. Due to the guide the rod is capable of moving along the y-axis. Its motion is given by the equation:

$$N\cos\alpha - m_2 g = m_2 a_y$$

Here a_y is the acceleration of the rod along the y axis.

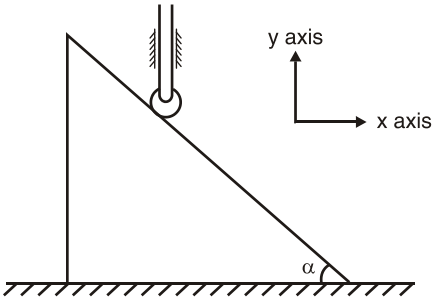


Fig. 2.44a

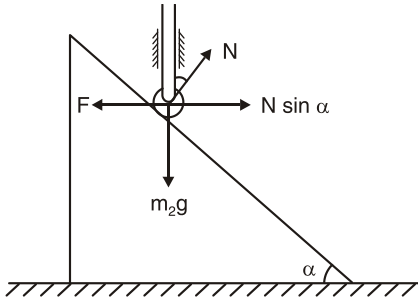


Fig. 2.44b

In Fig. 2.44c the forces acting on the wedge are shown: (i) N' = the force of reaction of the smooth horizontal plane on which the wedge rests, (ii) N = the force of reaction of the rod, and (iii) the weight $m_1 g$ of the wedge.

The equation of motion of wedge along x-axis is given by

$$-N\sin\alpha = m_1 a_x$$

Here a_x is the acceleration of the wedge along the x-axis.

Further the displacement of the rod along the y-axis is given by

$$dy = \frac{1}{2} a_y t^2$$

and of the wedge along x-axis by

$$dx = \frac{1}{2} a_x t^2$$

Now $\dfrac{dy}{dx} = \tan\alpha = \dfrac{a_y}{a_x}$

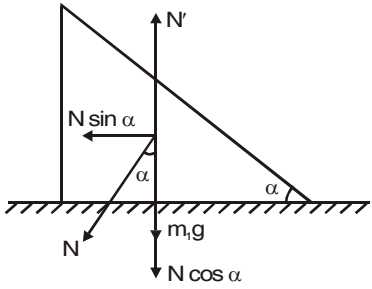


Fig. 2.44c

On dividing the equation $N\cos\alpha - m_2 g = m_2 a_y$ by $-N\sin\alpha = m_1 a_x$, we have

$$-\cot\alpha + \frac{m_2 g}{N\sin\alpha} = \frac{m_2}{m_1}\tan\alpha$$

$$\text{or } \frac{m_2 g}{N\sin\alpha} = \frac{m_2}{m_1}\tan\alpha + \frac{1}{\tan\alpha}$$

$$\text{or } \frac{m_1 m_2 g}{N\sin\alpha} = \frac{\left(m_2\tan^2\alpha + m_1\right)\cos\alpha}{\sin\alpha}$$

$$\text{or } N = \frac{m_1 m_2 g \cos\alpha}{m_2 \sin^2\alpha + m_1 \cos^2\alpha}$$

Substituting the values of various terms, we have

$$N = \frac{4 \quad 1 \quad 9.8 \quad \frac{\sqrt{3}}{2}}{\frac{1}{4} \quad 4 \quad \frac{3}{4}} = 10.43\ N$$

$$a_x = -\frac{N \sin\alpha}{m_1} = -\frac{m_2 g \sin\alpha \cos\alpha}{m_2 \sin^2\alpha + m_1 \cos^2\alpha}$$

$$= -\frac{1 \quad 9.8 \quad \frac{1}{2} \quad \frac{\sqrt{3}}{2}}{\frac{1}{4} \quad 4 \quad \frac{3}{4}} = -\frac{4.9\sqrt{3}}{6.5} = -1.3\ \text{m/s}^2$$

The negative sign indicates that the wedge will move along the negative side of the x-axis.

$$a_y = a_x \tan\alpha = -\frac{m_2 g \tan}{m_2 \tan \quad m_1 \cot}$$

$$= -\frac{1.3}{\sqrt{3}} = -0.75\ \text{m/s}^2$$

The negative sign indicates that the rod moved along the negative side of the y-axis.

2.71. A wedge of mass $M = 8.6$ kg and wedge angle $\alpha = 30°$ lies on a smooth horizontal plane. The tip of a rod of mass $m = 3.6$ kg being properly fixed between two couplings touches it (see Fig. 2.45a). Due to coupling the rod can move freely only along y-axis. Initially the rod and the wedge were both at rest. Calculate (i) the velocity of the wedge at the time the rod lowers through the height $h = 0.2$ m (ii) the velocity of the rod relative to the wedge.

Solution: (i) The velocities of the rod v_r and of the wedge v_w are related by the equation.

$v_r / v_w = \tan\alpha$ (Fig. 2.45b)

As the rod is lowered through a distance h the loss in its potential energy is balanced by the kinetic energy of wedge and rod both, i.e.,

$$\frac{1}{2} m v_r^2 + \frac{1}{2} M v_w^2 = mgh$$

$$\text{or } m v_w^2 \tan^2\alpha + M v_w^2 = 2mgh$$

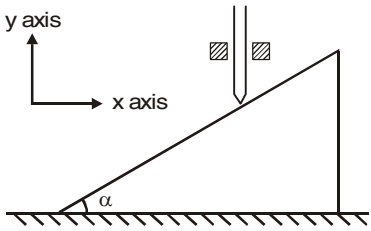


Fig. 2.45a

or $v_w = \sqrt{2mgh/M + m\tan^2\alpha}$

On substituting the values of various terms occurring in the above equation, we have

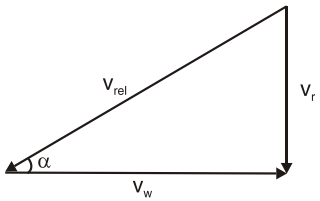


Fig. 2.45b

$$v_w = \sqrt{\frac{2 \quad 3.6 \quad 9.8 \quad 0.2}{8.6 \quad 3.6 \quad 1/3}}$$

$$= \sqrt{\frac{2 \quad 3.6 \quad 9.8 \quad 0.2}{9.8}} = 1.2 \text{ m/s}$$

(ii) From Fig. 2.45b, we have

$$v_{rel} = v_w \sec\alpha = \sec\alpha \times 1.2$$

$$= \frac{2 \quad 1.2}{\sqrt{3}} = 0.8 \times \sqrt{3}$$

$$= 1.384 \text{ m/s}$$

2.72. Two blocks of mass $m_1 = 0.7$ kg and $m_2 = 0.5$ kg connected by a spring of force constant $k = 1152$ N/m lie on a horizontal plane. Of the two, the block of mass m_1 is attached to the wall (see Fig. 2.46). The spring is compressed by displacing the block of mass m_2 towards the left through distance of $x = 10$ cm and then released. Calculate the velocity of the centre of mass of the system when the block of mass m_1 breaks off the wall.

Solution: Potential energy of compression = $\frac{1}{2}$ kx^2. When the block of mass m_1 breaks of the wall the spring will come back to its original length. The velocity of the block of mass m_2 can be determined on the basis of the law of conservation of energy.

Now $\frac{1}{2}$ m_2 $v_2^2 = \frac{1}{2}$ kx^2

or $v_2 = x\sqrt{\frac{k}{m_2}}$

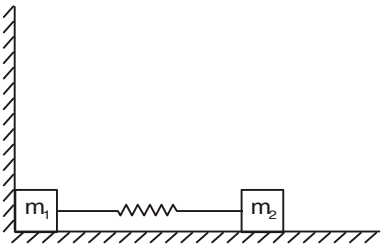


Fig. 2.46

Let us assume that the horizontal plane on which the two blocks lie is along x-axis. The x-coordinate of the centre of mass is given by

$X = \frac{m_1x_1 \quad m_2x_2}{m_1 \quad m_2}$; x_1 and x_2 are taken with reference to the wall.

On differentiating the above equation with repect to t we have

$$\frac{dX}{dt} = \frac{m_1}{m_1 \quad m_2}\frac{dx_1}{dt} + \frac{m_2}{m_1 \quad m_2}\frac{dx_2}{dt}$$

$$\frac{dx_1}{dt} = 0 \text{ and } \frac{dx_2}{dt} = v_2 = x\sqrt{k/m_2}$$

$$\therefore \quad \text{Velocity of centre of mass} = v = \frac{dX}{dt} = \frac{xm_2}{m_1 \quad m_2}\sqrt{\frac{k}{m_2}} = \frac{x\cdot\sqrt{m_2 k}}{m_1 + m_2}$$

On substituting the values of various terms, we have

$$v = \frac{0.1}{1.2}\sqrt{0.5 \quad 1152} = \frac{\sqrt{576}}{12} = 2 \text{ m/s}$$

2.73. Block B and pan P of equal mass M are connected through a spring of force constant k as shown in Fig. 2.47. The pan rests horizontally. A block B' having the same mass as that of B is dropped from a height of H above the pan and sticks to it. Find the minimum value of H at which the block B will bounce off the ground.

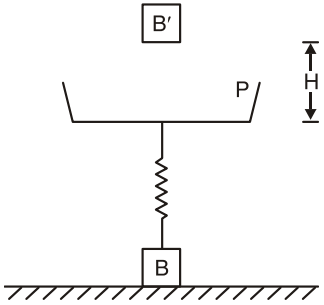


Fig. 2.47

Solution: As the mass B' falls the spring undergoes a compression Mg/k and from that an extension x. The tension in the spring = kx.

When $kx > Mg$ the block B will leave the ground.

Let H be the minimum height of the block B' above the pan when the block B leaves the ground. Then at that instant, Loss in potential energy of

$$B' = Mg \quad H \quad \frac{2Mg}{k}$$

Increase in potential energy of the pan = $Mg \cdot \frac{2Mg}{k}$

Now the block B will bounce off from the ground when

$$Mg \quad H \quad \frac{2Mg}{k} > Mg \quad \frac{2Mg}{k} \quad \text{or } H > \frac{4Mg}{k}$$

2.74. A homogeneous and inextensible chain of length l and mass m lies on a smooth table. A small portion of chain of length x hangs from the table. Initially the part of the chain lying on the table is held and then released. Calculate the velocity v with which the chain will leave the table.

Solution: With respect to the plane of the table,

Potential energy = $-\frac{m}{l}\, x \cdot g\, x/2$

As the chain slides we come across a situation in which the chain is exactly vertical and leaving the table with velocity v. Then Kinetic energy is $\frac{1}{2}\, mv^2$ and Potential energy = $\frac{mgl}{2}$.

In view of conservation of energy, we have

$$-\frac{m}{l} \times g \frac{x^2}{2} = \frac{1}{2} mv^2 - \frac{mgl}{2}$$

or $$v = \sqrt{\frac{g \; l^2 \quad x^2}{l}}$$

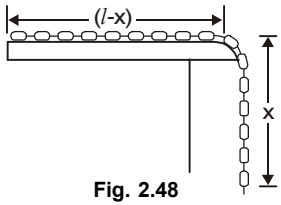


Fig. 2.48

2.75. Two blocks of mass m and $2m$ connected through a massless rod slide on a plane inclined at an angle $\alpha = 45°$ to the horizontal. The rod is parallel to the plane (see Fig. 2.49). The coefficients of friction k_1 and k_2 between the two masses and the plane, respectively, are 0.2 and 0.1. Find the common acceleration of the two masses.

Solution: The equation of motion of the two masses along the plane is given by

$$2ma = 2mg (\sin \alpha - k_2 \cos \alpha) - T;$$

$$k_2 N_2 = 2mg \, k_2 \cos \alpha$$

$$ma = T + mg (\sin \alpha - k_1 \cos \alpha);$$

$$k_1 N_1 = mg \, k_1 \cos \alpha$$

Here T stands for the tension in the rod and a for common acceleration.

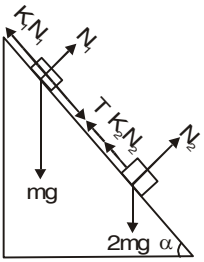


Fig. 2.49

On adding the two equations, we have

$$3ma = 3mg \sin \alpha - mg (k_1 + 2k_2) \cos \alpha$$

$$\therefore \; a = g \sin \alpha - \frac{g}{3} (k_1 + 2k_2) \cos \alpha$$

On substituting the values of the various terms, we have

$$a = \frac{g}{\sqrt{2}} - \frac{g}{\sqrt{2}} \times \frac{0.4}{3}$$

$$= \frac{g}{\sqrt{2}} \quad \frac{3 \quad 0.4}{3}$$

$$= \frac{9.8 \quad 2.6}{1.41 \quad 3} = 6.02 \text{ m/s}^2$$

2.76. A block of mass m slides in a right-angled trough inclined at an angle $\alpha = 30°$ with the horizontal (see Fig. 2.50a). Calculate the acceleration of the block if the coefficient of friction k between the trough and the block is 0.141.

Solution: A right-angled trough has two walls at an angle of 90° (see Fig. 2.50b). During motion there will be friction at the two walls. The total frictional force $f_s = 2 \; k.mg \cos \alpha \cos 45 = \sqrt{2} \; k \; mg \cos \alpha$.

The equation of motion of the block is given by

$ma = mg \sin \alpha - f_s$

$= mg \sin \alpha - \sqrt{2}\, k\, mg \cos \alpha$

$\therefore\ a = g \quad \sin \quad \sqrt{2}\, k \cos$

On substituting the value of g, α and k, we have

$a = 9.8 \quad \frac{1}{2} \quad \frac{\sqrt{2} \quad 0.141 \quad \sqrt{3}}{2}$

$= 9.8 \quad 0.5 \quad \frac{0.141 \quad 1.73}{1.41}$

$= 9.8\ [0.5 - 0.173]$

$= 9.8 \times 0.327 = 3.2\ \text{m/s}^2$

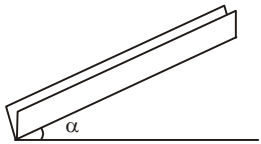


Fig. 2.50a

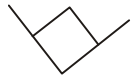


Fig. 2.50b

2.77. Two blocks are tied through a string as shown in Fig. 2.51. Find (i) the force F exerted by the two blocks in equilibrium. (ii) the force F' required to carry up the two masses with an acceleration of 2.2 m/s^2 and (iii) the tension in the string lying between the two blocks.

Solution:

(i) $F = (2 + 4) \times 9.8 = 58.8$ N

(ii) The equation of motion of the two blocks is given by

$F' - 2g - T = 2a$

and $T - 4g = 4a$ (a is the acceleration)

On addition of the two we have

$F' = 6\ (a + g) = 6\ (2.2 + 9.8) = 6 \times 12 = 72$ N

(iii) $T = 4\ (a + g) = 4\ (2.2 + 9.8) = 4 \times 12 = 48$ N

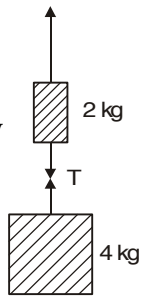


Fig. 2.51

2.78. A piece of ice slides down a plane inclined at an angle $\theta = 30°$ with the horizontal in twice the time it takes to slide down a similar frictionless plane. Find the coefficient of friction k between the ice and the plane.

Solution: Various forces are shown in Fig. 2.52. When friction is present the equation of motion of ice of mass m is given by

$mg \sin \theta - kmg \cos \theta = ma_2$; a_2 is the acceleration of ice along the plane or $a_2 = g\ (\sin \theta - k \cos \theta)$

If x be the length of the inclined plane, then

$$x = \frac{1}{2}\, a_2\, t_2^2 = \frac{1}{2}\, g\, (\sin \theta - k \cos \theta)\, t_2^2$$

where t_2 is the total time taken by the ice to slide down the plane.

In the absence of friction the equation of motion is given by

$ma_1 = mg \sin \theta$; a_1 is the acceleration of ice along the plane.

or $a_1 = g \sin\theta$

The distance travelled in time t_1 is given by

$$x = \frac{1}{2} g \sin\theta \cdot t_1^2$$

On dividing the two equations of x we have

$$1 = \frac{\sin\theta - k\cos\theta}{\sin\theta} \left(\frac{t_2}{t_1}\right)^2 ; \frac{t_2}{t_1} = 2$$

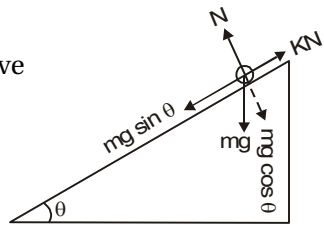


Fig. 2.52

or $\frac{1}{4} = \frac{\sin\theta - k\cos\theta}{\sin\theta}$

or $0.25 \sin\theta = \sin\theta - k\cos\theta$

or $k\cos\theta = 0.75 \sin\theta$

or $k = 0.75 \tan\theta$

$= 0.75 \tan 30$

$= \frac{0.75}{\sqrt{3}} = \frac{0.75}{1.73} = 0.4336$

2.79. In the Fig. 2.53 the mass of the blocks *A*, *B* and *C* is 2 kg each. Neglecting the mass of the pulley and the string find the tension in the string connecting the two blocks *B* and *C*.

Solution: The equation of motion of block *C* is given by

$2g - T = 2a$; T is the tension in the string connecting the blocks *B* and *C*. Similarly the equation of motion of block *B* and *A* is given by

$2g + T - T_1 = 2a$; a is the common acceleration.

and $T_1 - 2g = 2a$; T_1 is the tension in the string connecting the blocks *A* and *B*.

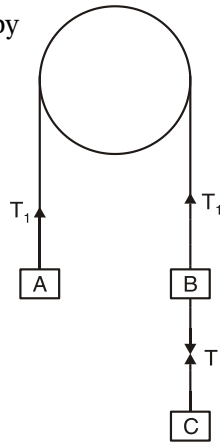


Fig. 2.53

On adding the three equations, we have

$2g = 6a$ or $a = \frac{g}{3}$

Now $T = 2g - 2a = 2g - \frac{2g}{3}$

$= \frac{4}{3} \frac{9.8}{} = \frac{39.2}{3} = 13.07$ N

2.80. A massless thread passing over a frictionless and massless pulley contained two masses m_1 and m_2 at its end as shown in Fig. 2.54. Find the tension in the string if the pulley as a whole moves up with an acceleration g.

Solution: Let a be the acceleration of the two masses m_1 and m_2. Let us assume that m_1 is going down and m_2 is going up. Now m_1 will go

downwards with a resultant acceleration $a - g$ and m_2 will go up with a resultant acceleration $a + g$. The equation of motion of the two masses is given by

$m_1 (a - g) = m_1 g - T$; T is the tension in the string

and $m_2 (a + g) = T - m_2 g$

On adding the two equations, we have

$(m_1 + m_2)\, a = 2 (m_1 - m_2)\, g$

or $a = \dfrac{2\ m_1 \quad m_2\ g}{m_1 \quad m_2}$

Now $T = 2m_1 g - m_1 a$

$$= 2m_1 g - 2m_1 \frac{m_1 \quad m_2}{m_1 \quad m_2}\, g$$

or $\quad T = \dfrac{4m_1 m_2\, g}{m_1 \quad m_2}$

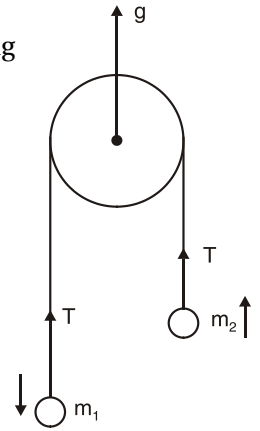


Fig. 2.54

2.81. In the arrangment shown in Fig. 2.55a the plane on which the block A moves is inclined at an angle $\theta = 45°$ with the horizontal. The coefficient of friction k between the block A and the plane is 0.141. The ratio of the masses of the blocks B and A $\dfrac{m_B}{m_A} = \eta = \dfrac{3}{2}$. Initially the system is stationary. Find the acceleration of the block B when the masses start moving.

Solution: $m_B = \eta\ m_A$ (given)

The various forces acting on the mass m_A are shown in Fig. 2.55b. Let a be the common acceleration of the system. The equation of motion of the two blocks is given by

$T - m_A g \sin\theta - k\ m_A\ g \cos\theta = m_A\ a$; T is the tension in string and $\eta\ m_A\ g - T = \eta\ m_A\ a$

On adding the two equations, we have

$m_A\ g\ (\eta - \sin\theta - k \cos\theta) = (\eta + 1)\ m_A\ a$

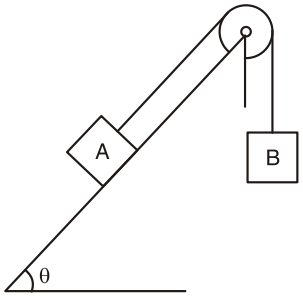


Fig. 2.55a

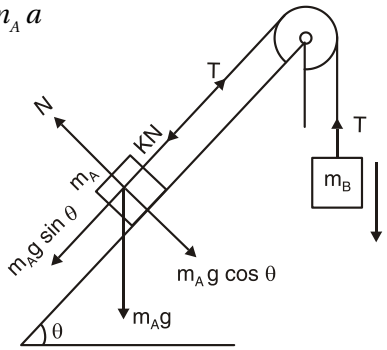


Fig. 2.55b

or $a = \dfrac{g}{\eta + 1}\ (\eta - \sin\theta - k\cos\theta)$

On substituting the values of θ, k, η and g we have

$$a = \frac{9.8}{\frac{3}{2}\ \ 1}\ \ \frac{3}{2}\ \ \frac{1}{\sqrt{2}}\ \ 0.141\ \ \frac{1}{\sqrt{2}}$$

$$= \frac{9.8\ \ 2}{5}\ [1.5 - 0.707 - 0.1]$$

$$= \frac{9.8\ \ 2\ \ 0.693}{5} = 2.71\ \text{m/s}^2$$

2.82. In the arrangement shown in Fig. 2.56, block A moves on a frictionless plane inclined at an angle $\alpha = 30°$ with the horizontal. Neglecting the masses of the pulley and the connecting strings find the acceleration of block B whose mass is $\eta = 2$ times the mass of block A.

Solution: Let m and ηm be the masses of the two blocks A and B respectively. The equation of motion of two blocks is given by mg sin $\alpha - T = ma_1$; a_1 is the acceleration of block A.

$\eta mg - T_1 = \eta ma$; a is the acceleration of block B.

On substituting $a_1 = \dfrac{a}{2}$ in the first equation, we have

$$-mg\sin\alpha + T = \frac{ma}{2}$$

or $\quad ma = 2T - 2mg\sin\alpha$

and $\eta\ ma = \dfrac{T}{2} + \eta\ mg$ since $T = 2T_1$

On solving the two equations, we have

$$a = \frac{2g\,(2\eta - \sin\alpha)}{4\eta + 1}$$

On substituting the values of α and η, we have

$$a = \frac{2g\ \ 3.5}{9} = \frac{7}{9}g$$

$$= 7.62\ \text{m/s}^2$$

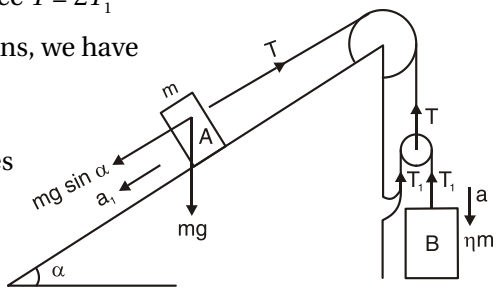


Fig. 2.56

2.83. Calculate the acceleration of the system and the tension in the strings for the system shown in Fig. 2.57.

Solution: The equation of motion of mass 8 kg is given by

$$8g - T_1 = 8a \qquad \text{(i)}$$

Similarly equation of motion of mass 2 kg and 4 kg is given by

$T_1 - T_2 - 2g\cos 60 = 2a$ (ii)

and $T_2 - 4g\cos 60 = 4a$ (iii)

On adding the three equations we have

$8g - 6g\cos 60 = 14a$

or $14a = 5g$

$$\therefore \quad a = \frac{5g}{14}$$

$$= \frac{5 \quad 9.8}{14} = 3.5 \text{ m/s}^2$$

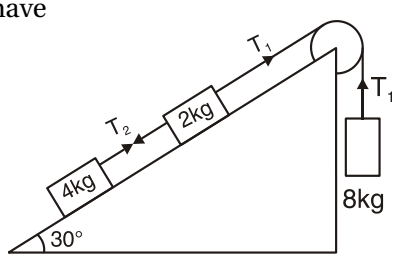


Fig. 2.57

Using equation (i), we have

$T_1 = 8\,(9.8 - 3.5) = 8 \times 6.3 = 50.4$ N

Using equation (iii) we have

$T_2 = 4a + 4g\cos 60 = 4a + 2g$

$= 4 \times 3.5 + 2 \times 9.8 = 33.6$ N

2.84. In the system shown in Fig. 2.58 block of mass m_1 slides on a frictionless plane inclined at an angle θ with the horizontal. Neglecting the mass of the pulley and connecting strings, find the acceleration of block of mass m_1 moving up along the plane.

Solution: In the arrangement shown in the Fig. 2.58 mass m_1 has a displacement twice that of m_2. The acceleration of m_1 will therefore be twice that of m_2.

Let a be the acceleration of m_1. Then the acceleration of m_2 is $a/2$.

The equation of motion of mass m_2 is given by

$$m_2 g - 2T = \frac{m_2 a}{2} \qquad \text{(i)}$$

Similarly the equation of motion of mass m_1 is given by

$T - m_1 g \sin\theta = m_1 a$ (ii)

One eliminating T from these two equations

$$a = \frac{2g\left(m_2 - 2m_1 \sin\theta\right)}{4m_1 + m_2}$$

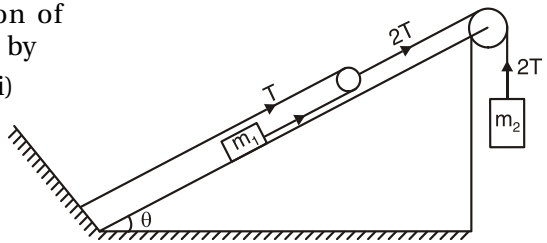


Fig. 2.58

2.85. In the system shown in Fig. 2.59 blocks B and C of mass m_1 and m_2 rest on a horizontal plane. The coefficient of friction between the blocks and the plane is k. Neglecting the mass of the pulley and the connecting strings find (i) the acceleration with which block A of mass m_0, comes down and (ii) the tension T in the string connecting the two blocks B and C.

Solution: Let a be the acceleration of the system. Equation of motion of block C of mass m_2 is given by

$$T - m_2 gk = m_2 a \quad \text{(i)}$$

Equation of motion of block B of mass m_1 is given by

$$T_1 - T - m_1 gk = m_1 a \quad \text{(ii)}$$

Equation of motion of block A of mass m_0 is given by

$$m_0 g - T_1 = m_0 a \quad \text{(iii)}$$

On adding the three equations, we have

$$g\,[m_0 - k\,(m_1 + m_2)] = (m_0 + m_1 + m_2)\,a$$

or $$a = \frac{g \quad m_0 \quad k \quad m_1 \quad m_2}{m_0 \quad m_1 \quad m_2}$$

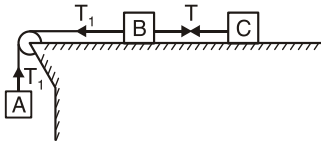


Fig. 2.59

On substituting the value of a in equation (i) we have

$$T = m_2 a + m_2 gk = m_2\,[a + gk]$$

$$= m_2 \quad \frac{gm_0 \quad gk \quad m_1 \quad m_2 \quad g\,k m_0 \quad g\,k m_1 \quad g\,k m_2}{m_0 \quad m_1 \quad m_2}$$

$$= \frac{gm_2 m_0 (1 \quad k)}{m_0 \quad m_1 \quad m_2}$$

2.86. A frictionless cart of mass M carries two other frictionless carts connected by a string passing over a pulley as shown in Fig. 2.60. Calculate the force which, when applied to the cart of mass M, will prevent the two carts of mass m_1 and m_2 to move relative to it.

Solution: Let a be the acceleration when the two carts of mass m_1 and m_2 are moving. Now we have

$$m_1 a = T = m_2 g \text{ or } a = \frac{m_2}{m_1}\, g$$

Rrequired force $F = (m_1 + m_2 + M)\,a$

$$= (m_1 + m_2 + M)\,\frac{m_2}{m_1}\, g$$

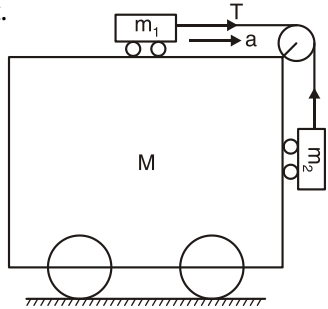


Fig. 2.60

2.87. In the arrangment shown in Fig. 2.61, the bodies have masses m_0, m_1 and m_2, friction is absent, and the mass of the pulleys and the threads is negligible. Find the acceleration of body m_1.

Solution: Equation of motion of mass m_0 is given by

$$m_0\, a_3 = T \quad \text{(i)}$$

Equation of motion of masses m_1 and m_2 is given by

$$m_1 g - T_1 = m_1 a_1 \qquad \text{(ii)}$$

$$T = 2T_1 \text{ and } \frac{a_1 \quad a_2}{2} = a_3$$

$$T_1 - m_2 g = m_2 a_2 \qquad \text{(iii)}$$

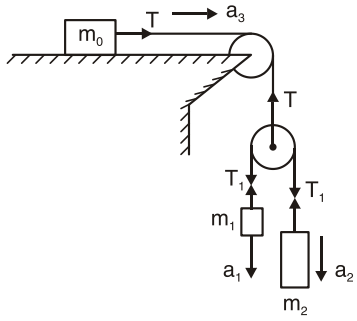


Fig. 2.61

On substituting $T_1 = T/2 = m_0 a_3/2$ in equation (iii), we have

$$m_0 a_3 - 2 m_2 g = 2 m_2 a_2 \qquad \text{(iv)}$$

Similarly on substituting $T_1 = m_0 a_3/2$ in equation (ii), we have

$$2m_1 g - m_0 a_3 = 2m_1 a_1 \qquad \text{(v)}$$

On substituting a_3 as $(a_1 - a_2)/2$ in equations (iv) and (v) we have

$$m_0 (a_1 - a_2) - 4m_2 g = 4 m_2 a_2$$

and $4 m_1 g - m_0 (a_1 - a_2) = 4m_1 a_1$

On solving these two equations, we get

$$a_1 = g \left[\frac{4m_1 m_2 + m_0 (m_1 - m_2)}{4m_1 m_2 + m_0 (m_1 + m_2)} \right]$$

2.88. In the system shown in Fig. 2.62 the mass ratio of the ball and the rod $(m_1/m_2) = \eta = 1.5$. The length of the rod $l = 1$ m. The ball is released after setting it at the level of the lower end of the rod. Find: (i) the acceleration of the ball, and (ii) the time after which the ball will be at the other end of the rod.

Solution: In view of the arrangement shown the mass m_2 is displaced twice as fast as m_1. Therefore its acceleration will be twice that of m_1. Let a be the acceleration of m_1. Then the acceleration of m_2 will be $2a$. The equation of motion of the ball and the rod is given by

$$2T - m_1 g = m_1 a$$

and $m_2 g - T = 2m_2 a$

On solving the two equations, we get

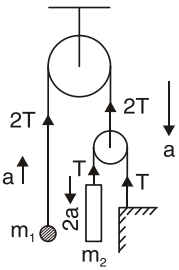


Fig. 2.62

$$a = \frac{2m_2 \quad m_1}{4m_2 \quad m_1} \quad g$$

$$= \left(\frac{2 - \eta}{4 + \eta} \right) g$$

$$= \frac{0.5}{5.5} \times 9.8 = 0.89 \text{ m/s}^2$$

In view of the equation of motion $s = ut + \frac{1}{2} a_1 t^2$

We have $l = 0 + \frac{3}{2}\ \frac{(2-\eta)}{4+\eta}\ gt^2$ (here a_1 is the relative acceleration = $3a$)

$$\therefore\ t = \sqrt{\frac{2l(4+\eta)}{3(2-\eta)g}} = \sqrt{\frac{2\ \ 1\ \ 5.5}{3\ \ 0.5\ \ 9.8}} = \sqrt{\frac{22}{3\ \ 9.8}} = 0.865 \text{ s}$$

2.89. Two blocks A and B are connected to each other by a string and a spring; the string passes over a frictionless pulley as shown in Fig. 2.63*a*. Block B slides over the horizontal top surface of a stationary block C and the block A slides along the vertical side of C, both with the same uniform speed. The coefficient of friction between the surfaces of the blocks is 0.2. The force constant of the spring is 1960 N/m. If the mass of block A is 2 kg, calculate the mass of block B and the energy stored in the spring.

Solution: As the speed of the two blocks is the same, there is no net force acting on the two bodies, i.e.,

$T = \mu N = \mu$ mg (for block B); μ is the coefficient of friction (see Fig. 2.63b).

For block A, we have $2g = kx = T = \mu mg$; k is the force constant.

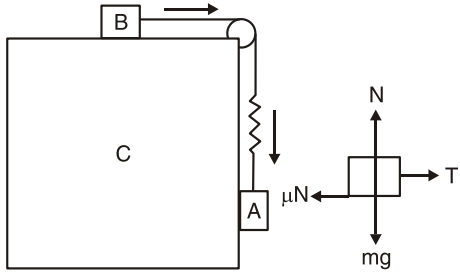


Fig. 2.63a

Fig. 2.63b

$$\text{or } m = \frac{2g}{\mu g} = \frac{2}{\mu} = \frac{2}{0.2} = 10 \text{ kg}$$

$$\text{Now } x = \frac{2\ \ 9.8}{1960} = \frac{1}{100} \text{ m}$$

$$\text{Energy stored} = \frac{1}{2}\ kx^2 = \frac{1}{2} \times 1960 \times 10^{-4} = 0.098 \text{ J}$$

2.90. The bodies B and C of equal mass m are arranged on a block A as shown in Fig. 2.64. The coefficient of friction between A and either B or C is μ. Assuming the pulley and the string to be massless and the pulley to be smooth, calculate the minimum horizontal acceleration with which block A should move so that B and C both remain stationary, relative to it.

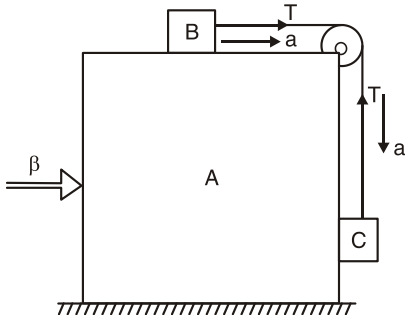


Fig. 2.64

Solution: Let an acceleration β is applied to the body A.

The equation of motion of body B is given by

$T - \mu mg = m(\beta + a)$ [$(\beta + a)$ is the resultant acceleration of body B.]

The equation of motion of body C is given by

$mg - \mu m\beta - T = ma$ [($\mu m\beta$ is the frictional force acting on the body C.)]

On adding the two equations, we have

$mg(1-\mu) - \mu m\beta = m(\beta + 2a)$

Since the bodies B and C have to be stationary, i.e., $a = 0$, we have

$mg(1-\mu) = m(1+\mu)\beta$

$$\therefore \beta = \frac{g(1-\mu)}{1+\mu}$$

2.91. A cycle chain AB of length l is placed inside a hollow horizontal smooth pipe in such a way so that a part of the chain of length h hangs freely with the end B touching a table (see Fig. 2.65). The end of the tube A is set free. The chain starts sliding inside the pipe. Find the velocity of the chain when end A slips out of the pipe.

Solution: Let T be the tension at the highest point of the hanging part of the chain. The equation of the motion of the chain is given by

$mgh - T = mha$; m = mass per unit length of the chain.

Let x be the length of the chain inside the pipe at some instant, then

$$T = x \cdot ma$$

$$= xm \cdot \frac{dv}{dt}$$

$$T = xm \cdot \frac{dv}{dx} \cdot \frac{dx}{dt}$$

$$= x \cdot mv \cdot \frac{dv}{dx}$$

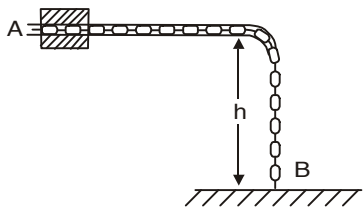


Fig. 2.65

Now $mgh - xmv\frac{dv}{dx} = mh \cdot v \cdot \frac{dv}{dx}$

or $gh = \frac{vdv}{dx}(h+x)$

or $\int_0^{l-h} \frac{gh}{h+x}\,dx = \int_0^V v dv$

$$gh\,|\log_e(h\quad x)|_0^{l\ \ h} = \frac{V^2}{2}$$

or $V = \sqrt{2gh\log_e l/h}$

2.92. Two similar buggies a and b each of mass M carrying one man of mass m moves without friction towards each other. When they are opposite to each other the men exchange their positions by way of jumping in a direction perpendicular to their direction of motion. As a result buggy

a stops whereas buggy *b* keeping going in the same direction with velocity *v*. Find the initial velocities of the two buggies.

Solution: Let v_1 and v_2 be the initial velocities of the two buggies. Due to jump there is a change in momentum.

Change in momentum of buggy $a = -(mv_1 + mv_2)$

For buggy *a* we have

$(M + m)\, v_1 - m\,(v_1 + v_2) = 0$; momentum mv_1 and mv_2 are exchanged but in opposite direction.

$\therefore\ Mv_1 = mv_2$

For buggy *b* we have

$$(M + m)v_2 - m\,(v_1 + v_2) = (M + m)\, v$$

or $$Mv_2 - mv_1 = (M + m)\, v$$

or $$Mv_2 - \frac{m^2 v_2}{M} = (M + m)\, v$$

$$(M - m)\, v_2 = Mv$$

or $$v_2 = \frac{Mv}{M \quad m}$$

$$v_1 = \frac{mv}{M \quad m}$$

2.93. A plank supporting a bar of mass *m* is placed on a horizontal plane. One end of a light elastic non-deformed cord of length *l* is attached to the bar while its other end to a rigid support *O* as shown in Fig. 2.66a. The plank is slowly moved towards the right by applying a force. At the moment when the cord deviates through an angle θ the bar starts sliding over the plank. Assuming the coefficient of friction between the bar and plank as μ find the work performed by the frictional force.

Solution: The forces acting on the bar is shown in Fig. 2.66b. Change in length of the cord

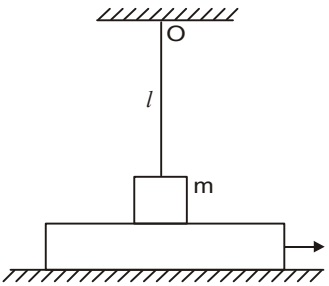


Fig. 2.66a

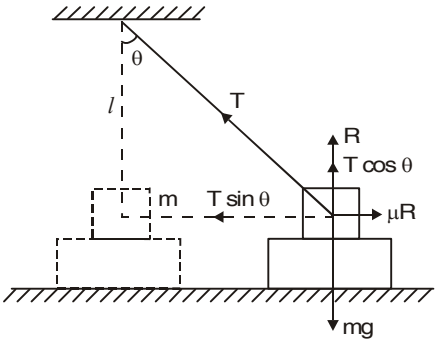


Fig. 2.66b

$\Delta l = l\,(\sec\theta - 1)$

$T\sin\theta = \mu R$

and $R + T\cos\theta = mg$

$\therefore\ R = (mg - T\cos\theta)$

With this value of R we have

$T\sin\theta = \mu\,(mg - T\cos\theta)$

or $T = \dfrac{\mu mg}{\sin\theta + \mu\cos\theta}$

$T = K\cdot\Delta l$; K = force constant

$$\therefore K = \frac{\mu\, mg}{l(\sin\theta + \mu\cos\theta)(\sec\theta - 1)}$$

$$\text{Work performed} = \frac{1}{2}\,K\cdot\Delta l^2$$

$$= \frac{1}{2}\cdot\frac{\mu mg}{l(\sin\theta + \mu\cos\theta)(\sec\theta - 1)}\times l^2\,(\sec\theta - 1)^2$$

$$= \frac{l\cdot\mu mg\,(1 - \cos\theta)}{2\cos\theta(\sin\theta + \mu\cos\theta)}$$

2.94. Bar A of mass m placed on a horizontal plane as shown in Fig. 2.67a is attached to its left by thread to a point P and to its right by a thread attached to a particle B of mass m passing over a weightless pulley. In addition to this the bar is attached to a point O by means of spring of length l and force constt. $K = 5$ mg/l. On cutting the thread PA the bar starts moving to the right. Find its velocity when it is breaking off the horizontal plane.

Solution: When the bar moves towards the right the spring becomes elongated. When the bar breaking off the plane we have

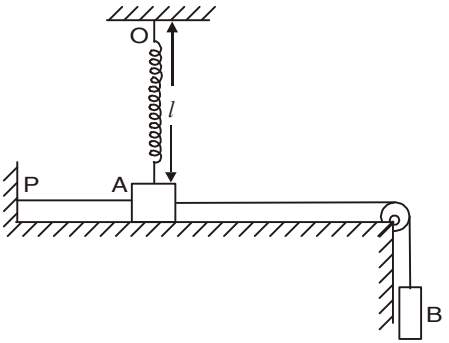


Fig. 2.67a

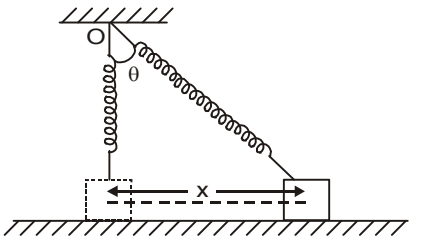


Fig. 2.67b

$KL\cos\theta = mg$; $L = \sqrt{l^2 - x^2} \quad l$ (see Fig. 2.67b)

or $\frac{5mg}{l} \cdot \sqrt{l^2 - x^2} \quad l \cdot \frac{l}{\sqrt{l^2 - x^2}} = mg$

or $5 \sqrt{l^2 - x^2} \quad l = \sqrt{l^2 - x^2}$

or $4 \cdot \sqrt{l^2 + x^2} = 5l$

or $16l^2 + 16x^2 = 25l^2$

or $x = \frac{3l}{4}$

According to conservation of energy.

Gain in *K.E* of *A* + gain in *K.E* of *B* + gain in *P.E* of spring = Loss in *P.E* of *B*

$$\frac{1}{2} mv^2 + \frac{1}{2} mv^2 + \frac{1}{2} KL^2 = mgx$$

or $mv^2 + \frac{1}{2} \cdot \frac{5mg}{l} \quad \sqrt{l^2 - x^2} \quad l^{\,2} = mg \cdot \frac{3l}{4}$

or $mv^2 + \frac{5mg}{2l} \quad \sqrt{l^2 - \frac{9l^2}{16}} \quad l^{\,2} = \frac{3mgl}{4}$

or $mv^2 + \frac{5mg}{2l} \times \frac{l^2}{16} = \frac{3mgl}{4}$

or $v^2 = \frac{3gl}{4} - \frac{5gl}{32}$

$$= \frac{19gl}{32}$$

$$\therefore v = \sqrt{\frac{19gl}{32}}$$

3

CIRCULAR MOTION AND GRAVITATION

3.1. A particle moves around a circle of radius 15 cm with a constant tangential acceleration of $A_t = 2.5$ cm/sec^2. Calculate the time from the beginning of the motion at which the normal acceleration A_n will be equal to the tangential acceleration of the particle.

Solution:

$$A_t = \frac{v}{t}$$

$$A_n = \frac{v^2}{r}$$

$$t = \sqrt{\frac{r \cdot A_n}{A_t^2}} = \sqrt{\frac{0.15}{2.5 \quad 10^{\ 2}}} = 2.45 \text{ sec.}$$

3.2. A small particle moves around a circular track of radius 15 cm with a constant tangential acceleration A_t. Calculate A_t if the particle acquires a velocity $v = 88$ cm/sec after completing four complete revolutions.

Solution:

$$\frac{d\omega}{dt} = \frac{A_t}{r} \text{ ; } r = \text{radius of the path}$$

$$\omega = \frac{A_t}{r} \cdot t$$

$$\int d\theta = \frac{A_t}{r} \cdot \quad t \cdot dt$$

$$\theta = \frac{A_t \cdot t^2}{2r} + C$$

$$\text{At } t = 0, \theta = 0$$

$$\therefore \quad C = 0$$

$$\therefore \quad \theta = \frac{A_t \cdot t^2}{2r}$$

$$2\pi n = \frac{A_t}{2r} \cdot t^2$$

On putting $t = \dfrac{v}{A_t}$

$$2\pi n = \frac{v^2}{2r \cdot A_t}$$

$$\therefore \quad A_t = \frac{v^2}{2\pi n \cdot 2r}$$

$$= \frac{88^2 \quad 10^4}{2 \quad 3.14 \quad 4 \quad 2 \quad 0.15}$$

$$= 0.103 \text{ m/sec.}^2$$

3.3. A small particle moves around a circular path of radius r = 15 cm with constant tangential acceleration A_t. Calculate the normal acceleration of the particle in time t = 15 sec in which the particle has completed 5 revolutions and has acquired a linear velocity of 20 cm/sec.

Solution: In the light of question No. 3.2:

$$\omega = \frac{A_t}{r} \cdot t \text{ and } \theta = \frac{A_t \cdot t^2}{2r}$$

$$A_n = \frac{A_t^2 \cdot t^2}{r} = \left(\frac{4\pi nr}{t^2}\right)^2 \cdot \frac{t^2}{r}$$

Further $v = r\omega = A_t \cdot t$

$$\frac{(vt)^4}{(4\pi nr)^2} = (4\pi nr)^2$$

$$\therefore \quad A_n = \frac{v^4 \cdot t^2}{16\pi^2 n^2 r^3}$$

$$= \frac{(0.2)^4 \quad 225}{16 \quad 3.14^2 \quad 25 \quad 0.15^3}$$

$$= 0.027 \text{ m/sec.}^2$$

3.4. A small ball of mass m is attached to one end of a massless elastic string of unstretched length l_0 and elastic constant K. The ball is rotated in a horizontal circle with an angular velocity ω_1 by holding the other end of the string. What amount of work will be done in increasing its angular velocity to ω_2?

Solution:

$$K(l - l_0) = ml\omega^2$$

$$l\left[K - m\omega_1^2\right] = Kl_0$$

$$\therefore \quad l = \frac{Kl_0}{K - m\omega_1^2}$$

$$E_1 = \text{Initial energy} = \frac{1}{2}\, m\,(l\omega_1)^2 + \frac{1}{2}\, K\,(l - l_0)^2$$

$$= \frac{1}{2}\, ml^2\, \omega_1^2 + \frac{1}{2}\, K \cdot \frac{m^2 l^2 \omega_1^4}{K^2}$$

$$= \frac{1}{2}\, \frac{m\omega_1^2 l^2}{K}\, \left[K + m\omega_1^2\right]$$

$$= \frac{1}{2}\, \frac{m\omega_1^2}{K} \cdot \frac{K^2 l_0^2 \left(K + m\omega_1^2\right)}{\left(K - m\omega_1^2\right)^2}$$

$$= \frac{1}{2}\, mK\omega_1^2\, l_0^2\, \frac{\left(K + m\omega_1^2\right)}{\left(K - m\omega_1^2\right)^2}$$

Further $K(l_2 - l_0) = ml_2\omega_2^2$

$$E_2 = \text{Final energy} = \frac{1}{2}\, m\,(l_2\,\omega_2)^2 + \frac{1}{2}\, K\,(l_2 - l_0)^2$$

$$= \frac{1}{2}\, mKl_0^2\, \omega_2^2\, \frac{\left(K + m\omega_2^2\right)}{\left(K - m\omega_2^2\right)}$$

$$\text{Work done} = E_2 - E_1 = \frac{1}{2}\, mKl_0^2 \left[\omega_2^2 \frac{\left(K + m\omega_2^2\right)}{K - m\omega_2^2} - \omega_1^2 \cdot \frac{K + m\omega_1^2}{\left(K - m\omega_1^2\right)^2}\right]$$

3.5. A particle travelled $\frac{3}{4}$ th of a circle at a mean speed $\langle v\rangle$ during a certain time. Find the magnitude of the mean velocity $\langle V\rangle$ of the particle during the same time.

Solution: Suppose the particle is moving on the circumference of a circle of radius R and completing one revolution in time τ.

$$\text{Mean speed } \langle v\rangle = \frac{3}{4} \cdot \frac{2\pi R}{3\tau} = \frac{2\pi R}{\tau}$$

$$\text{Magnitude of mean velocity } \langle V\rangle \text{ in that time} = \frac{R\sqrt{2}}{\frac{3\tau}{4}} = \frac{4R\sqrt{2}}{3\tau}$$

$$= \frac{4\sqrt{2}}{3}\, \frac{\langle v\rangle}{2\pi} = \frac{2\sqrt{2}}{3\pi}\, \langle v\rangle$$

3.6. A small body of mass m slides without friction from the top of a hemisphere of radisu R as shown in the Fig. 3.1. At what height will the body be detached from the surface of the hemisphere ?

Solution: Let us assume that the body is detached from the hemisphere after traversing a vertical distance h. At B the velocity is given by

$v^2 = u^2 + 2gh$ since $u = 0$

$v = \sqrt{2gh}$

At the point B a component of the weight, i.e., $mg\cos\theta$ will be equal to the centripetal force since the reaction force at B will be equal to zero.

$$\text{or} \quad mg\cos\theta = \frac{mv^2}{R}$$

$$\text{or} \quad g\frac{OC}{R} = \frac{v^2}{R} = \frac{2gh}{R}$$

$$\text{or} \quad R - h = 2h$$

$$\text{or} \quad h = \frac{R}{3}$$

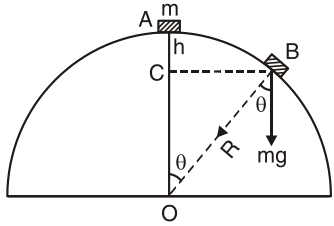


Fig. 3.1

3.7. A small sphere of mass m attached to a string of length l is suspended from O (see Fig. 3.2). A peg P is located at a distance x below the point of suspension O. What should be the value of x if the ball starting from A swings full circle with the peg at its centre.

Solution: In order to swing completely in a circle centred on the peg P the sphere must have a speed $\sqrt{g(l - x)}$ at the point C. In view of conservation of energy the total energy at C is given by the potential energy mgl. Therefore,

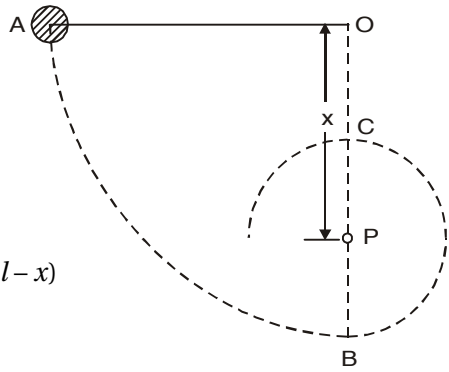


Fig. 3.2

$$mgl = \frac{1}{2}mg(l - x) + 2mg(l - x)$$

$$\text{or} \quad 2l = (l - x) + 4(l - x)$$

$$\text{or} \quad 5x = 3l$$

$$\text{or} \quad x = \frac{3}{5}l = 0.6l$$

3.8. A small sphere of mass m attached to one end of a rubber cord of length l revolves in a horizontal circle centred at the other end of the cord with a frequency n. The tension in the cord is no longer constant but varies directly with elongation. Find (i) the radius of the circular path (ii) the tension T' in the cord by taking tension per unit elongation as k.

Solution: The sphere revolves in a horizontal circle. The tension in the cord is balanced by the centripetal force according to the following equation:

Tension $t = mr\omega^2 = mr \cdot 4\pi^2 n^2$

where r is the radius of the circular path.

$dt = 4\pi^2 mn^2\, dr$

Let T be the tension in unstretched condition, i.e., $T = 4\pi^2\, mn^2\, l$

$$\frac{T\ \ dt}{dr} = \frac{\text{Tension}}{\text{elongation}} = k$$

or $\quad \dfrac{T}{dr} + \dfrac{dt}{dr} = k$

or $\quad T = (k - 4\pi^2 mn^2)\, dr$

$$\therefore\ dr = \frac{T}{k - 4\pi^2 mn^2}$$

$$\therefore\ T' = T + dt = k \cdot dr = \frac{kT}{k - 4\pi^2 mn^2} = \frac{k \cdot 4\pi^2 mn^2 l}{k - 4\pi^2 mn^2}$$

Further $T' = T + dt = 4\pi^2 mn^2 l + 4\pi^2 mn^2 dr = 4\pi^2 mn^2 R$

where R is the radius of the circular path $= l + dr$

$$\therefore\ R = \frac{T'}{4\pi^2 mn^2} = \frac{kl}{k - 4\pi^2 mn^2}$$

3.9. A small sphere of mass m slides down a plane from point A at height H and then moves in a circular loop of radius r as shown in Fig. 3.3. Calculate (i) height H in terms of r, and (ii) pressure exerted by the sphere at P.

Solution: If the ball has to cross C it must have a critical speed of $\sqrt{gr}$. Thus the total energy at C is given by

$$\frac{1}{2}\, mgr + mg \cdot 2r$$

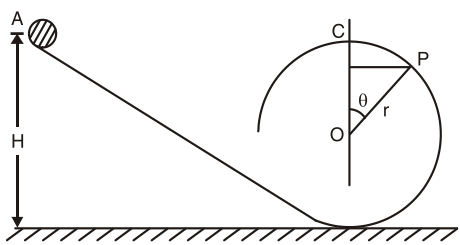


Fig. 3.3

In view of conservation of energy, $mgH = \dfrac{5}{2}\, mgr$

or $\quad H = \dfrac{5r}{2}$

The kinetic energy at P can be obtained as

$\dfrac{1}{2}\, mV_P^2 + mg(r + r\cos\theta) = mg \cdot \dfrac{5r}{2}$; V_P is the velocity of sphere at P.

or $\quad V_P^2 = 5gr - 2gr - 2gr\cos\theta$

or $\quad V_P^2 = (3 - 2\cos\theta)\, gr$

At P we have

$mg\cos\theta + F = \dfrac{mV_P^2}{r} = (3 - 2\cos\theta)\, mg$; F is the force of reaction at the point P

or $F = 3\, mg\,(1 - \cos\theta)$

3.10. A sphere starting from point A moves in a vertical loop from which a symmetrical section of an angle $POR = 2\theta$ has been taken out (see Fig. 3.4). From what maximum and minimum heights should the sphere be allowed to move so that after losing contact with the track at point P, it reaches point R via a parabolic path PQR in air?

Solution: Let v be the velocity of the sphere at point P. In view of the figure, PR will be the range of the parabolic path PQR.

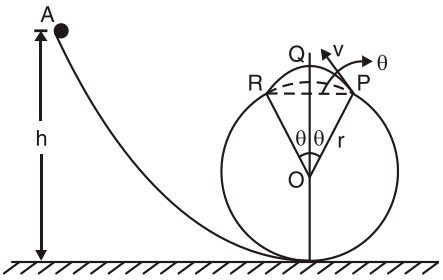


Fig. 3.4

$\therefore$ Range $= PR = 2r \sin\theta$

The time of flight for the parabolic path, $PQR = \dfrac{2v \sin\theta}{g}$

Therefore, $2r \sin\theta = \dfrac{2v \sin\theta}{g} \times v \cos\theta$

or $v^2 = \dfrac{gr}{\cos\theta}$

According to law of conservation of energy we have

$$mgh = mgr + mgr\cos\theta + \frac{1}{2}mv^2$$

Substituting the value of v^2 from the above we have

$$mgh = mgr(1 + \cos\theta) + \frac{mgr}{2\cos\theta}$$

$$\frac{h}{r}\cos\theta = \cos\theta + \cos^2\theta + \frac{1}{2}$$

Put $\dfrac{h}{r} = b$

Now $2\cos^2\theta - 2(b-1)\cos\theta + 1 = 0$

$$\cos\theta = \frac{2(b \quad 1) \quad \sqrt{4(b \quad 1)^2 \quad 8}}{4} = \frac{1}{2} \quad (b \quad 1) \quad \sqrt{(b \quad 1)^2 \quad 2}$$

for θ to be real, we must have $(b-1) \geq \sqrt{2}$

or $b \geq 1 + \sqrt{2}$

For $\cos\theta$, we have the identify $0 < \cos\theta < 1$, i.e.

$$b - 1 + \sqrt{(b \quad 1)^2 \quad 2} \leq 2 \text{ and } b \leq 2.5$$

i.e., $1 + \sqrt{2} \leq b \leq 2.5$ or $1 \quad \sqrt{2} \quad r \leq h \leq 2.5r$

3.11. A small sphere of mass m starts sliding from height H down a frictionless inclined groove passing into a half circle of radius $H/2$ as shown

in Fig. 3.5. Find the velocity of the sphere at the highest point of its path after breaking off the groove.

Solution: Suppose at point P the body breaks off the groove. At this point, then, the reaction force on the groove will vanish. A component of its weight will provide the centripetal force, i.e.,

$$mg\cos\theta = \frac{mv^2}{H/2}$$

or $$v^2 = \frac{1}{2}\, gH\cos\theta$$

With velocity v the sphere will now describe a parabolic trajectory. The velocity at the highest point of the trajectory will be $v\cos\theta$.

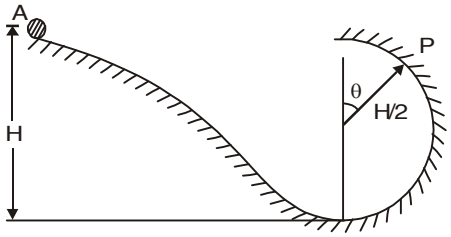


Fig. 3.5

In view of the principle of conservation of energy the total energy at P will be obtained at the cost of potential energy mgH, i.e.,

$$mgH = mg\ \frac{H}{2} + mg\ \frac{H}{2}\ \cos\theta + \frac{1}{2}\ mv^2$$

or $$mg\ \frac{H}{2} = mg\ \frac{H}{2}\ \cos\theta + mg\ \frac{H}{4}\ \cos\theta$$

or $$\frac{1}{2} = \frac{3}{2}\cos\theta$$

or $$\cos\theta = \frac{2}{3}$$

The velocity at the highest point of the trajectory after breaking off the groove equals

$$v\cos\theta = \sqrt{v^2\cos^2\theta} = \sqrt{\frac{1}{2}gH\cos^3\theta}$$

$$= \sqrt{\frac{1}{2}gH\ \frac{8}{27}} = \sqrt{\frac{4gH}{27}} = \frac{2}{3}\ \sqrt{\frac{gH}{3}}$$

3.12. A track shown in Fig. 3.6 consists of a straight portion of AB and a semi-circle BCD of radius r. A particle of mass m moves to the right along the straight path with a constant velocity $\sqrt{\frac{22gr}{5}}$. After covering the straight track it moves up and after some time it reaches point O where it loses the contact with the track. How far from B will the particle strikes the straight track AB?

Solution: At O we have $\frac{mv_0^2}{r} = mg\cos\theta$

or $v_0 = \sqrt{gr\cos\theta}$

According to conservation of energy we have

$$\frac{1}{2}\, m \cdot \frac{22gr}{5} = \frac{1}{2}\, mgr\cos\theta + mgr\,(1+\cos\theta)$$

On solving $\cos\theta = 4/5$ and $\sin\theta = 3/5$

$$x = r\cos\theta = 0.8\, r$$

For height EF we have

$$\left(\sqrt{gr\cos\theta}\cdot\sin\theta\right)^2 = 2gh$$

or $h = EF = \dfrac{gr\cos\theta}{2g}\cdot\sin^2\theta$

$$= \frac{18r}{125}$$

$$\therefore FB = \frac{18r}{125} + 1.8\, r = \frac{243r}{125}$$

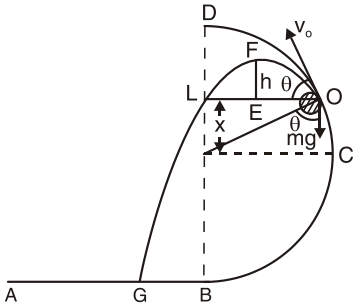


Fig. 3.6

Time for path $OF = \dfrac{\sqrt{gr\cos\theta}\cdot\sin\theta}{g} = t$

Distance $OE = \dfrac{\sqrt{gr\cos\theta}\cdot\cos\theta\cdot\sqrt{gr\cos\theta}\cdot\sin\theta}{g}$

$$= r\cos^2\theta\cdot\sin\theta$$

Distance $EL = r\sin\theta - r\cos^2\theta\sin\theta = r\sin^3\theta = \dfrac{27r}{125}$

Time of flight for path FB

$$\frac{243r}{125} = \frac{1}{2}\, g\cdot t_0^2$$

$$\therefore t_0 = \sqrt{\frac{2\quad 243r}{125g}}$$

Distance $EG = \sqrt{gr\cos\theta}\cdot\cos\theta\cdot\sqrt{\dfrac{2\times 243r}{125g}}$

$$= \frac{2r}{\sqrt{5}} \times \frac{4}{5} \times \sqrt{\frac{2\quad 243}{125}}$$

$$= \frac{8r}{125}\sqrt{2\quad 243}$$

Distance $BG = \dfrac{8r\sqrt{2\quad 243}}{125} - \dfrac{27r}{125}$

$$= 1.1949\, r$$

3.13. A simple pendulum is constructed by suspending a sphere of mass m through a massless string of length l. The pendulum is suspended at O (see Fig. 3.7). What minimum velocity should be imparted to the point of suspension O horizontally for the ball to move along the circle about that point ? What will be the tension in the string when the sphere is at the point B ?

Solution: In the given figure, O is the point of suspension. In the equilibrium position the ball of the simple pendulum is at the point A. Let us assume that a velocity $u_{\min}$ is given to the point of suspension in the direction BO.

With respect to the point of suspension O the sphere will have a velocity $u_{\min}$ and with this velocity it will describe a circular path in the clockwise direction. In order to cross the point C it must have a speed of $\sqrt{gl}$ at that point.

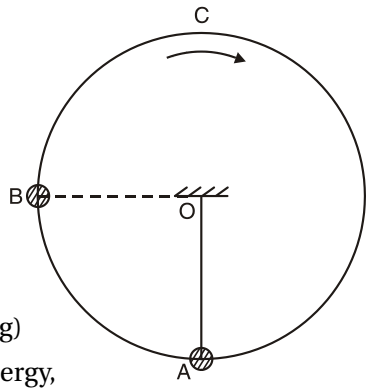


Fig. 3.7

In the view of the principle of conservation of energy

$$\frac{1}{2}\, m u_{\min}^2 = mg \cdot 2l + \frac{1}{2}\, mgl$$

or $$u_{\min} = \sqrt{5gl}$$

At the point B the string will be horizontal, and we have the equation

$$\frac{mv_B^2}{l} = T \text{ (T is the tension in the string)}$$

Further, in view of conservation of energy,

$$\frac{1}{2}\, m \left(\sqrt{5gl}\right)^2 = mgl + \frac{1}{2}\, m v_B^2$$

or $$\frac{3}{2}\, mgl = \frac{1}{2}\, m v_B^2$$

or $$v_B = \sqrt{3gl}$$

In the light of this the tension in the string is given by

$$T = \frac{m}{l} \times 3gl = 3\, mg$$

3.14. A particle of mass m is placed inside a hemispherical bowl which rotates about its vertical axis with constant angular velocity ω. It is just prevented from sliding down when OP is inclined 45° with the axis (O is the centre of the sphere and P is the position of the particle). The radius of the bowl is $10\sqrt{2}$ m, coefficient of friction μ between the particle and the bowl is 0.5. Find ω given $g = 10$ m/s^2.

Solution:

Frictional force $f = \mu N$

At P sum of the forces $f \sin 45 + N \sin 45$ will be balanced by the weight of the particle, i.e.,

$$f \sin 45 + N \sin 45 = mg$$

$$\text{or} \quad (\mu N + N) = mg\sqrt{2}$$

$$\text{or} \quad N(1 + \mu) = mg\sqrt{2}$$

For the purpose of rotation the necessary centripetal force is obtained from the force $N \cos 45 - f \cos 45$, i.e.,

$$N \cos 45 - f \cos 45 = mr\omega^2$$

or $N(1 - \mu) = \sqrt{2} \cdot m\omega^2 R \cos 45$ ($r = R \cos 45$ where R is the radius of the bowl) $= m\omega^2 R$

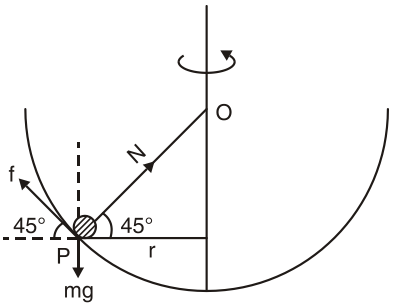


Fig. 3.8

On dividing the two equations, we have

$$\frac{\omega^2 R}{g\sqrt{2}} = \left(\frac{1-\mu}{1+\mu}\right)$$

$$\text{or} \quad \omega = \sqrt{\frac{(1-\mu)\, g\sqrt{2}}{(1+\mu)\, R}}$$

On substituting the values of the various terms, we have

$$\omega = \sqrt{\frac{0.5 \quad 10\sqrt{2}}{1.5 \quad 10\sqrt{2}}} = \frac{1}{\sqrt{3}} \text{ rad/s}$$

3.15. A particle of mass 100 g is suspended from one of the end of a weightless string of length 1 m and is allowed to swing in the vertical plane. The speed of the mass is 2 m/s when the string is inclined at angle $\theta = 60°$ with the vertical. Determine (i) the tension in the string at $\theta = 60°$, and (ii) the speed of the particle when it is in the lowest position.

Solution: At the point B (Fig. 3.9) the equation of motion of the particle is given by

$$T - mg \cos 60 = \frac{mv_B^2}{r}$$ (r is the radius of circular path = length of the string, T is the tension in the string, v_B is the speed of the particle)

$$\therefore \; T = \frac{mv_B^2}{r} + mg \cos 60$$

$$= 0.1 \times 4 + \frac{0.1 \quad 9.8}{2}$$

$$= 0.40 + 0.49 = 0.89 \text{ N}$$

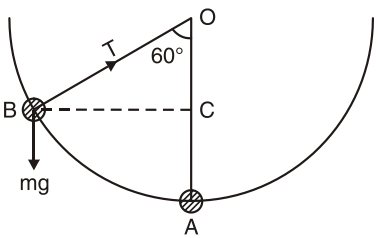


Fig. 3.9

The total energy at A is kinetic and is given by

$\frac{1}{2} m v_A^2 = mg(r - r\cos\theta) + \frac{1}{2} m v_B^2$; v_A is the speed of the particle at A

$$v_A = \sqrt{2gr(1-\cos\theta) + v_B^2}$$

$$\therefore v_A = \sqrt{2 \quad 9.8 \quad \frac{1}{2} \quad 4} = \sqrt{13.8} = 3.715 \text{ m/s}$$

3.16. A light metallic chain of mass m = 50 gm and length l = 60 cm is fitted on an ebonite disc after its ends have been soldered (see Fig. 3.10a). Find the tension in the chain when the disc rotates with a speed of n = 60 rps about an axis passing through its centre.

Solution: Mass per unit length of the chain = $\frac{m}{l}$

Consider an element of the chain of length $R \cdot 2\theta$; R = radius of the disc

Mass of the length sought = $\frac{m}{l} R \cdot 2\theta$

Let T be the tension in the chain. On resolving T in two mutually perpendicular directions, the component $T\cos\theta$ will vanish (see Fig. 3.10b). Thus we have

$\frac{m}{l} R\, 2\theta\, \omega^2 R = 2T\sin\theta = 2T\theta$ ($\sin\theta = \theta$ when θ is small)

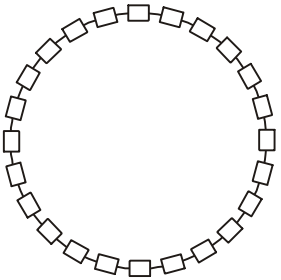


Fig. 3.10a

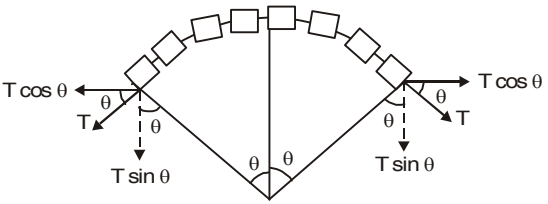


Fig. 3.10b

or $T = \frac{m}{l} R^2 4\pi^2 n^2 = \frac{m}{l} n^2 l^2$; $4\pi^2 R^2 = l^2$

$= ml \cdot n^2$

On substituting the values of the terms m, l and n, we have

$T = 50 \times 10^{-3} \times 60 \times 10^{-2} \times 60 \times 60 = 108$ N

3.17. A sphere of mass m is suspended from a string of length l from the point O (see Fig. 3.11). The sphere rotates in a circular path in a horizontal

plane. The string makes an angle α with the vertical. Find the time period of rotation.

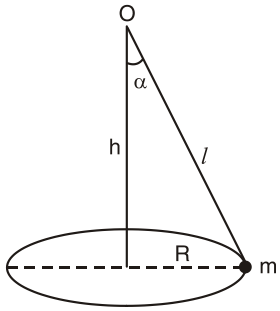


Fig. 3.11

Solution: For the rotation of the sphere in the circular path a centripetal force $m\ R\omega^2$ is required. Further,

$$\frac{mR\omega^2}{mg} = \tan\alpha$$

or $$\omega^2 = \frac{g}{R}\tan\alpha = \frac{g}{R} \times \frac{R}{h} = \frac{g}{h}$$

or $$\omega = \frac{2\pi}{T} = \sqrt{\frac{g}{h}} \text{ or } T = 2\pi\sqrt{\frac{h}{g}}$$

3.18. An object is placed at the top of a hemisphere of radius r. What horizontal velocity should be imparted to it so that it may detach from the surface of the hemisphere at the initial point of its motion?

Solution: Let v_1 be the velocity imparted to the object. Let its velocity be v_2 after it has decend through h.

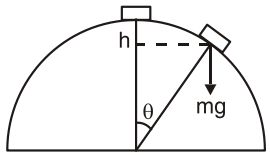


Fig. 3.12

According to conservation of energy we have

$$\frac{1}{2}\ m_1 v_1^2 + mgh = \frac{1}{2}\ m v_2^2 \text{ (see Fig. 3.12)}$$

or $$v_2^2 = v_1^2 + 2gh$$

At the break off we have

$$\frac{mv_2^2}{r} = mg\cos\theta$$

$$\therefore\ v_2^2 = g\,r\cos\theta = g\,r\ \ \frac{r\ \ h}{r}\ \ = v_1^2 + 2gh$$

or $$gr - 3gh = v_1^2$$

At the initial point $h = 0$

$$\therefore\ v_1 = \sqrt{gr}$$

3.19. A particle of mass m is projected from the point A with a velocity of $v_0 = \frac{1}{5}\sqrt{95gr}$ along the inner surface of a smooth vertical circle of radius r as shown in Fig. 3.13. At what angle θ from the vertical it will detach from the track and what will be its velocity at that instant?

Solution:

(i) According to conservation of energy, we have

$$\frac{1}{2}\ m \cdot \frac{1}{25} \cdot 95\ gr = mg\ [r + r\cos\theta] + \frac{1}{2}\ mv^2$$

Further $mg\cos\theta = \frac{mv^2}{r}$

$$\frac{19}{10} = 1 + \cos\theta + \frac{1}{2}\cos\theta$$

$$\frac{9}{10} = \frac{3}{2}\cos\theta$$

$$\theta = \cos^{-1}\frac{3}{5}$$

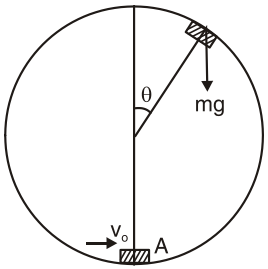


Fig. 3.13

(ii) $v^2 = gr\cos\theta = \frac{3}{5}gr$

$$v = \frac{1}{5}\sqrt{15gr}$$

3.20. An insect is sitting on a gramophone record rotating on its base at $\frac{100}{3}$ revolution per min. The insect is at a distance of 9.8 cm. from the centre of the record. Calculate the coefficient of friction between the record and the insect.

Solution:

$$\mu mg = mr\omega^2$$

$$\therefore \mu = \frac{r\omega^2}{g} = \frac{4\pi^2 \cdot n^2 r}{g}$$

On substituting the valves we have

$$\mu = \frac{4\pi^2}{9.8} \times \frac{25}{81} \times 9.8 \times 10^{-2}$$

$$= \frac{\pi^2}{81}$$

3.21. A ball of mass $m = 250$ gm. is fixed at one end of a 1 m long string. By holding the other end of the string the ball is rotated in a horizontal circle. Assuming the greatest weight that the string can support is $M = 4$ kg. Calculate the maximum number of revolutions/sec that the ball will make without breaking the string.

Solution:

$$T = Mg = ml\omega^2$$

$$2\pi n = \left(\frac{Mg}{ml}\right)^{1/2}$$

$$\therefore n = \frac{1}{2\pi} \cdot \left(\frac{Mg}{ml}\right)^{1/2}$$

On substituting the values we have

$$n = \frac{1}{2\ \ 3.14}\ \ \frac{4\ \ 9.8}{0.25\ \ 1}^{\frac{1}{2}}$$

$= 1.99$

$= 2$ rev/sec.

3.22. Two identical balls of mass m are tied to two weightless strings of equal length l as shown in the Fig. 3.14. The system thus formed is rotated in a horizontal circle with A as centre at a constant angular speed ω. Find the ratio of the tension in the two strings.

Solution: The equation of motion of ball C is given by

$$T_2 = m \cdot 2l \cdot \omega^2$$

The equation of motion of ball B is given by

$$T_1 - T_2 = ml \cdot \omega^2$$

or $$T_1 = T_2 + ml\omega^2$$

$$= m \cdot 2l \cdot \omega^2 + m \cdot l \cdot \omega^2$$

$$= 3\ ml \cdot \omega^2$$

$$\frac{T_1}{T_2} = \frac{3ml\omega^2}{2ml\omega^2} = 3 : 2$$

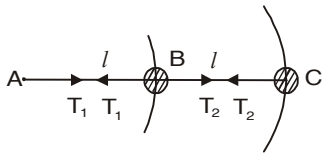


Fig. 3.14

3.23. A car is moving on a circular road of radius 500 m at a speed of 25 m/sec. It is increasing in speed at a rate of 3 m/sec^2. Calculate its acceleration.

Solution:

$$\text{Radial acceleration } a_r = \frac{v^2}{r} = \frac{25\ \ 25}{500} = 1.25 \text{ m/sec.}^2$$

$$\text{Tangential acceleration } a_T = \frac{dv}{dt} = 3 \text{ m/sec}^2$$

$$\therefore \text{ Resultant acceleration } a = \sqrt{a_r^2\ \ a_T^2} = \sqrt{1.5625\ \ 9} = 3.25 \text{ m/sec.}^2$$

3.24. At what speed earth should rotate about its own axis so that a person on the equator should have his weight as $\frac{2}{5}$ of its present value?

Solution:

$$\frac{2}{5}\ mg = mg - mr\omega^2;\ r = \text{radius of the earth}$$

$$\frac{3g}{5} = r\omega^2$$

$$\omega = \sqrt{\frac{3g}{5r}}$$

3.25. A man of mass $m = 50$ kg is sitting on a conventional swing of length 4 m. supported by two chains. At the lowest point each chain has a tension of 400 N. Calculate (i) the speed of the man at the lowest point. (ii) the force of the seat on the man at the lowest point.

Solution:

(i) Let T be the tension in each chain then the equation of motion at the lowest point is given by

$$2T - mg = \frac{mv^2}{r}$$

$$\therefore v = \sqrt{\frac{r}{m}(2T \quad mg)}$$

On substituting the values we have

$$v = \sqrt{\frac{4(2 \quad 400 \quad 50 \quad 9.8)}{50}}$$

$$= \sqrt{\frac{4 \quad 310}{50}}$$

$$= \sqrt{24.8} = 4.98 \text{ m/sec.}$$

(ii) Force of the seat on the man $= mg + \frac{mv^2}{r}$

$$= 50 \times 9.8 + \frac{50 \quad 4.98^2}{4}$$

$$= 490 + 310 = 800 \text{ N}$$

3.26. In an aircraft a pilot of mass 50 kg executes loop the loop in which the craft moves in a vertical circle of radius 2.5 km at a constant speed of 200 m/sec. Calculate the force of the seat on the pilot when the craft is at, B, T and A (see Fig. 3.15) respectively.

Solution:

$$N_B = mg + \frac{mv^2}{r}$$

$$= 50 \times 9.8 + \frac{50 \quad 200^2}{2.5 \quad 10^3}$$

$$= 490 + 800$$

$$= 1290 \text{ N}$$

$$N_T + mg = \frac{mv^2}{r}$$

$$N_T = \frac{mv^2}{r} - mg$$

$$= 800 - 490$$

$$= 310 \text{ N}$$

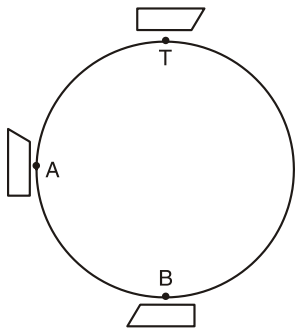


Fig. 3.15

$$N_A = \frac{mv^2}{r} = 800 \text{ N}$$

3.27. A simple pendulum is constructed by using a bob of mass $m = 500$ gm and a weightless string of length $l = 1$ m. Initially the bob of the pendulum is at A (Fig. 3.16) and the string is horizontal. The pendulum released from rest swings downwards under the influence of earth's gravity and acquires the position B by describing an angle $\theta = 30°$. At B find (i) the velocity of the bob (ii) the tension in the string.

Solution:

(i) At B we have

$$mgl \sin\theta = \frac{1}{2} mv^2$$

or $v = \sqrt{2gl\sin\theta}$

On substituting the values we have

$$v = \sqrt{2 \quad 9.8 \quad 1 \quad \frac{1}{2}}$$

$$= \sqrt{9.8}$$

$$= 3.13 \text{ m/sec.}$$

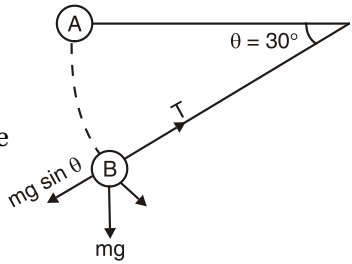


Fig. 3.16

(ii) $T - mg \sin\theta = \frac{mv^2}{l}$

$$T = mg \sin\theta + 2mg \sin\theta$$

$$= 3\, mg \sin\theta$$

On putting the valves we have

$$T = 3 \times 0.5 \times 9.8 \times \frac{1}{2}$$

$$= 7.35 \text{ N}$$

3.28. In Fig. 3.17 the bob of the simple pendulum is at A and the string is horizontal. The bob falls under the influence of earth's gravitational field and strikes a vertical rigid wall W at B. Assuming the coefficient of restitution $e = \frac{1}{2}$ find the angle θ described by the bob after the rebound.

Solution: Let m be the mass of the bob and l be the length of the string, then in view of conservation of energy we have.

$mgl = \frac{1}{2} m v_B^2$; v_B = velocity of the bob when it strikes the wall.

or $v_B = \sqrt{2gl}$

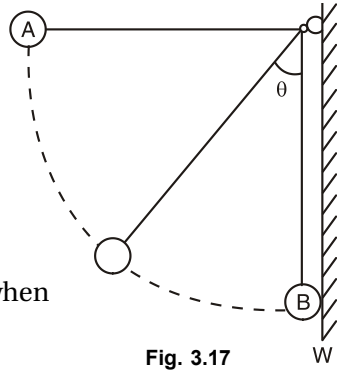


Fig. 3.17

The velocity at rebound $= v_B = \frac{1}{2} v_B$

$$= \frac{1}{2}\sqrt{2gl}$$

Now in view of conservation of energy we have

$$mg\,[l - l\cos\theta] = \frac{1}{2}\, m \cdot \frac{1}{4}\, 2gl$$

$$1 - \cos\theta = \frac{1}{4}$$

$$\cos\theta = 0.75$$

$$\theta = 41°25'$$

3.29. In the simple pendulum shown in the Fig. 3.18 the mass of the bob $m = 170$ gm and the length of the string $l = 1$ m. The bob initially at A is at rest. When the bob is released it swing through an angle $\theta = 30°$ and strikes a spring of force constant $K = 1960$ N/m at B. Find the compression in the spring.

Solution: According to conservation of energy

$mg\,[l - l\cos 30] = \frac{1}{2}\, Kx^2$; x is the compression in the spring

$$mgl\ \ 1\ \ \frac{\sqrt{3}}{2}\ \ = \frac{1}{2}\, Kx^2$$

$$mgl\ \ 2\ \ \sqrt{3}\ \ = Kx^2$$

$$\therefore\ x = \sqrt{\frac{mgl\,[0.268]}{K}}$$

On substituting the values of various terms we have

$$x = \sqrt{\frac{170\ \ 10^{-3}\ \ 9.8\ \ 1\ \ 0.268}{1960}}$$

$$= 1.5 \text{ cm}$$

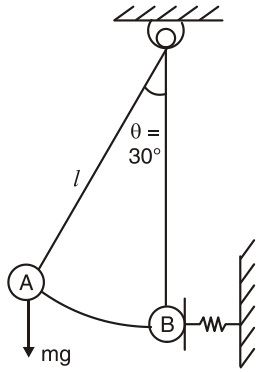


Fig. 3.18

3.30. A particle travels along a parabolic path given by $x^2 = 4ay$ with a constant speed v. Calculate the value of the maximum acceleration acting on the particle.

Solution:

$$y = \frac{x^2}{4a}$$

$$\frac{dy}{dx} = \frac{x}{2a}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2a}$$

The radius of curvature ρ is given by

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{1 \quad \frac{dy}{dx}^{2}^{3/2}} = \frac{\frac{1}{2a}}{1 \quad \frac{x}{2a}^{2}^{3/2}}$$

For acceleration to be maximum ρ should be minimum. The same will be obtained by putting $x = 0$.

$$\therefore \frac{1}{\rho} = \frac{1}{2a}$$

or acceleration $= \frac{v^2}{\rho} = \frac{v^2}{2a}$

3.31. A particle is projected with an initial velocity v at an angle θ with the horizontal in the earth's gravitational field. Calculate the radius of curvature of the path when the particle is at the highest point of its trajectory.

Solution: The particle will describe a parabolic path given by

$$y = x \tan\theta - \frac{1}{2} g \frac{x^2}{v^2 \cos^2\theta}$$

$$\frac{d^2y}{dx} = \tan\theta - \frac{gx}{v^2 \cos^2\theta}$$

$$\frac{d^2y}{dx^2} = -\frac{g}{v^2 \cos^2\theta}$$

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{1 \quad \frac{dy}{dx}^{2}^{3/2}}$$

The value of $\frac{dy}{dx}$ at the highest point is obtained by putting $x = \frac{1}{2}$ Range

$$= \frac{v^2}{g} \cdot \sin\theta \cdot \cos\theta$$

Now $\frac{dy}{dx} = \tan\theta - g \cdot \frac{v^2}{g} \cdot \frac{\sin\theta \cdot \cos\theta}{v^2 \cos^2\theta} = 0$

$$\therefore \frac{1}{\rho} = \frac{d^2y}{dx^2}$$

$$\text{or} \quad \rho = \frac{1}{\frac{d^2y}{dx^2}} = \frac{v^2 \cos^2\theta}{g}$$

3.32. A particle moves in xy plane with a velocity v whose magnitude is constant and describes an ellipse with semi major and semi minor axes as a & b respectively. Find the acceleration of the particle and the radius of curvature of the ellipse at the point $x = 0$.

Solution: The equation of ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = \pm \frac{b}{a} \sqrt{a^2 \quad x^2}$$

Let us take $y = -\frac{b}{a} \sqrt{a^2 \quad x^2}$

$$\frac{dy}{dt} = -\frac{b}{a} \times \frac{1}{2} \quad a^2 \quad x^2 \quad ^{1/2} \times -2x \cdot \frac{dx}{dt} = \frac{b}{a} \quad a^2 \quad x^2 \quad ^{1/2} x \cdot \frac{dx}{dt}$$

$$\frac{d^2y}{dt^2} = \left\{ \frac{b}{a} \times -\frac{1}{2} \left(a^2 - x^2\right)^{-3/2} \times -2x \cdot \frac{dx}{dt} \times \frac{xdx}{dt} \right.$$

$$\left. + \frac{b}{a} \left(a^2 - x^2\right)^{-1/2} \left[\frac{xd^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 \right] \right\}$$

At $x = 0$

$$\frac{d^2y}{dt^2} = \frac{b}{a} \cdot a^{-1} \quad \frac{dx}{dt} \quad ^2 = \frac{bv^2}{a^2}$$

$$\text{Radius of curvature } r = \frac{v^2}{\text{acceleration}} = \frac{v^2 \cdot a^2}{bv^2} = \frac{a^2}{b}$$

3.33. A particle initially at the origin moves in xy plane with a velocity $V = ai + bxj$. Where a and b are constants. Find (i) the path of the particle (ii) the radius of curvature of the trajectory.

Solution:

(i) $V = ai + bxj$

$$\frac{dx}{dt} = \text{velocity along } x \text{ axis} = a$$

$$\therefore dx = adt$$

or $x = at$

$$\frac{dy}{dt} = bx$$

$$dy = b \cdot at \cdot dt$$

or $y = \dfrac{abt^2}{2}$

On combining the two we have

$$y = \frac{ab}{2} \cdot \frac{x^2}{a^2}$$

or $y = \dfrac{b}{2a} \cdot x^2$

Thus the particle will describe a parabolic path.

(ii) $$\frac{dy}{dx} = \frac{b}{a} \cdot x$$

$$\frac{d^2y}{dx^2} = \frac{b}{a}$$

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{\dfrac{b}{a}}{\left[1+\left(\dfrac{bx}{a}\right)^2\right]^{\frac{3}{2}}}$$

or $$\rho = \frac{a}{b}\left[1+\left(\frac{bx}{a}\right)^2\right]^{\frac{3}{2}}$$

3.34. A scooter moves along a sine curve represented by $y = b \sin (x/\beta)$ where b and β are constants (see Fig. 3.19). Assuming the coefficient of friction between the road and the tyre as k determine the velocity with which the scooter rides without sliding.

Solution:

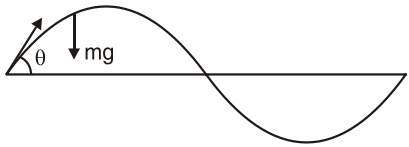


Fig. 3.19

$$y = b \sin\left(\frac{x}{\beta}\right)$$

$$\frac{dy}{dx} = \frac{b}{\beta}\cos\left(\frac{x}{\beta}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{b}{\beta^2}\sin\left(\frac{x}{\beta}\right)$$

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{-\dfrac{b}{\beta^2}\sin\left(\dfrac{x}{\beta}\right)}{\left[1+\dfrac{b^2}{\beta^2}\cdot\cos^2\left(\dfrac{x}{\beta}\right)\right]^{\frac{3}{2}}}$$

The scooter will ride without sliding when

$$Kmg\cos\theta = \frac{mv^2}{\rho}$$

or $\quad v^2 = K\rho g\cos\theta;\ \frac{dy}{dx} = \tan\theta$

$$\text{or}\quad v^2 = \frac{Kg\left[1+\frac{b^2}{\beta^2}\cdot\cos^2\left(\frac{x}{\beta}\right)\right]^{\frac{3}{2}}}{\frac{b}{\beta^2}\sin\left(\frac{x}{\beta}\right)}\cdot\frac{1}{\left[1+\frac{b^2}{\beta^2}\cdot\cos^2\left(\frac{x}{\beta}\right)\right]^{\frac{1}{2}}}$$

$$\text{or}\quad v^2 \le \frac{Kg\cdot\beta^2}{b}$$

$$\text{or}\quad v \le \beta\cdot\sqrt{\frac{Kg}{b}}$$

3.35. A ball A of mass m moves along a parabolic path $y = \frac{4a}{b^2}x^2$ as shown in Fig. 3.20. Calculate the reaction force of the track on the ball when it is at the point O.

Solution: As the ball decends through a its velocity v is given by

$$v = \sqrt{2ga}$$

$$\frac{dy}{dx} = \frac{8ax}{b^2}$$

$$\frac{d^2y}{dx^2} = \frac{8a}{b^2}$$

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{\frac{8a}{b^2}}{\left[1+\left(\frac{8ax}{b}\right)^2\right]^{\frac{3}{2}}}$$

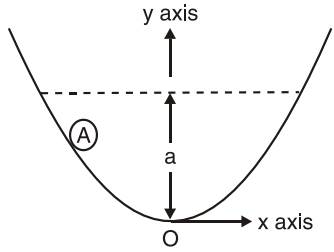


Fig. 3.20

At the point O $x = 0$

$$\therefore\ \frac{1}{\rho} = \frac{8a}{b^2}$$

At the point O we have

$$R - mg = \frac{mv^2}{\rho} = m\cdot 2ga\times\frac{8a}{b^2}$$

$$R = mg\ \ 1\ \ \frac{16a^2}{b^2}$$

3.36. A 3 kg ball is attached at the middle of a 2.828 m long weightless string as shown in Fig. 3.21. The ends of the string are attached to the two points of a rod *A* and *B* which are 2 m apart. The ball is rotated in a vertical circle at a speed of 4 m/sec. Find the tension in the string when the ball is (i) at the lowest point (ii) in the horizontal position (iii) at the highest point.

Solution:

(i) In the lowest position, we have

$$2T_1 \cos\theta - mg = \frac{mv^2}{r}$$

$$\therefore T_1 = \frac{mg + \frac{mv^2}{r}}{2\cos\theta}$$

On substituting the values of various terms we have

$$T_1 = \frac{3 \quad 9.8 \quad \frac{3 \quad 4^2}{1}}{2 \quad 1.414}$$

$$= 27.37 \text{ N.}$$

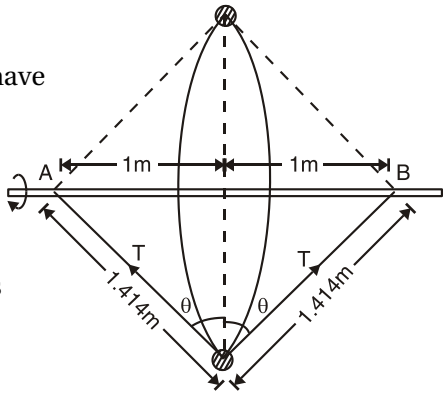


Fig. 3.21

(ii) In the horizontal position we have

$$2T_2 \cos\theta = \frac{mv^2}{r}$$

$$\therefore T_2 = \frac{mv^2}{2r\cos\theta}$$

On substituting the values we have

$$T_2 = \frac{3 \quad 4^2}{2 \quad 1 \quad 1.414}$$

$$= 16.97 \text{ N}$$

(iii) At the highest point we have

$$2T_3 \cos\theta = \frac{mv^2}{r} - mg$$

$$\therefore T_3 = \frac{\frac{mv^2}{r} - mg}{2\cos\theta}$$

On substituting the values we have

$$T_3 = \frac{3 \quad 4^2 \quad 3 \quad 9.8}{2 \quad 1.414}$$

$$= 6.57 \text{ N}$$

3.37. A thin circular wire of radius r containing a bead of mass m rotates about a vertical axis with an angular velocity ω. Find (i) the angle θ at which the bead rests relative to the wire (ii) the normal reaction force on the bead.

Solution:

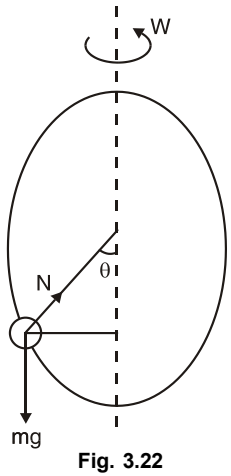


Fig. 3.22

(i) Let N be the normal force then in view of Fig. 3.22 we have

$$N \sin\theta = m\omega^2 r \sin\theta$$

$$\therefore N = mr\omega^2$$

$$\tan\theta = \frac{m\omega^2 r \sin\theta}{mg} = \frac{\omega^2 r}{g} \cdot \sin\theta$$

$$\cos\theta = \frac{g}{\omega^2 r}$$

$$\theta = \cos^{-1}\frac{g}{\omega^2 r}$$

(ii) $$N \sin\theta = m\omega^2 r\left[1 - \frac{g^2}{\omega^4 r^2}\right]^{\frac{1}{2}}$$

3.38. Two balls of masses m and 2 m suspended through two strings of same length l forms two pendulums as shown in the Fig. 3.23. The pendulum of mass m is kept at rest at an angle of $\theta_1 = 45°$ with the vertical. When the ball is released from its rest position it strikes the ball of the second pendulum elastically. Calculate the angle θ_2 through which the second pendulum will be deflected.

Solution: Let v_1 and v'_1 and v_2 and v'_2 be the velocities of the two balls before and after impact then we have

$$mv_1 + 2m \cdot 0 = mv'_1 + 2mv'_2$$

or $$v'_1 + 2v'_2 = v_1 \qquad \ldots (1)$$

According to conservation of energy we have

$$\frac{1}{2} m v_1^2 + \frac{1}{2} \cdot 2m \cdot 0 = \frac{1}{2} m v_1'^2 + \frac{1}{2} \cdot 2m v_2'^2$$

or $$v_1^2 = v_1'^2 + 2v_2'^2 \qquad \ldots (2)$$

Now $$v_1^2 - v_1'^2 = 2v_2'^2$$

and $$v_1 - v'_1 = 2v'_2$$

On dividing the two we have

$$v_1 + v'_1 = v'_2 \qquad \ldots (3)$$

$$v_1 + v_1 - 2v'_2 = v'_2$$

or $$2v_1 = 3v'_2$$

or $$v'_2 = \frac{2}{3} v_1 \qquad \ldots (4)$$

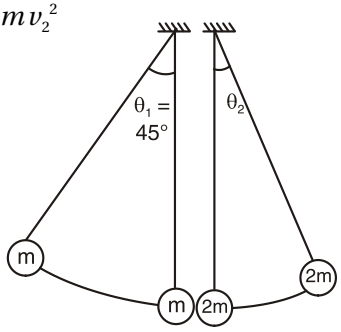


Fig. 3.23

In view of conservation of energy we have

$$\frac{1}{2} m v_1^2 = mgl\,[1 - \cos 45] = mgl \quad 1 \quad \frac{1}{\sqrt{2}}$$

or $$v_1 = \sqrt{gl \quad 2 \quad \sqrt{2}}$$

Again in view of conservation of energy we have

$$\frac{1}{2} \cdot 2m \cdot v_2^2 = 2m \cdot gl\,[1 - \cos \theta_2]$$

or $$\frac{1}{2} \cdot \frac{4}{9} \cdot v_1^2 = gl\,[1 - \cos \theta_2]$$

or $$\frac{2}{9} \times gl\left[2 - \sqrt{2}\right] = gl\,[1 - \cos \theta_2]$$

$$\frac{2}{9} \times 0.586 = 1 - \cos \theta_2$$

$$0.13 = 1 - \cos \theta_2$$

$$\cos \theta_2 = 0.87$$

$$\theta_2 = 29° 30'$$

3.39. A smooth hollow tube *PQ* as shown in Fig. 3.24a is one fourth arc of a circle of mean radius *R*. The tube is filled with a chain of length $\frac{\pi R}{2}$ and mass *m*. The chain is released from rest. Find the velocity with which it will move along the frictionless horizontal plane *QG* after emerging out of the tube.

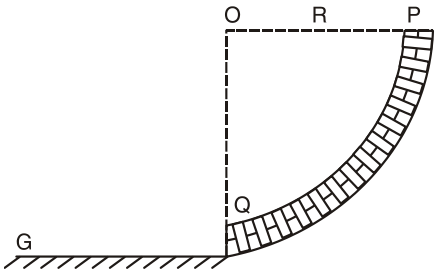


Fig. 3.24a

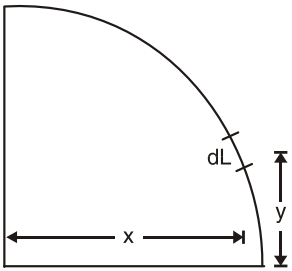


Fig. 3.24b

Solution: In Fig. 3.24b, consider an element of length *dL*, such that

$dL^2 = dx^2 + dy^2$

For a circle $x^2 + y^2 = R^2$

$\therefore\ x\,dx = -y\,dy$

$$\therefore\ dy = -\frac{x}{y} \cdot dx$$

Now $dL = \sqrt{dx^2 + \frac{x^2}{y^2} \cdot dx^2} = \frac{R}{y} \cdot dx$

y component of centroid is given by

$$Y = \frac{\int_0^R y dL}{\int_0^R dL} = \frac{\int_0^R y \cdot \frac{R}{y} \cdot dx}{\frac{\pi R}{2}} = \frac{2R}{\pi}$$

Thus the y component of centroid is at a distance of $\frac{2R}{\pi}$ from the centre of the arc.

Now in Fig. 3.24a, the centroid is at a height of $\left(R - \frac{2R}{\pi}\right)$ above Q. In view of conservation of energy we have

$$\frac{1}{2} mv^2 = mg \cdot R\left(1 - \frac{2}{\pi}\right)$$

or
$$v = \sqrt{gR\left(2 - \frac{4}{\pi}\right)}$$
$$= \sqrt{0.727\, gR}$$
$$= 0.85\ \sqrt{gR}$$

3.40. A smooth hollow semi-circular tube of mean radius R completely filled with a chain of mass m is placed in a vertical plane as shown in the Figure 3.25. A slight disturbance causes the chain to move. Find the velocity with which the chain will emerge from end Q of the tube.

Solution: The y component of centroid is at a height $\frac{2R}{\pi}$ above O (see Question No. 3.39).

According to conservation of energy we have

$$\frac{1}{2} mv^2 = mg \cdot \frac{2R}{\pi} + mg \cdot \frac{\pi R}{2}$$

$$v = \sqrt{g\left[\frac{4}{\pi} + \pi\right]R}$$

$$= \sqrt{4.414\, gR}$$

$$v = 2.1\ \sqrt{gR}$$

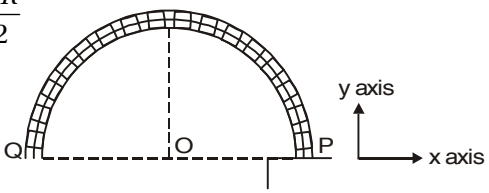


Fig. 3.25

3.41. A conical pendulum of length $l = 40$ cm rotates at constant speed v in a horizontal circle of radius $r = 20$ cm as shown in the Fig. 3.26. How much of the string must be pulled through the tube to double the speed of ball?

Solution: In view of the forces acting on the ball we have

$$T\cos\alpha = mg \text{ and } T\sin\alpha = \frac{mv^2}{r}$$

$$\therefore \tan\alpha = \frac{v^2}{rg} = \frac{1}{\sqrt{3}}$$

Similarly $\tan\alpha' = \frac{v'^2}{rg}$; v' is the velocity of the ball when it moves in a circular path of radius r'.

According to conservation of angular momentum we have

$$I_1\omega_1 = I_2\omega_2$$

$$\text{or } mr^2 \cdot \frac{v}{r} = mr'^2 \cdot \frac{v}{r}$$

$$rv = r' \cdot v' = r' \cdot 2v \ (v' = 2v \text{ given})$$

$$\therefore r' = \frac{r}{2}$$

$$\tan\alpha' = \frac{2}{r \cdot g}\frac{4v^2}{} = 8\tan\alpha$$

$$= \frac{8}{\sqrt{3}}$$

$$\sin\alpha' = \frac{r}{l} = \frac{r}{2l} = \frac{8}{\sqrt{67}}$$

$$\therefore l' = \frac{r\sqrt{67}}{16}$$

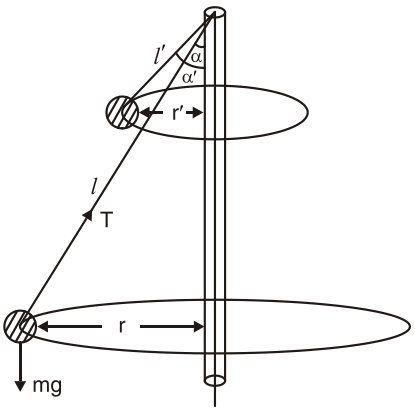


Fig. 3.26

On putting the value of $r = 20$ cm

$$l' = \frac{20\sqrt{67}}{16} = \frac{5\sqrt{67}}{4} = 10.2 \text{ cm}$$

Length of the string pulled $= l - l' = 40 - 10.2 = 29.8$ cm.

3.42. A ball of mass m is resting on the straight portion of the track shown in the Fig. 3.27 at a height h about the lowest point A. When the ball is released it enters into the circular part of the track of radius r after covering the linear portion of the track. Find (i) the kinetic energy of the ball when it is at the point O' (ii) the radial and the tangential accelerations when the ball is at O'.

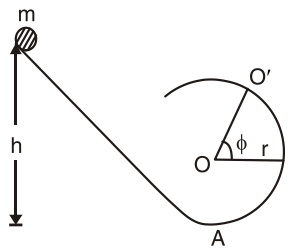


Fig. 3.27

Solution:

(i) The kinetic energy of the ball at O' is equal to the change in potential energy.

$\therefore$ *K.E.* of the ball at $O' = mgh - mg\,[r + r\sin\phi]$

$= mg\,[h - r - r\sin\phi]$

(ii) Total kinetic energy of the ball $= \frac{1}{2}\,mv^2 + \frac{1}{5}\,mv^2 = \frac{7}{10}\,mv^2$

$$\frac{7}{10}\,mv^2 = mg\,[h - r - r\sin\phi]$$

$$\therefore \text{ radial acceleration} = \frac{v^2}{r} = \frac{10g}{7}\left[\frac{h}{r} - 1 - \sin\phi\right]$$

On differentiating the above equation we have

$$\frac{2va}{r} = -\frac{10g}{7}\cos\phi \cdot \frac{d\phi}{dt}$$

$$\frac{va}{r} = -\frac{5g}{7}\cos\phi \cdot \omega;\ \frac{d\phi}{dt} = \omega$$

$$\therefore \text{ tangential acceleration } a = -\frac{5g}{7}\cos\phi$$

3.43. On imparting an initial velocity v_0 to the ball shown in the figure it begins to move in a horizontal circle of radius R on the horizontal plane *abcd* as in Fig. 3.28. Assuming the coefficient of friction between the ball and the plane as μ determine the time required for the ball to come to rest.

Solution:

$I\alpha = -\mu mgR$; m = mass of the ball

$$\text{or} \quad \frac{m \cdot R^2}{R^2} \cdot \frac{dv}{dt} = -\mu mg$$

$$\text{or} \quad \frac{dv}{dt} = -\mu g$$

$$\int_{v_0}^{0} dv = -\mu g \int_0^t dt$$

$$-v_0 = -\mu g \cdot t$$

$$\text{or} \quad t = \frac{v_0}{\mu g}$$

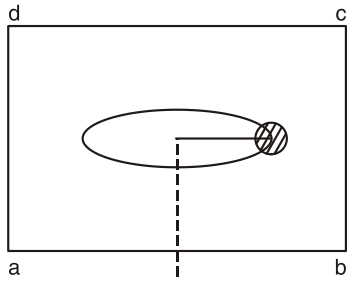


Fig. 3.28

3.44. A small object of mass m placed at the point O on the surface of a smooth cylinder of radius R slides down along OA. At A it detaches from the surface and after travelling freely in air it strikes at horizontal plane at B (see Fig. 3.29). Find the length BC.

Solution: At A we have $v^2 = 2gR\,(1 - \cos\theta)$

and $\frac{mv^2}{R} = mg\cos\theta$

$\therefore \quad v^2 = Rg\cos\theta = 2gR(1-\cos\theta)$

or $3\cos\theta = 2$

or $\cos\theta = \frac{2}{3}$

Velocity at $A = \sqrt{\frac{2gR}{3}}$

Horizontal component of this velocity $v_x = \sqrt{\frac{2gR}{3}} \cdot \cos\theta$

$= \frac{2}{3} \cdot \sqrt{\frac{2gR}{3}}$

Vertical component of this velocity $v_y = \sqrt{\frac{2gR}{3}} \cdot \sin\theta$

$= \frac{\sqrt{5}}{3} \cdot \sqrt{\frac{2gR}{3}} = \frac{\sqrt{\frac{10gR}{3}}}{3}$

Point A is at a height $R + R\cos\theta$ above the horizontal plane.

Time required to fall from this height is obtained by the equation.

$$R + \frac{2R}{3} = \frac{5R}{3} = \sqrt{\frac{10gR}{3}} \cdot t + \frac{1}{2} gt^2$$

or $\frac{10R}{3g} = \sqrt{\frac{40R}{3g}} \times \frac{t}{3} + t^2$

$$t = -\frac{\sqrt{\frac{40R}{3g}} \times \frac{1}{3} \pm \sqrt{\frac{40R}{27g} + \frac{40R}{3g}}}{2}$$

$$= -\frac{\sqrt{\frac{40R}{3g}} \times \frac{1}{3} \pm \frac{20\sqrt{R}}{3\sqrt{3g}}}{2}$$

$$= \frac{1}{3}\sqrt{\frac{R}{3g}} \quad 10 \quad \sqrt{10}$$

$$= \frac{1}{3}\sqrt{\frac{10R}{3g}} \quad \sqrt{10} \quad 1$$

$$= \sqrt{\frac{10R}{3g}} \times \frac{2.162}{3}$$

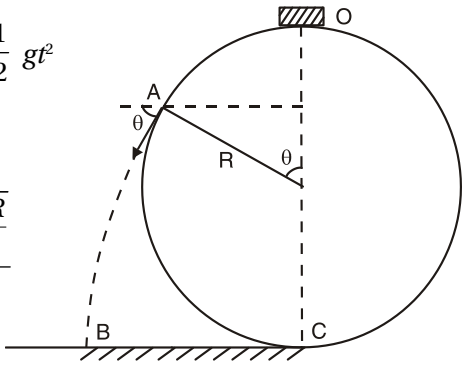


Fig. 3.29

$$\text{Distance } BC = \frac{2}{3}\sqrt{\frac{2gR}{3}} \times \sqrt{\frac{10R}{3g}} \times \frac{2.162}{3} + R\sin\theta$$

$$= \frac{4R\sqrt{5}}{27}\ \frac{2.162}{} + \frac{R\sqrt{5}}{3}$$

$$= R\,[0.7162 + 0.7453]$$

$$= 1.461\,R$$

3.45. A smooth sphere of radius R is made to translate in a straight line with a constant acceleration a. A particle A kept on the top of the sphere is released from there at zero velocity with respect to the sphere (see Fig. 3.30). Find the speed of the particle with respect to the sphere as a function of angle θ.

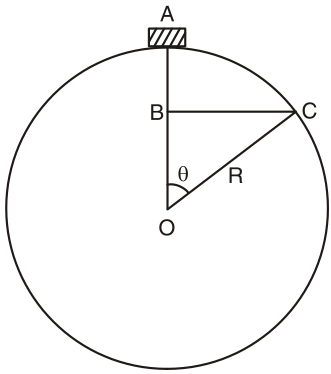


Fig. 3.30

Solution: The sphere moves through a distance $BC = R\sin\theta$ with an acceleration a hence velocity v_1 is given by

$$v_1^2 = 2aR\sin\theta$$

Now $AB = R\,(1 - \cos\theta)$

Velocity v_2 in the vertical direction is given by

$$v_2^2 = 2gR\,(1 - \cos\theta)$$

$$\therefore\ v^2 = v_1^2 + v_2^2 = 2R\,[a\sin\theta + g - g\cos\theta]$$

$$\text{or } v = \left\{2R\,(a\sin\theta + g - g\cos\theta)\right\}^{\frac{1}{2}}$$

3.46. A sleev A can slide freely along a smooth rod bent in the shape of a half circle of radius R as shown in Fig. 3.31a. The system is set in rotation with a constant angular velocity ω about a vertical axis OO'. Find the angle θ corresponding to the steady position of the sleeve.

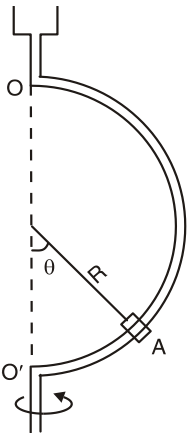


Fig. 3.31a

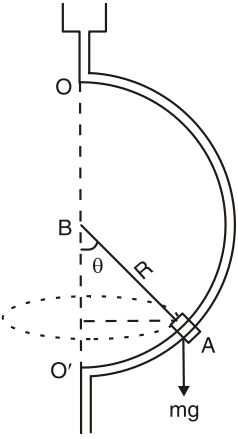


Fig. 3.31b

Solution: Normal force N will act along AB (see Fig. 3.31b).

$$N\sin\theta = m\omega^2 R\sin\theta$$

and $N\cos\theta = mg$

$$\frac{mg}{\cos\theta} \cdot \sin\theta = m\omega^2 R\sin\theta$$

or $\sin\theta\,(\omega^2 R\cos\theta - g) = 0$

when $\sin\theta = 0$

$\theta = 0$

when $\omega^2 R\cos\theta - g = 0$

$$\cos\theta = \frac{g}{\omega^2 R}$$

When $\omega^2 R > g$, there are two positions of equilibrium one is $\theta = 0$ and the other $\theta = \cos^{-1} g/\omega^2 R$.

When $\omega^2 R < g$ then $\sin\theta = 0$ or $\theta = 0$ is possible.

3.47. A particle is projected along the inside of a vertical hoop from its lowest point with such a velocity that it leaves the hoop and returns to the point of projection again. Find the velocity of projection and determine where the particle leaves the hoop.

Solution: At point P of Fig. 3.32, we have $mg\cos\theta = \frac{mv^2}{r}$; r = radius of the hoop

$$\therefore v = \sqrt{gr\cos\theta}$$

$$v^2\sin^2\theta = 2gh$$

$$\therefore h = \frac{gr\cos\theta}{2g}\cdot\sin^2\theta = \frac{r}{2}\cos\theta\cdot\sin^2\theta$$

Time of flight for $OO' = T = \sqrt{\left[2 + 2\cos\theta + \cos\theta\cdot\cos^2\theta\right]\frac{r}{g}}$

Time t to cover a height h is given by

$$t = \frac{v\sin\theta}{g} = \frac{\sqrt{g}}{g}\cdot\sqrt{r\cos\theta}\cdot\sin\theta$$

$$= \sqrt{\frac{r\cos\theta\cdot\sin^2\theta}{g}}$$

$$PP' = \sqrt{gr\cos\theta}\cdot\cos\theta\cdot\sqrt{\frac{r\cos\theta\cdot\sin^2\theta}{g}}$$

$$= r\cos^2\theta\cdot\sin\theta$$

$$a = r\sin\theta - r\cos^2\theta\cdot\sin\theta = r\sin^3\theta$$

$$r\sin^3\theta = Tv\cos\theta = \sqrt{\frac{r}{g}\left(2 + 2\cos\theta + \cos\theta\cdot\sin^2\theta\right)}\cdot\sqrt{gr\cos\theta}\cdot\cos\theta$$

$$\sin^6\theta = 2\cos^3\theta + 2\cos^4\theta + \cos^4\theta\cdot\sin^2\theta$$

$$2\cos^3\theta + 2\cos^4\theta + \sin^2\theta\,(\cos^4\theta - \sin^4\theta) = 0$$

$$2\cos^3\theta + 2\cos^4\theta + \sin^2\theta\,(\cos^2\theta - \sin^2\theta) = 0$$

$2\cos^3\theta + 2\cos^4\theta + (1-\cos^2\theta)(2\cos^2\theta - 1) = 0$

$2\cos^3\theta + 3\cos^2\theta - 1 = 0$

$\cos\theta = -1$ or $\cos\theta = \frac{1}{2}$

$\frac{1}{2}mv^2 = mg(r + r\cos\theta) + \frac{1}{2}mgr\cos\theta$

$= mg\ \frac{3r}{2}\ \frac{r}{4}$

$= \frac{7}{4}mgr$

$\therefore V = \sqrt{\frac{7gr}{2}}$

$H = r + r\cos\theta = \frac{3r}{2}$

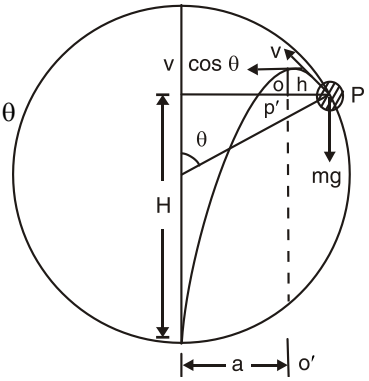


Fig. 3.32

3.48. A solid cylinder of radius R rolls on a plane ABC as shown in figure. Find the maximum value of the velocity v_0 which will allow the cylinder to roll on the inclined plane section without a jump.

Solution: With respect to the line AB' of Fig. 3.33, we have energy equation as

$\frac{1}{2}mv_0^2 + \frac{1}{2}\frac{mR^2}{2}\ {}^{2}_{0} + mgR = \frac{1}{2}mv^2 + \frac{1}{2}\frac{mR^2}{2}\omega^2 + mgR\cos\theta;$

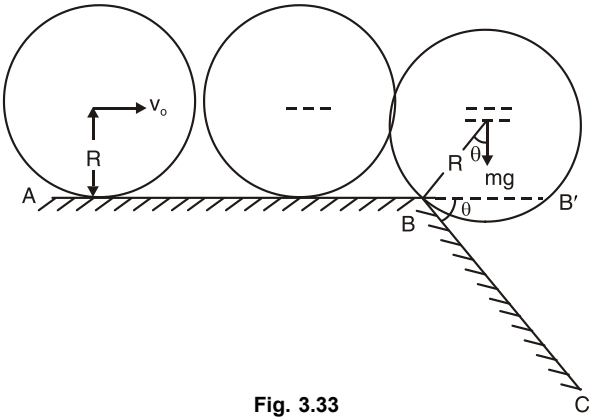


Fig. 3.33

ω_0 = angular velocity when cylinder moves along AB

ω = angular velocity when cylinder moves along BC

v = linear velocity along BC

or $\frac{1}{2}mv_0^2 + \frac{1}{4}mv_0^2 + mgR = \frac{1}{2}mv^2 + \frac{1}{4}mv^2 + mgR\cos\theta$

$$\therefore v^2 = v_0^2 + \frac{4}{3} gR(1 - \cos\theta)$$

For no jump $\frac{mv^2}{R} = mg\cos\theta$

or $v^2 = gR\cos\theta$

hence $gR\cos\theta - \frac{4}{3} gR(1 - \cos\theta) = v_0^2$

or $v_0 = \sqrt{\frac{gR}{3}(7\cos\theta - 4)}$

3.49. A mass point of mass m is situated at the origin of x-axis. A bar of length L and mass M lies on the x-axis. The end near the mass is at a distance a from it (see Fig. 3.34). Find the gravitational force of attraction between the two masses.

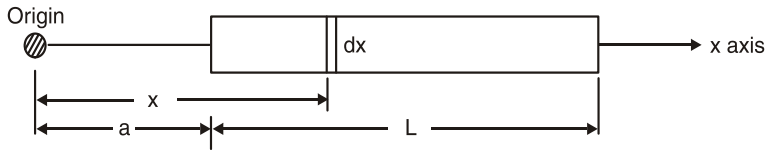


Fig. 3.34

Solution: Let us consider an element of length dx within the rod lying at a distance x from the origin.

Mass of the element = $\frac{M}{L} dx$

Gravitational force of attraction between the element and mass m is given by

$$dF = G \cdot m \cdot \frac{M}{L} \frac{dx}{x^2}$$

Net force on the bar is given by $F = \int dF = \frac{GmM}{L} \int_a^{L+a} \frac{dx}{x^2} = \frac{GmM}{a(L \quad a)}$

3.50. A thin wire of mass m_1 bent in the form of an arc of radius R subtends an angle ϕ at its centre (see Fig. 3.35). Calculate the force of attraction of the wire on a point mass m_2 placed at the centre of the arc.

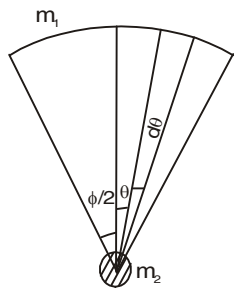


Fig. 3.35

Solution: Mass per unit length of the wire = $\frac{m_1}{R\phi}$

Consider an element subtending an angle $d\theta$ at the centre.

Mass of the element = $\frac{m_1}{R\phi} \cdot R \cdot d\theta$

Gravitational field due to the element at the centre $= \dfrac{G \cdot m_1 R d\theta}{R \cdot \phi \cdot R^2}$

$= \dfrac{G \cdot m_1}{R^2 \phi}\, d\theta$

The effective component of the gravitational field $= \dfrac{G m_1 \cos\theta \cdot d\theta}{R^2 \phi}$

Net gravitational field at the centre $= 2\,\dfrac{G m_1}{R^2 \phi} \int_0^{\phi/2} \cos\theta\, d\theta$

$= \dfrac{2 G m_1}{R^2 \phi} \cdot \sin\dfrac{\phi}{2}$

$\therefore$ Force of attraction of the wire on mass $m_2 = \dfrac{2 G m_1 m_2}{R^2 \phi} \sin\dfrac{\phi}{2}$

3.51. Find the force of attraction of a solid sphere of mass M and radius R on a particle of mass m placed at a distance r ($r < R$) from its centre.

Solution: The mass of the sphere which will cause the force is given by

$$\frac{M}{\frac{4}{3}\pi R^3} \cdot \frac{4}{3}\pi r^3 = \frac{M r^3}{R^3}$$

$$\therefore F = \frac{G \cdot M r^3}{R^3} \cdot \frac{m}{r^2}$$

$$= \frac{G M m r}{R^3}$$

3.52. A solid sphere of mass M and radius R has a non-uniform density that varies with r, the distance from the centre by relation $\rho = Ar$. Determine the force on a particle of mass m placed inside the sphere.

Solution: Consider a shell of radius r and $r + dr$

The mass of the shell $= \dfrac{4}{3}\pi \quad (r \quad dr)^3 \quad r^3 \quad Ar$

$= \dfrac{4}{3}\ \pi \cdot 3r^2\ dr \cdot Ar$

$= 4\pi A\ r^3 dr$

Total mass $M = \int_O^R 4\pi A\ r^3 dr$

$M = \pi AR^4$

$\therefore A = \dfrac{M}{\pi R^4}$

Let the particle be at a distance r from the centre of the sphere. Consider a shell of radius r and $r + dr$.

Mass of the shell $= \dfrac{4}{3}\pi \quad (r \quad dr)^3 \quad r^3 \quad Ar = 4\pi A r^3 dr$

Mass of the sphere which will cause the force

$$M' = 4\pi A \int_0^r r^3 dr$$
$$= \pi A r^4$$
$$= \pi r^4 \cdot \frac{M}{\pi R^4}$$
$$= \frac{Mr^4}{R^4}$$

$$\text{Gravitational force} = \frac{G \cdot Mr^4}{R^4} \cdot \frac{m}{r^2} = \frac{GMmr^2}{R^4}$$

3.53. A hemispherical thin shell of mass M and radius R has a small particle of mass m placed at its centre. Find the force of attraction between them.

Solution: Consider an element of width $Rd\theta$ (see Fig. 3.36).

$$\text{Mass of the element} = \frac{M}{2\pi R^2} \cdot 2\pi R \sin\theta \cdot Rd\theta$$

Gravitational force between the particle and element $= \dfrac{GmM \sin\theta \cdot d\theta}{R^2}$

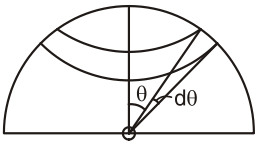


Fig. 3.36

$$\text{Net gravitational force} = \int_0^{\pi/2} \frac{GMm}{R^2} \sin\theta \cos\theta \, d\theta$$
$$= \frac{GMm}{2R^2}$$

3.54. A thin rod AB of mass M and length $2l$ and a particle of mass m is placed as shown in the Fig. 3.37. Calculate the gravitational force of attraction between the rod and the particle.

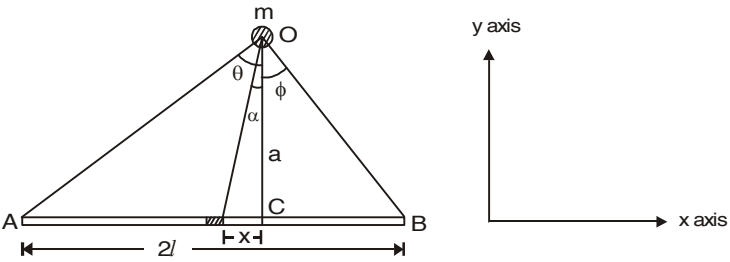


Fig. 3.37

Solution: Consider an element of length dx situated at a distance x from the point C in the AC portion of the rod.

$$\text{Gravitational field due to this element at } O = \frac{G \cdot M}{2l} \cdot \frac{dx}{a^2 \quad x^2}$$

Gravitational force between the element and particle

$$= \frac{GMm}{2l} \cdot \frac{dx}{a^2 \quad x^2}$$

Net gravitational force along –ve side of x axis $= \int_0^{AC} \frac{GMm \cdot x dx}{2l\left(a^2 + x^2\right)^{3/2}}$

$$= \frac{GMm}{2l \cdot a} (1 - \cos\theta)$$

Net gravitational force along –ve side of y axis $= \int_0^{AC} \frac{GM}{2l} \cdot \frac{a \cdot dx \cdot m}{a^2 \quad x^2 \quad ^{3/2}}$

$$= \frac{GMm}{2l \cdot a} \cdot \sin\theta$$

Similarly net gravitational force between BC portion of the rod and particle along +ve side of x axis $= \frac{GMm}{2l \cdot a} (1 - \cos\phi)$

Net gravitational force between BC portion of the rod and the particle along –ve side of y axis $= \frac{GMm}{2l \cdot a} \sin\phi$

Total gravitational force along –ve side of y axis

$$= F_y = \frac{GMm}{2l \cdot a} (\sin\theta + \sin\phi)$$

Total gravitational force along +ve side of x axis

$$= \frac{GMm}{2l \cdot a} (\cos\theta - \cos\phi)$$

Result gravitational force between the rod AB and particle.

$$F = \frac{GMm}{2l \cdot a} \sqrt{(\sin\theta + \sin\phi)^2 + (\cos\theta - \cos\phi)^2}$$

$$= \frac{GMm}{l \cdot a} \sin\left(\frac{\theta + \phi}{2}\right)$$

3.55. A homogenous rod of mass m_1 is turned in the form of semi-circle of radius R as shown in Fig. 3.38. Calculate the force experienced by a particle of mass m_2 placed at the centre of the semi-circle.

Solution: Mass per unit length of the semi-circle $= \frac{m_1}{\pi R}$

Consider an element of length dL

mass of the element $= \frac{m_1 dL}{\pi R}$

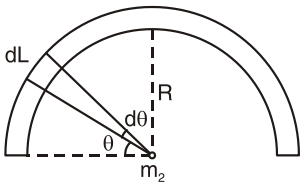


Fig. 3.38

Gravitational force between the element and the particle of mass m_2 is given by $G \cdot \frac{m_1 dL}{\pi R} \cdot \frac{m_2}{R^2}$

Out of the two components of the above force the effective component is given by $\frac{G \cdot m_1 dL}{\pi R} \cdot \frac{m_2}{R^2} \cdot \sin\theta$

Net gravitational force $= F = 2 \int_0^{\pi/2} \frac{G \cdot m_1 m_2 \sin\theta \cdot R d\theta}{\pi R^3}$

$$= \frac{2 G \cdot m_1 m_2}{\pi R^2} \int_0^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{2 G m_1 m_2}{\pi R^2}$$

3.56. A rod of length L is placed along x axis keeping its one end at a distance l from the origin. The mass per unit length λ varies with distance x by the relation $\lambda = k_0 + k_1 x^2$ where k_0 and k_1 are positive constants. Find the force experienced by a particle of mass m placed at the origin.

Solution: Consider an element of length dx situated at a distance x from the origin (see Fig. 3.39). Force between the particle and the element is given by

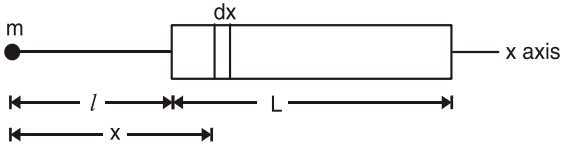


Fig. 3.39

$$dF = \frac{G \cdot m \cdot \left(k_0 + k_1 x^2\right) \cdot dx}{x^2}$$

Net force $F = dF = G \cdot m \cdot k_0 \int_l^{l+L} \frac{dx}{x^2} + G \cdot m \cdot k_1 \int_l^{l+L} dx$

$$= G \cdot m \cdot k_0 \left[\frac{1}{l} - \frac{1}{l + L} \right] + G \cdot m \cdot k_1 \cdot L$$

$$= \frac{G m \cdot k_0 \cdot L}{l (l + L)} + G m \cdot k_1 L$$

3.57. A ring is made of a thin wire of radius r. Find the force of attraction between the mass point of mass m placed along the axis of the ring of radius R at a distance x from its centre.

Solution: Consider an element of length dl on the circumference of the

circle of radius R. The mass of this element = $\pi r^2 dl \cdot \rho$ where ρ is the density of the material of the wire.

The force of attraction between the element chosen and the point mass is $dF = \dfrac{Gm\pi r^2 dl\rho}{\left(R^2 + x^2\right)}$

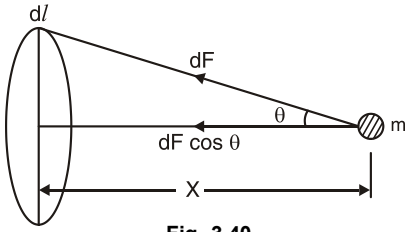


Fig. 3.40

This force will have two components $dF \sin\theta$ and $dF \cos\theta$. From considerations of symmetry, $dF \sin\theta$ vanishes. Therefore the net effective force is given by

$$F = \int dF \cos\theta = \frac{\pi m G r^2 \rho x}{\left(R^2 + x^2\right)^{\frac{3}{2}}} \int_0^{2\pi R} dl$$

$$= \frac{2\pi^2 r^2 R \rho G m x}{\left(R^2 + x^2\right)^{\frac{3}{2}}}$$

3.58. A small particle of mass m' is placed at the axis of a circular wire of mass m and radius r at a small distance from the centre as shown in Fig. 3.41. Show that the particle will execute simple harmonic motion. Find the period of oscillation.

Solution: The gravitational field at O at a distance x from the centre is given by

$$E_G = \frac{G \cdot m \cdot x}{\left(r^2 + x^2\right)^{\frac{3}{2}}}$$

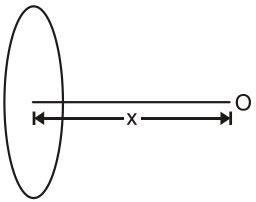


Fig. 3.41

Equation of motion of the particle of mass m' for small values of x is given by

$$m' \cdot \text{acceleration} = -\frac{G \cdot m x \cdot m'}{r^3}$$

$$\text{acceleration} = -\frac{Gm}{r^3} \text{ displacement.}$$

The acceleration is proportional to displacement hence the particle will execute simple harmonic motion.

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{Gm}{r^3}}$$

$$\text{or } T = 2\pi \sqrt{\frac{r^3}{Gm}}$$

3.59. Calculate the intensity of gravitational field at P due to an infinitely long straight wire of mass per unit length σ.

Solution: Consider an element of length dx placed at a distance x from O as shown in Fig. 3.42. The gravitational field at P due to the element is given by

$$dI = \frac{G\sigma dx}{\left(x^2 + r^2\right)}$$

This field will be in the direction PA. This has two components: $dI\cos\theta$ along PO and $dI\sin\theta$ perpendicular to it. If we consider a similar element at a distance x below O then we come across similar fields. The field in the perpendicular direction will vanish. The net field will be along PO.

$$\therefore I = \int dI = 2\int_0^{\infty} \frac{G\sigma dx}{\left(x^2 + r^2\right)} \cos\theta$$

$$= 2G\sigma r \int_0^{\infty} \frac{dx}{\left(x^2 + r^2\right)^{\frac{3}{2}}}$$

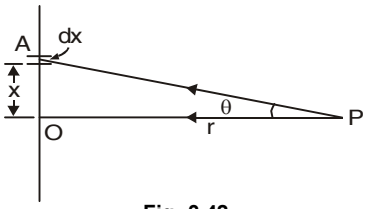


Fig. 3.42

Let $x = r\tan\theta$

$$dx = r\sec^2\theta d\theta$$

$$\text{Now } I = 2G\sigma r \int_0^{\frac{\pi}{2}} \frac{r}{r^3} \cos\theta d\theta = \frac{2G\sigma}{r} \left|\sin\theta\right|_0^{\pi/2} = \frac{2G\sigma}{r}$$

3.60. A solid cylinder of mass M, radius a and height h and a particle of mass m are placed as shown in the Fig. 3.43. Find the force of attraction between them.

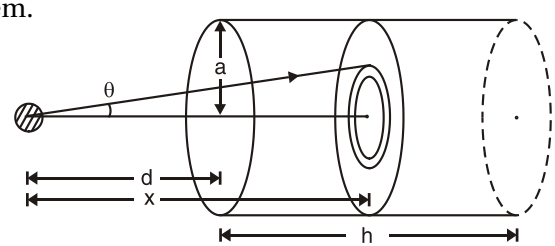


Fig. 3.43

Solution: Consider an element of thickenss dx situated at a distance x from mass m. The element will be in the form of disc of radius a and width dx. Upon this consider an element enclosed between circles of radii r and $r + dr$.

Volume of the element $= 2\pi r dr \cdot dx$

$$\text{Mass of the element} = \frac{M}{\pi a^2 h} \cdot 2\pi r dr dx$$

The gravitational field due to this element at the particle of mass m is given by

$$dE_0 = 2G \cdot \frac{M}{a^2 h} \cdot \frac{r\,dr \cdot dx}{x^2 \quad r^2}$$

Gravitational field due to the element along the axis of cylinder

$$dE_1 = 2G \cdot \frac{M}{a^2 h} \cdot \frac{r\,dr \cdot dx}{x^2 \quad r^2} \cos\theta$$

$$= 2G \frac{M}{a^2 h} \cdot x dx \cdot \frac{r\,dr}{\left(x^2 + r^2\right)^{\frac{3}{2}}}$$

Gravitational field due to the disc considered

$$dE_1 = dE = \frac{2GM}{a^2 h} \cdot x dx \int_0^a \frac{r dr}{\left(x^2 + r^2\right)^{\frac{3}{2}}}$$

$$dE = \frac{2GM}{a^2 h} x dx \cdot \frac{1}{x} \frac{1}{\sqrt{a^2 \quad x^2}}$$

Net gravitational field $E = dE = \frac{2GM}{a^2 h} \left[\int_d^{h+d} dx - \int_d^{h+d} \frac{x dx}{\sqrt{a^2 + x^2}} \right]$

$$= \frac{2GM}{a^2 h} \quad h \quad \sqrt{a^2 \quad d^2} \quad \sqrt{a^2 \quad (h \quad d)^2}$$

Force of attraction along the axis of cylinder is given by

$$F = mE = \frac{2GMm}{a^2 h} \quad h \quad \sqrt{a^2 \quad d^2} \quad \sqrt{a^2 \quad (h \quad d)^2}$$

3.61. A particle of mass m is at the centre of one end of a solid cylinder of mass M, radius a and height h as shown in Fig. 3.44. The volume of the cylinder is constant. For what value of a/h the force of attraction between the particle and cylinder is maximum?

Solution: $V = \pi a^2 h$

$$dV = 0 = \pi\,[2a \cdot dah + a^2 \cdot dh]$$

$$\frac{da}{dh} = -\frac{a}{2h}$$

Consider an element of thickness dx situated at a distance x from the particle of mass m. The element will be in the form of disc of radius a and width dx. Upon this consider an element enclosed between circles of radii r and $r + dr$

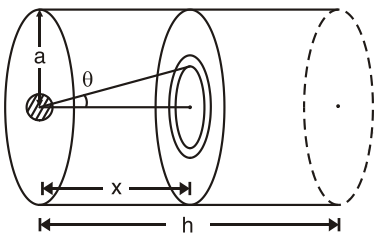


Fig. 3.44

Volume of the element = $2\pi r dr \cdot dx$

Mass of the element = $\dfrac{M}{\pi a^2 h} \cdot 2\pi r dr \cdot dx$

Effective gravitational field due to element at the particle.

$$dE_1 = 2 \cdot \frac{GM}{a^2 h} \frac{r dr \cdot x dx}{\left(x^2 + r^2\right)^{\frac{3}{2}}}$$

Gravitational field at the particle due to the disc

$$dE_1 = dE = \frac{2GM}{a^2 h} x dx \int_0^a \frac{r dr}{\left(x^2 + r^2\right)^{\frac{3}{2}}}$$

$$= \frac{2GM}{a^2 h} \quad dx \quad \frac{x dx}{\sqrt{a^2 \quad x^2}}$$

Net gravitational field due to the cylinder

$$E = \quad dE = \frac{2GM}{a^2 h} \left[\int_0^h dx - \int_0^h \frac{x dx}{\sqrt{a^2 + x^2}} \right]$$

$$= \frac{2GM}{a^2 h} \quad h \quad \sqrt{a^2 \quad h^2} \quad a$$

Force of attraction between the particle and cylinder is given by

$$F = \frac{2GMm}{a^2 h} \quad h \quad \sqrt{a^2 \quad h^2} \quad a$$

$$F = \frac{2\pi GMm}{V} \quad h \quad \sqrt{a^2 \quad h^2} \quad a$$

Now $$dF = \frac{2\pi GMm}{V} \quad dh \quad \frac{ada \quad hdh}{\sqrt{a^2 \quad h^2}} \quad da \quad = 0$$

$$dh \left[\sqrt{a^2 + h^2} - h \right] = da \quad a \quad \sqrt{a^2 \quad h^2}$$

or $$\left[\sqrt{a^2 + h^2} - h \right] = - \frac{a}{2h} \quad a \quad \sqrt{a^2 \quad h^2}$$

$$\sqrt{a^2 \quad h^2} \quad [2h - a] = (2h^2 - a^2)$$

On simplification we have

$$4a^2 - 9ah + 4h^2 \; = 0$$

$$\frac{a}{h} = \frac{9 \quad \sqrt{17}}{8}$$

3.62. Three thin uniform rods marked as 1, 2 & 3 having same mass M and length a are placed in the same plane and form an equilateral triangle. Find the force acting on a particle of mass m placed at the intersection of the medians of the triangle.

Solution: Consider an element of length dx placed at a distance x from the centre of the rod 1 (see Fig. 3.45).

Gravitational field at O due to the element

$$G \cdot \frac{M}{a} \cdot \frac{dx}{l^2 \quad x^2}$$

The effective gravitational field due to the element $= G \dfrac{M}{a} \cdot \dfrac{dx \cdot \cos\theta}{\left(l^2 + x^2\right)}$

$$= \frac{GMl\,dx}{a\left(l^2 + x^2\right)^{\frac{3}{2}}}$$

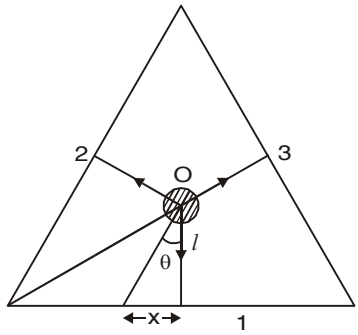


Fig. 3.45

Net gravitational field due to the rod 1 $= \dfrac{2GMl}{a} \int_0^{a/2} \dfrac{dx}{\left(l^2 + x^2\right)^{\frac{3}{2}}}$

$$= \frac{2GM}{l\sqrt{a^2 \quad 4l^2}}$$

The gravitational field due to the other two rods will have the same value. The resultant field at O is zero. Therefore force acting on the particle of mass m is equal to zero.

3.63. Two bodies of masses m_1 and m_2 are d apart. Find the gravitational potential at a point in between the masses where the field is zero.

Solution: Let us choose a point O in between the two masses at a distance x from m_1.

$$\frac{Gm_1}{x^2} = \frac{Gm_2}{(d \quad x)^2}$$

or $$x = \frac{\sqrt{m_1} \cdot d}{\sqrt{m_1} + \sqrt{m_2}}$$

Gravitational potential at

$$O = -\frac{Gm_1}{x} - \frac{Gm_2}{d \quad x}$$

$$= -\frac{Gm_1}{\sqrt{m_1} \cdot d} \quad \sqrt{m_1} \quad \sqrt{m_2} \quad - \frac{Gm_2}{\sqrt{m_2} \cdot d} \quad \sqrt{m_1} \quad \sqrt{m_2}$$

$$= -\frac{G}{d} \quad m_1 \quad m_2 \quad 2\sqrt{m_1 m_2}$$

3.64. An astronant of mass $m = 60$ kg is connected to a satellite of mass M through $a\ L = 70$ m long cord. The satellite is moving around earth in the close vicinity of earth. Calculate the tension in the cord when the satellite and the astronaut are arranged as in Fig. 3.46.

Solution: For the satellite the equation of motion is

$$\frac{GM_e \cdot M}{R^2} = M \cdot R \cdot \frac{4\pi^2}{\tau^2}; \qquad \tau = \text{period of revolution}$$

R = radius of earth

For astronaut the equation of motion is

$$\frac{m4\pi^2}{\tau^2}(R+L) = \frac{GM_e \cdot m}{(R+L)^2} + T;\ T = \text{tension in the cord}$$

$$\frac{4\pi^2 R}{\tau^2} + \frac{4\pi^2 L}{\tau^2} = \frac{GM_e}{R^2} \quad 1 \quad \frac{2L}{R} \quad + \frac{T}{m}$$

$$\frac{4\pi^2 L}{\tau^2} = -\frac{2GM_e \cdot L}{R^3} + \frac{T}{m}$$

$$\frac{3GM_e \cdot L}{R^3} \cdot m = T$$

or $T = \dfrac{3mgL}{R}$; $GM_e = gR^2$

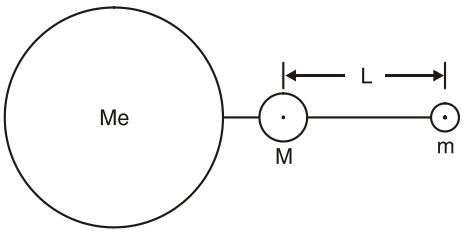


Fig. 3.46

On putting the values

$$T = \frac{3 \quad 60 \quad 9.8 \quad 70}{6400 \quad 10^3}$$

$$= 19.3 \times 10^{-3}\ \text{N}$$

3.65. A particle of mass m is dropped from a height ($h << R$) above a hole drilled along the vertical diameter of earth. Calculate the speed of the particle at the centre of the earth.

Solution: According to conservation of energy

$$-\frac{GM_e m}{R_e} + mgh = -\frac{3}{2}\ \frac{GM_e m}{R_e} + \frac{1}{2}\ mv^2$$

$$+\frac{1}{2}\ \frac{GM_e}{R_e} + gh = \frac{1}{2}\ v^2$$

$$\text{or } v = \sqrt{\frac{GM_e}{R_e} \quad 2gh}$$

3.66. A body falls to the ground from a great height h. Neglecting the resistance of the air, find (i) the velocity v with which the body will strike the surface of earth (ii) the time that the body will take to reach to the surface of earth.

Solution:

(i) $m\dfrac{dv}{dt} = \dfrac{G \cdot M_e \cdot m}{(R_e + h - x)^2}$; x is the vertical height through which the body falls

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot v = \frac{g R_e^2}{R_e \quad h \quad x^{\,2}}$$

$$\int_0^v v\,dv = g R_e^2 \int_0^x \frac{dx}{(R_e + h - x)^2}$$

$$v^2 = 2g R_e^2 \quad \frac{1}{R_e \quad h \quad x} \quad \frac{1}{R_e \quad h}$$

$$v = \frac{\sqrt{2g} \cdot R_e}{\sqrt{R_e \quad h}} \cdot \sqrt{\frac{x}{R_e \quad h \quad x}}$$

Velocity at the surface of earth will be obtained by putting $x = h$

$$v = \sqrt{\frac{2gR_e h}{R_e \quad h}}$$

(ii) $$\frac{dx}{dt} = \frac{\sqrt{2g} \cdot R_e}{\sqrt{R_e \quad h}} \cdot \sqrt{\frac{x}{R_e \quad h \quad x}}$$

$$\int_0^T dt = \sqrt{\frac{R_e \quad h}{2g}} \cdot \frac{1}{R_e} \int_0^h \sqrt{\frac{R_e + g - x}{x}}\, dx$$

$$T = \sqrt{\frac{R_e \quad h}{2g}} \cdot \frac{1}{R_e} \cdot \left[\frac{R_e + h}{2} \cdot \cos^{-1}\left(\frac{R_e - h}{R_e + h} \right) + \sqrt{R_e h} \right]$$

3.67. Under the influence of a central force at O a particle moves in a circular orbit passing through O. Find the law of force.

Solution: In polar coordinate the equation of circle of radius a is given by

$r = 2a \cos \theta$

If the orbit is given by $r = r(\theta)$ and $u = u(\theta)$ where $u = \dfrac{1}{r}$ the central force can be found by using the relation.

$$f(r) = \frac{mh^2}{r^4} \left[\frac{d^2 r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta} \right)^2 - r \right]; \; r^2 \dot{\theta} = h$$

or $$f\left(\frac{1}{u}\right) = -m h^2 u^2 \left[\frac{du^2}{d\theta^2} + u \right]$$

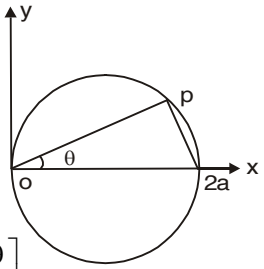


Fig. 3.47

$$\text{Now} \quad u = \frac{1}{r} = \frac{\sec\theta}{2a}$$

$$\frac{du}{d\theta} = \frac{\sec\theta\tan\theta}{2a}$$

$$\frac{d^2u}{d\theta^2} = \frac{\sec^3\theta + \sec\theta\cdot\tan^2\theta}{2a}$$

$$f\ \frac{1}{u} = -mh^2u^2\left[\frac{\sec^3\theta+\sec\theta\cdot\tan^2\theta}{2a}+\frac{\sec\theta}{2a}\right]$$

$$= -\frac{mh^2u^2\cdot\sec^3\theta}{a}$$

$$= -\frac{mh^2u^2}{a}\cdot 8u^3a^3$$

$$= -\frac{8a^2mh^2}{r^5}$$

3.68. A particle of mass m approaches earth from a large distance with an initial speed v_0. Calculate (i) the impact parameter b of the particle so that it passes off tangentially to the earth surface (ii) the ecentricity of the orbit.

Solution: Initial angular momentum, $L_1 = mv_0b$

Initial energy = $\frac{1}{2}\ mv_0^2$

Angular momentum at $B = mvR$; v = velocity

Final energy = $\frac{1}{2}\ mv^2 - \frac{GMm}{R}$; R = radius of earth

In view of conservation of angular momentum we have

$$v = \frac{v_ob}{R}$$

$$\frac{1}{2}\ mv_0^2 = \frac{1}{2}\ mv^2 - \frac{GMm}{R}$$

$$\text{or } v_0^2 = v_0^2\ \frac{b^2}{R^2} - \frac{2GM}{R}$$

$$b = R\left[1+\frac{2GM}{v_0^2R}\right]^{\frac{1}{2}}$$

$$= R\left[1+2\left(\frac{v_c}{v_0}\right)^2\right]^{\frac{1}{2}};\ v_c = \sqrt{\frac{GM}{R}}$$

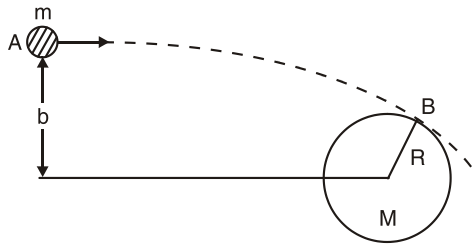


Fig. 3.48

B is perigee of the orbit at $\frac{1}{r_{\min}} = \frac{1}{p}\ \frac{e}{p}$; $p = \frac{v_0^2b^2}{GM}$

Eccentricity $e = \frac{p}{r_{\min}} - 1 = \frac{v_0^2 b^2}{GMR} - 1 = 1 + \left(\frac{v_o}{v_c}\right)^2$

Since $v_0 > v_c$

$\therefore\ e > 1$

The path is therefore hyperbola.

3.69. A space vehicle travels along a parabolic path with the earth at its focus. Calculate its total mechanical energy.

Solution: When the vehicle is at the apex of the parabola, its total energy

$W = \frac{mv_0^2}{2} - \frac{GmM}{r_0}$; r_0 is the distance between earth and vehicle

For curved path we have $\frac{GmM}{r_0^2} = \frac{mv_0^2}{R_0}$; R_0 = radius of curvature.

$$\frac{GMm}{r_0} = \frac{mv_0^2}{R_0} \cdot r_0$$

$$W = \frac{mv_0^2}{2} - \frac{mv_0^2 r_0}{R_0} = m v_0^2 \quad \frac{1}{2} \quad \frac{r_0}{R_0}$$

$$r_0 = \frac{R_0}{2}$$

$$\therefore\ W = O$$

3.70. A particle of mass m is in a force field of the type $F = -\frac{}{r^2}\ e_r$. Where α is a positive constant, and e_r is the unit vector of the particles position vector. The particle was placed at a point where the position vector is r_0 and the initial velocity v_0 perpendicular to r_0 was imparted to it. Find the trajectory of the particle.

Solution: Taking $r = \frac{1}{u}$ the differential equation for the path of the particle in a central field is

$$\frac{d^2u}{d\theta^2} + u = -\frac{f\ \frac{1}{u}}{mh^2u^2} \text{ where } h = r^2\dot{\theta}$$

Potential energy $V = -\quad f(r)dr = -\frac{}{r} = -\alpha u$

General equation of conic is

$$r = \frac{p}{1 + \in \cos\theta} = \frac{1}{\frac{1}{p} + \frac{\in}{p}\cos\theta}$$

Let $\frac{1}{p} = \frac{\alpha}{mh^2}$; $\frac{\in}{p} = C$

$$u = \frac{1}{r} = \frac{\alpha}{mh^2} + \cos\theta$$

From conservation of energy $\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E \quad V)}{mh^2}$

On putting the values we have

$$(-C\sin\theta)^2 + \left(\frac{\alpha}{mh^2} + C\cos\theta\right)^2 = \frac{2E}{mh^2} + \frac{2\alpha}{mh^2}\left(\frac{\alpha}{mh^2} + \cos\theta\right)$$

$$C = \sqrt{\frac{\alpha}{m^2h^4} + \frac{2E}{mh^2}}$$

$$u = \frac{1}{r} = \frac{\alpha}{mh^2}\left\{1 + \sqrt{1 + \frac{2Emh^2}{\alpha^2}}\cos\theta\right\}$$

Eccentricity $\in = \sqrt{1 + \frac{2Emh^2}{\alpha^2}}$ and $E = \frac{1}{2}\, m\, v_0^2 - \frac{\alpha}{r_0}$

$$\in = \sqrt{1 + \left(mv_0^2 - \frac{2\alpha}{r_0}\right)mr_0^4\dot{\theta}^2}$$

$$= \sqrt{1 + \left(mv_0^2 - \frac{2\alpha_0}{r_0}\right)\frac{mr_0^2v_0^2}{\alpha^2}}$$

For hyperbola $mv_0^2 - \frac{2\alpha}{r_0} > 0$ since $\in > 1$ or $v_0 > \sqrt{\frac{2\alpha}{mr_0}}$

For parabola $\in = 1 \;\therefore\; v_0 = \sqrt{\frac{2\alpha}{mr_0}}$

For ellipse $\in < 1 \;\therefore\; v_0 < \sqrt{\frac{2\alpha}{mr_0}}$

3.71. A body of mass m is thrown vertically up with an initial velocity u from the north pole of the earth. To what height will the body ascend?

Solution: Let us consider a body of mass m. Let it be thrown vertically up from the north pole N of the earth with an initial velocity u and let it ascend up to A such that $AN = h$ (see Fig. 3.49). In view of conservation of energy,

$$\frac{GM_em}{R_e \quad h} - \frac{GM_em}{R_e} = \frac{1}{2}\, mu^2$$

where M_e = mass of earth
R_e = radius of earth
G = gravitational const.

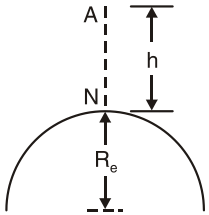


Fig. 3.49

or $\frac{GM_e mh}{R_e \quad h \quad R_e} = \frac{1}{2} mu^2$

We know that $GM_e = gR_e^2$

Therefore,

$$\frac{gR_e h}{R_e \quad h} = \frac{1}{2} u^2$$

or $\frac{2gR_e h}{u^2} = R_e + h$

or $h \quad \frac{2gR_e}{u^2} \quad 1 \quad = R_e$

or $h = \frac{R_e}{\frac{2gR_e}{u^2} \quad 1}$

3.72. Three particles each of mass m are situated at the vertices of an equilateral triangle of side length a. The force acting on the particles is their mutual gravitational force. It is desired that each particle moves in a circle by maintaining their original separation. Find (i) the velocity of each particle (ii) the period of the circular motion.

Solution: (i) From Fig. 3.50 $AD = a \sin 60 = \frac{a\sqrt{3}}{2}$

$$r = OA = \frac{2}{3} AD = \frac{a}{\sqrt{3}}$$

The gravitational force acting between the two particles = $\frac{Gm^2}{a^2}$

Net gravitational force along the centre of the circle = $\frac{2Gm^2}{a^2} \cdot \cos 30$

$$= \frac{\sqrt{3}\, G \cdot m^2}{a^2}$$

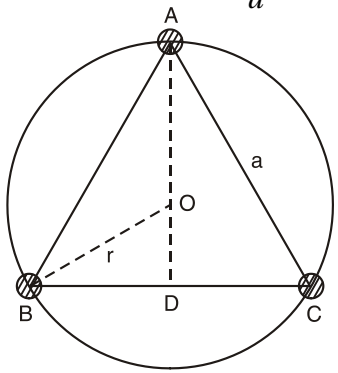


Fig. 3.50

In order to describe a circular path we must have

$$\frac{\sqrt{3} \cdot G \cdot m^2}{a^2} = \frac{mv^2}{r} = \frac{mv^2\sqrt{3}}{a}$$

$$\therefore v = \sqrt{\frac{Gm}{a}}$$

(ii) Time period $= \dfrac{2\pi r}{v}$

$$= \frac{2\pi}{\sqrt{Gm}} \cdot \sqrt{a} \cdot \frac{a}{\sqrt{3}} = \frac{2\pi a^{\frac{3}{2}}}{(3Gm)^{\frac{1}{2}}}$$

3.73. Calculate the magnitude and the direction of the gravitational field at a point P on the perpendicular bisector of the equal masses separated by a distance $2l$ as shown in the Fig. 3.51. For what value of r the field will be maximum?

Solution: The gravitational field at a point P due to one mass is given by

$$= \frac{G \cdot m}{\left(l^2 + r^2\right)}$$

Net gravitational field due to both the masses $I = \dfrac{2Gm\cos\theta}{\left(l^2 + r^2\right)}$

$$= \frac{2Gmr}{\left(l^2 + r^2\right)^{\frac{3}{2}}}$$

This will be in the direction PO.

For maxima or minima

$$\frac{dI}{dr} = O = 2Gm\left[\frac{1}{\left(l^2 + r^2\right)^{\frac{3}{2}}} - \frac{3}{2} \cdot \frac{r \cdot 2r}{\left(l^2 + r^2\right)^{\frac{5}{2}}}\right]$$

$$= \frac{2Gm\left(l^2 - 2r^2\right)}{\left(l^2 + r^2\right)^{\frac{3}{2}}}$$

$$\therefore r = \frac{l}{\sqrt{2}}$$

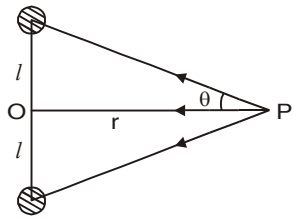


Fig. 3.51

With this value of r, $\dfrac{d^2I}{dr^2}$ assumes a negative value hence for $r = \dfrac{l}{\sqrt{2}}$ the gravitational field is maximum.

3.74. Calculate the force of attraction on a particle of mass m placed at the vertex of a solid cone of mass M, height h and radius R as shown in Fig. 3.52a.

Solution: Consider an element of thickness dy at a depth y from the vertex. The element will be in the form of disc. Upon this disc consider an element of radius x and $x + dx$.

Volume of the element $= 2\pi\, xdxdy$

Mass of the element $= \dfrac{3M}{\pi R^2 h} \cdot 2\pi x dx \cdot dy$

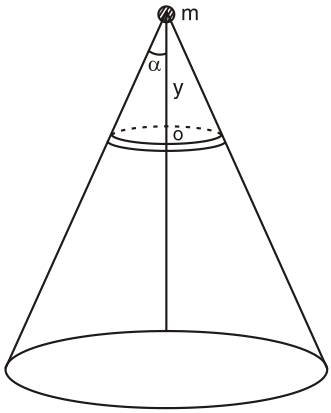


Fig. 3.52a

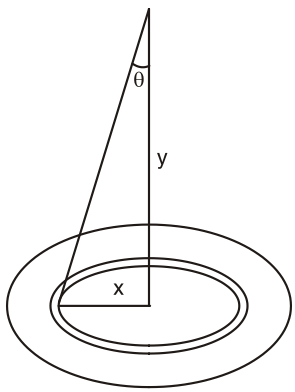


Fig. 3.52b

Gravitational force between the element and particle of mass m

$= \frac{G \cdot 6M \cdot m}{R^2 h} \, dy \cdot \frac{xdx}{x^2 \quad y^2}$

Effective gravitational force = $\frac{G \cdot 6Mm}{R^2 h} \cdot dy \cdot \frac{xdx}{x^2 \quad y^2} \cdot \cos\theta$ (see Fig. 3.52b)

Net gravitational force due to the disc = $\int_0^{y\tan\alpha} \frac{G \cdot 6Mm}{R^2 h} \cdot \frac{dy \cdot xdx \cdot y}{\left(x^2 + y^2\right)}$

$dF = \frac{6GMm}{R^2 h} \cdot dy \, (1 - \cos\alpha)$

$dF = \frac{6GMm}{R^2 h} \cdot dy \quad 1 \quad \frac{h}{\sqrt{h^2 \quad R^2}}$

Net gravitational force between the cone and particle is given

$F = \quad dF \; = \int_0^h \frac{6GMm}{R^2 h} \quad 1 \quad \frac{h}{\sqrt{h^2 \quad R^2}} \quad \cdot dy$

$= \frac{6GMm}{R^2} \quad 1 \quad \frac{h}{\sqrt{h^2 \quad R^2}}$

3.75. A solid hemisphere of mass M and radius R has a particle of mass m placed at its centre. Calculate the force of attraction between them.

Solution: Consider an element of thickness dy located at a height y above the centre (see Fig. 3.53). The element is in the form of disc of radius $a = \sqrt{R^2 \quad y^2}$. Upon this disc consider an element of radius x and $x + dx$.

Volume of the element $= 2\pi\, xdx \cdot dy$

Mass of the element $= \dfrac{3M}{2\pi R^3} \cdot 2\pi dy \cdot xdx$

$= \dfrac{3M}{R^3}\, dy \cdot xdx$

Gravitational force between the element and the particle of mass m

$= \dfrac{G \cdot 3M \cdot m}{R^3} \cdot dy \cdot \dfrac{xdx}{x^2 \quad y^2}$

Effective gravitational force $= \dfrac{G \cdot 3M \cdot m}{R^3} \cdot \dfrac{dy \cdot xdx}{x^2 \quad y^2} \cos\alpha$

$= \dfrac{G \cdot 3M \cdot m}{R^3} \cdot dy \cdot \dfrac{xdx}{\left(x^2 + y^2\right)^{\frac{3}{2}}} = \dfrac{G \cdot 3Mm \cdot y \cdot dy}{R^3} \quad \dfrac{xdx}{\left(x^2 + y^2\right)^{\frac{3}{2}}}$

Net gravitational force due to disc

$dF = \dfrac{G \cdot 3M \cdot m}{R^3} \cdot y \cdot dy \int_0^a \dfrac{xdx}{\left(x^2 + y^2\right)^{\frac{3}{2}}}$

$dF = \dfrac{3GMm}{R^3}\, ydy \quad \dfrac{1}{y} \quad \dfrac{1}{R}$

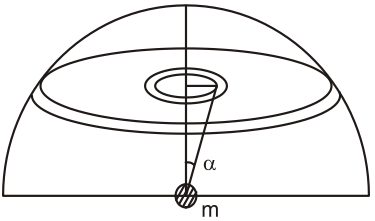


Fig. 3.53

Net gravitational force between hemisphere and particle

$F = \quad dF = \dfrac{3GMm}{R^3} \left[\int_0^R dy - \dfrac{1}{R}\int_0^R ydy\right]$

$F = \dfrac{3GMm}{R^3} \quad R \quad \dfrac{R}{2}$

$= \dfrac{3GMm}{2R^2}$

3.76. A solid hemisphere of mass M and radius R and a particle of mass m are placed as shown in the Fig. 3.54. Find the gravitational force of attraction between them.

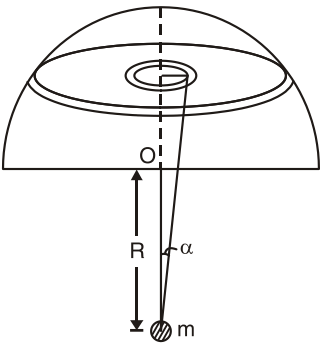


Fig. 3.54

Solution: Consider an element of thickness dy located at a height y above the centre O. The element is in the form of a disc of radius $\sqrt{R^2 \quad y^2}$. Upon this disc consider an element of radius x and $x + dx$.

Volume of the element $= 2\pi\, xdx \cdot dy$

Mass of the element = $\frac{3M}{2\pi R^3}\ 2\pi\, xdx \cdot dy$

Gravitational force between the element and the particle of mass m

$$= \frac{G \cdot 3M \cdot m}{R^3} \cdot dy \cdot \frac{xdx}{x^2 \quad (y \quad R)^2}$$

Effective gravitational force = $\frac{G \cdot 3Mm}{R^3} \cdot dy \cdot \frac{xdx \cdot \cos\alpha}{\left[x^2 + (y+R)^2\right]}$

Net gravitational force due to the disc

$$= \int_0^{\sqrt{R^2 - y^2}} \frac{G \cdot 3Mm}{R^3}\ (R+y)\ dy \cdot \frac{xdx}{\left[x^2 + (y+R)^2\right]^{\frac{3}{2}}}$$

$$dF = \frac{3GMm}{R^3}\ (R+y)\ dy \left[\frac{1}{y+R} - \frac{1}{\sqrt{2R}} \cdot \frac{1}{\sqrt{y+R}}\right]$$

Net gravitational force between the solid hemisphere and the particle.

$$F = \quad dF \ = \frac{3GMm}{R^3} \left[\int_0^R dy - \frac{1}{\sqrt{2R}} \int_0^R \sqrt{y+R} \cdot dy\right]$$

$$= \frac{GMm}{R^2} \quad \sqrt{2} \quad 1$$

3.77. In Fig. 3.55 a particle of mass m lies at a distance l from a thin homogenous disc of radius R and mass per unit area σ on a line perpendicular to its plane and passing through its centre. Calculate the force which the disc acts on it.

Solution: Consider an element of radius x and $x + dx$ on the disc of radius R.

Area of the element = $2\pi x dx$

Mass of the element = $2\pi x dx \sigma$

Effective gravitational force between the mass m and the element is given by

$$dF = \frac{G \cdot 2\pi x dx \sigma}{\left(x^2 + l^2\right)} \cdot m \cos\theta$$

$$= G \cdot 2\pi x \sigma m l \frac{xdx}{\left(x^2 + l^2\right)^{\frac{3}{2}}}$$

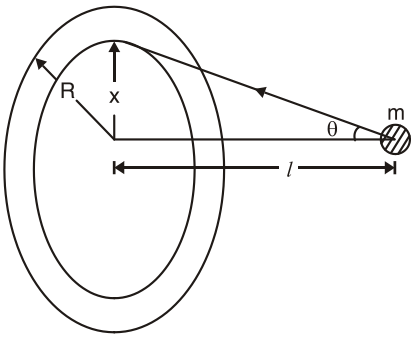


Fig. 3.55

$$\text{Net force } F = \int_0^R 2\pi\sigma G m l \frac{x\,dx}{\left(x^2 + l^2\right)^{\frac{3}{2}}}$$

$$= 2\pi\sigma G m l \frac{1}{l} \frac{1}{\sqrt{l^2 \quad R^2}}$$

$$= 2\pi\sigma G m \quad 1 \quad \frac{l}{\sqrt{l^2 \quad R^2}}$$

3.78. Calculate the gravitational force which a sphere of mass M and radius a exerts on a particle of mass m placed at a distance r from the centre of sphere.

Solution: Let ρ be the mass per unit area then

$$\rho = \frac{M}{4\pi a^2}$$

Consider a slice of thickness $ad\theta$ (see. Fig. 3.56).

Radius of the ring = $a \sin\theta$

Area of the strip = $2\pi a \sin\theta \cdot ad\,\theta$

Mass of the strip = $2\pi a \sin\theta \cdot ad\theta \cdot \rho$

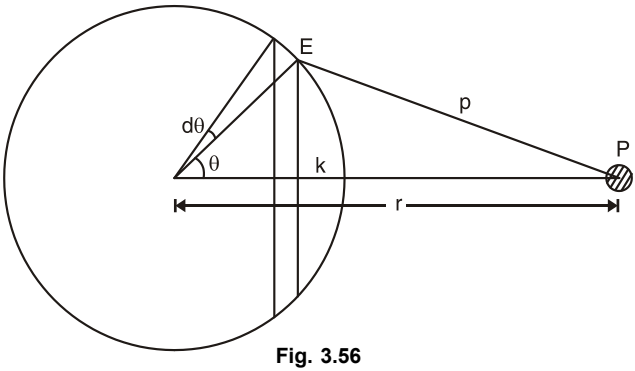


Fig. 3.56

Let $EP = p$ i.e. every point on the slice is at a distance p from the point P.

Gravitational potential at P is given by

$$dV = -\frac{2\pi a^2 \rho \sin\theta \cdot d\theta \cdot G}{p}$$

Further $p^2 = a^2 + r^2 - 2ar\cos\theta$

$2pdp = 2ar\sin\theta \cdot d\theta$

$$\therefore p = \frac{a \cdot r\sin\theta d\theta}{dp}$$

$$\therefore dV = -\frac{2\pi a^2 \rho \cdot p \cdot dp \cdot G}{p \cdot a \cdot r}$$

$$= -\frac{2\pi a \rho G \cdot dp}{r}$$

$$V = -\frac{2\pi a \rho G}{r}\int_{r-a}^{r+a} dp = -\frac{4\pi a^2 \rho G}{r} = -\frac{GM}{r}$$

Gravitational potential energy $U = -\frac{GMm}{r}$

Gravitational force $F = -\frac{dU}{dr} = -\frac{GMm}{r^2}$

3.79. Calculate the pressure of the gas in the middle of the radius of sun. The density is everywhere same and the acceleration due to gravity at this point is half of the value at that of the surface of sun. Mass of sun $M_S = 1.98 \times 10^{30}$ kg, radius of sun = $R_s = 7 \times 10^8$ m and $G = 6.67 \times 10^{-11}$ m^3/kg.s^2.

Solution: Assuming the gas to behave as a fluid the pressure can be determined by the equation.

$$p = h\rho g$$

$$= \frac{R_s}{2} \cdot \frac{M_s}{\frac{4}{3}\pi R_s^3} \cdot \frac{G \cdot M_s}{2R_s^2} = \frac{3G \cdot M_s^2}{16\pi \cdot R_s^4}$$

On substituting the values we have

$$p = \frac{3 \quad 6.67 \quad 10^{11} \quad 1.98 \quad 10^{30}{}^2}{16 \quad 3.14 \quad 7 \quad 10^8{}^4}$$

$$= 6 \times 10^{13} \text{ N/m}^2$$

3.80. At what height above the surface of earth the value of g will reduce by 1%. Radius of earth = 6400 km.

Solution: At a height h the weight of a body is defined by

$$mg' = \frac{GM_e \cdot m}{R_e \quad h^2} \; ; R_e = \text{radius of earth}$$

$$g' = \frac{g R_e^2}{R_e \quad h^2} = \frac{99g}{100}$$

$$10 R_e = \sqrt{99} \; (R_e + h)$$

$$10 R_e = 9.95 R_e + 9.95 h$$

$$0.05 R_e = 9.95 h$$

$$h = \frac{0.05 \times 6400}{9.95} = 32.2 \text{ km}$$

3.81. A body is thrown from the surface of earth with a velocity $\sqrt{1.6\,g\,R_e}$. How high it will rise.

Solution: When a body is projected with a velocity less than the escape velocity then it will describe as elliptical path.

According to conservation of angular momentum

$m \cdot \sqrt{1.6\,g\,R_e} \cdot R_e = mv\,(R_e + h)$; h is the height attained

$$\therefore\ v = \sqrt{1.6\,g\,R_e} \cdot \frac{R_e}{R_e \quad h}$$

According to conservation of energy we have

$$\frac{1}{2} \cdot m \cdot 1.6g\,R_e - \frac{GM_e m}{R_e} = \frac{1}{2}\ m \cdot 1.6\,g\,R_e \cdot \frac{R_e^2}{R_e \quad h^{\,2}} - \frac{GM_e m}{R_e \quad h}$$

or $\quad 0.8g\,R_e - g\,R_e = \dfrac{0.8\,g\,R_e^3}{(R_e + h)^2} - \dfrac{g\,R_e^2}{R_e \quad h}$

or $\quad 0.8\ R_e^2 - R_e\,(R_e + h) + 0.2\,(R_e + h)^2 = 0$

$$R_e^2 + 0.4\,R_e \cdot h + 0.2h^2 - R_e^2 - R_e h = 0$$

$$0.2h^2 - 0.6R_e h = 0$$

$$0.2h\,[h - 3R_e] = 0$$

$$h = 3R_e$$

3.82. A missile is fired from the surface of earth with a velocity of $\sqrt{g\,R_e}$ where R_e is the radius of the earth (see Fig. 3.57). At what angle θ it should be fired so that it may attain a maximum height $H = \dfrac{3R_e}{4}$?

Solution: According to conservation of energy we have

$$-\frac{GM_e m}{R_e} + \frac{1}{2}\ mgR_e = -\frac{GM_e m}{R_e + \frac{3}{4}R_e} + \frac{1}{2}\ m\ \left(\sqrt{g'R_e} \cdot \cos\theta\right)^2;$$

g' is the value of g at a height of $\dfrac{3R_e}{4}$.

Now $\quad mg' = \dfrac{G \cdot M_e m}{\left(R_e + \dfrac{3R_e}{4}\right)^2} = m \cdot \dfrac{16}{49} \cdot g$

$$\therefore\ g' = \frac{16g}{49}$$

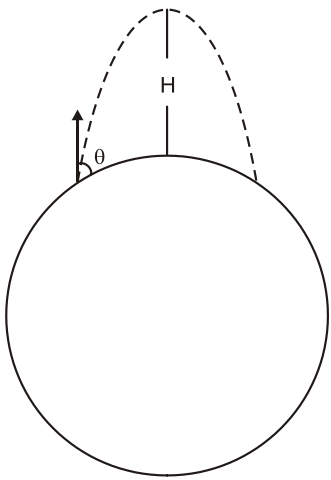


Fig. 3.57

$$\text{Now } -gR_e - 0.5\,gR_e = -\frac{4}{7}\,gR_e + \frac{1}{2} \times \frac{16g}{49} \cdot R_e \cdot \cos^2\theta$$

$$\frac{8}{49}\cos^2\theta = \frac{1}{14}$$

$$\text{or} \quad \frac{8}{49}(1 - \sin^2\theta) = \frac{1}{14}$$

$$\sin^2\theta = \frac{9}{16}$$

$$\sin\theta = \frac{3}{4}$$

$$\theta = 49°$$

3.83. Calculate the net gravitational force acting on a particle placed inside a hollow sphere.

Solution: Let P be any point in Fig. 3.58 inside a hollow sphere of surface density ρ. Through P draw two cones which are intersecting two areas S and S'. Let S and S' be at a distance r and r' and let the solid angle at P be ω.

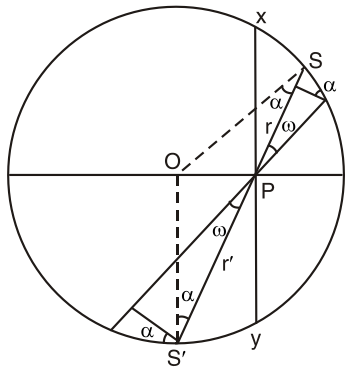


Fig. 3.58

Area of cone $S = r^2\omega$

Area of cone $S' = r'^2\omega$

$S\cos\alpha = r^2\omega$ and $S'\cos\alpha = r'^2\omega$

$$\text{Mass of area } S = \frac{\rho r^2 \omega}{\cos\alpha}$$

$$\text{Mass of area } S' = \frac{\rho r'^2 \omega}{\cos\alpha}$$

Intensity at P due to S in the direction of $PS = \dfrac{r^2\omega\rho G}{\cos\alpha \cdot r^2}$

$$= \frac{\omega\rho G}{\cos\alpha}$$

Similarly intensity at P due to $S' = \dfrac{r'^2\omega\rho G}{\cos\alpha \cdot r'^2} = \dfrac{\omega\rho G}{\cos\alpha}$

Net intensity at P is zero.

Hence force acting on a particle placed inside is equal to zero.

3.84. Calculate the gravitational potential and gravitational field due to a solid sphere of mass M and radius R at a point inside it at a distance r from its centre.

Solution: Let P be a point in Fig. 3.59 at a distance r at which the potential is to be determined. The solid sphere may be imagined to be made up of an inner solid sphere of radius r surrounded by number of hollow spheres.

Therefore the gravitational potential at P is equal to potential due to inner sphere + potential due to shells lying outside.

Potential at P due to inner sphere of radius $r = -\frac{4}{3}\pi\frac{r^3\sigma G}{r}$; σ = mass per unit volume.

Consider a spherical shell of radius x and $x + dx$.

Mass of the shell = $4\pi x^2 dx\,\sigma$

We know that potential at any point inside the shell is equal to potential at the surface, therefore, potential due to shell $= -\frac{4\pi x^2 dx\sigma G}{x}$

$= -4\pi x dx\sigma G$

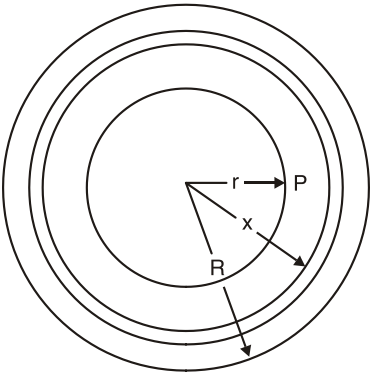


Fig. 3.59

Net potential due to shell lying outside $= -4\pi\sigma G\int_r^R x dx$

$= -2\pi\sigma G[R^2 - x^2]$

Total potential at $P = -\frac{4}{3}\pi r^2\sigma G - 2\pi\sigma G(R^2 - r^2)$

$= -\frac{4}{3}\pi\sigma G\ \frac{3R^2\ \ r^2}{2}$

$= -\frac{4}{3}\pi R^3\sigma G\ \frac{3R^2\ \ r^2}{2R^3}$

$V = -GM\ \frac{3R^2\ \ r^2}{2R^3}$

Gravitational field $= -\frac{dV}{dr} = -\frac{GM\vec{r}}{R^3}$

3.85. Calculate the pressure due to gravitational compression inside a sphere of radius R and mass M as a function of distance r from its centre.

Solution: Let p be the compression pressure at a distance r. Consider an element of thickness dr and area ΔA (see Fig. 3.60).

Outward force on the element = $\Delta p \cdot \Delta A$

Gravitational force = $-\frac{G\cdot\Delta A\cdot dr\rho\cdot\frac{4\pi}{3}r^3\rho}{r^2}$

$dp\cdot\Delta A = -G\cdot\Delta A\cdot dr\cdot\rho^2\cdot\frac{4\pi}{3}r$

$$p = -G\rho^2 \frac{4\pi}{3}\int r dr$$

$$p = -G\rho^2 \cdot \frac{4\pi}{3} \cdot \frac{r^2}{2} + C$$

at $r = R$, $p = 0$

$$\therefore C = G\rho^2 \frac{4\pi}{3} \cdot \frac{R^2}{2}$$

$$p = \frac{2\pi}{3} G \cdot \rho^2 [R^2 - r^2]$$

$$= \frac{2\pi}{3} \cdot G \cdot \frac{9M^2}{16\pi^2 R^6} [R^2 - r^2]$$

$$= \frac{3GM^2}{8\pi R^4} \quad 1 \quad \frac{r^2}{R^2}$$

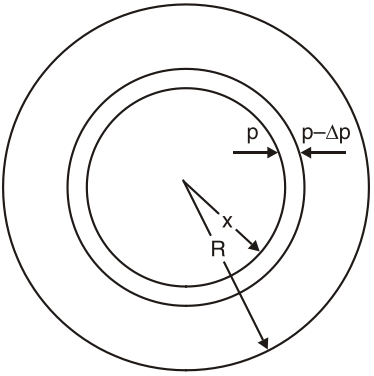


Fig. 3.60

3.86. A chain of length l and mass m is placed on the surface of a smooth solid sphere of radius R ($R > l$) with one end fixed at point O as shown in Fig. 3.61a. Find (i) the gravitational potential energy of the chain in reference to the centre of the sphere (ii) the velocity acquired by the chain after sliding through an angle θ (iii) the tangential acceleration of the chain as it begins to slide.

Solution: (i) Consider a small length dL of the chain.

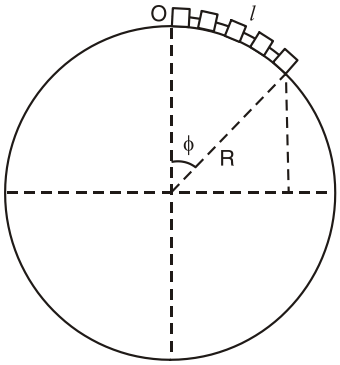


Fig. 3.61a

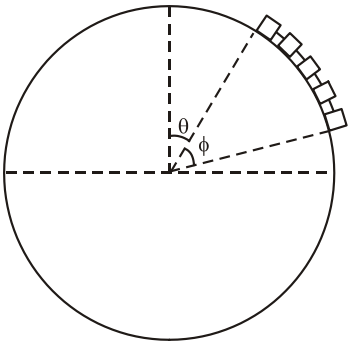


Fig. 3.61b

$dL^2 = dx^2 + dy^2$

or $dL = \sqrt{dx^2 \quad dy^2}$

In a circle $x^2 + y^2 = R^2$

$xdx + ydy = 0$

$$\therefore dy = -\frac{x}{y} dx$$

$$\therefore dL = \sqrt{1 - \frac{x^2}{y^2}} \cdot dx = \frac{R\,dx}{y}$$

$$y_c = \frac{y\,dL}{dL}$$

$$= \int_0^{R\sin\phi} y \cdot \frac{R}{y} \cdot \frac{dx}{l}$$

$$= \frac{R^2 \sin\phi}{l} = \frac{R^2 \sin\left(\frac{l}{R}\right)}{l}; \phi = \frac{l}{R}$$

Gravitational potential energy $= \frac{mg \cdot R^2}{l} \sin\left(\frac{l}{R}\right)$

(ii) After sliding through an angle θ (see Fig. 3.61b)

$$y'_c = \int_{R\sin\theta}^{R\sin\left(\theta+\frac{l}{R}\right)} \frac{R \cdot dx}{l}$$

$$= \frac{R^2}{l}\left[\sin\left(\theta + \frac{l}{R}\right) - \sin\theta\right]$$

Loss in potential energy

$$= \frac{mgR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]$$

$$= \text{gain in K.E.} = \frac{1}{2}\,mv^2$$

$$\therefore v = \sqrt{\frac{2gR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]}$$

(iii) $$\frac{dv}{dt} = \frac{1}{2}\,R \cdot \sqrt{\frac{2g}{l}} \cdot \frac{\left[\cos\theta - \cos\left(\theta + \frac{l}{R}\right)\right]\omega}{\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]^{\frac{1}{2}}};$$

$$\omega = \frac{d\theta}{dt} = \frac{v}{R}$$

$$= \frac{1}{2}\,R \cdot \sqrt{\frac{2g}{l}} \cdot \frac{1}{R} \cdot \frac{\left[\cos\theta - \cos\left(\theta + \frac{l}{R}\right)\right] R \cdot \sqrt{\frac{2g}{l}} \cdot \left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]^{\frac{1}{2}}}{\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]^{\frac{1}{2}}}$$

$$= \frac{gR}{l}\left[\cos\theta - \cos\left(\theta + \frac{l}{R}\right)\right]$$

As it begins to slide $\theta \to 0$

$$\therefore \text{ Tangential acceleration} = \frac{gR}{l}\left[1 - \cos\left(\frac{l}{R}\right)\right]$$

3.87. A tunnel is dug through the centre of the earth (see Fig. 3.62). A particle is projected into the tunnel from A with an initial speed of $\sqrt{gR_e}$. How long it will take to reach the other end of the tunnel.

Solution: According to conservation of energy

$\frac{1}{2}\ mgR_e - \frac{GM_e m}{R_e} = -\frac{GM_e m}{2R_e^3}\ \ 3R_e^2\ \ x^2\ \ + \frac{1}{2}\ mv_B^2$; v_B = velocity of the particle at B.

Further $GM_e = gR_e^2$

$$\therefore \frac{1}{2}\ gR_e - gR_e = -\frac{g}{2R_e}\ \ 3R_e^2\ \ x^2\ \ + \frac{1}{2}\ v_B^2$$

$$v_B = \sqrt{2gR_e\ \ 1\ \ \frac{x^2}{2R_e^2}}$$

Now $\int_{R_e}^{0} \frac{dx}{\sqrt{1-\frac{x^2}{2R_e^2}}} = -\sqrt{2gR_e}\ \ \int_0^t dt$; Velocity and displacement is opposite to each other

t = time taken by the particle to reach the centre of the earth.

$$\sqrt{2gR_e}\cdot t\ = \sqrt{2}\ R_e \cdot \sin^{-1}\frac{1}{\sqrt{2}}$$

$$= \sqrt{2}\ R_e \cdot \frac{\pi}{4}$$

$$\therefore t = \frac{\pi}{4}\cdot\sqrt{\frac{R_e}{g}}$$

Total required time $= 2t = \frac{\pi}{2}\ \sqrt{\frac{R_e}{g}}$

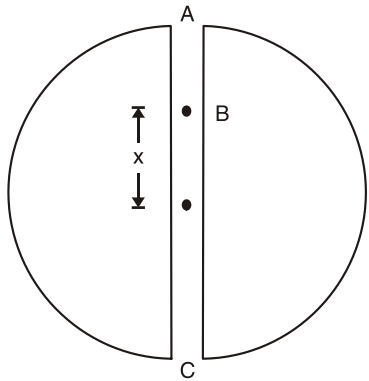


Fig. 3.62

3.88. A planet of mass m is moving in an elliptical orbit with the sun at its focus. The perihelion and aphelion distances are r and R respectively. Find the period of revolution of the planet considering Kepler's law of planetary motion.

Solution: Let a be the semi-major axis of the ellipse. Then $2a = r + R$ or $a = \frac{R \quad r}{2}$. Now consider that the planet is moving round in a circular orbit of radius $(r + R)/2$. The equation of motion of the planet in this orbit is given by

$$\frac{GM_s m}{(R \quad r)/2^{\ 2}} = m \quad \frac{R \quad r}{2} \quad \times \frac{4\pi^2}{T^2}$$

where M_s = mass of sun and
G = gravitational constant

$$\text{or } T^2 = \quad \frac{R \quad r}{2}^{\ 3} \quad \frac{4\pi^2}{GM_s}$$

$$\text{or } T = \pi \sqrt{\frac{(R \quad r)^3}{2GM_s}}$$

3.89. A planet of mass m is moving in an elliptical orbit with the sun of mass M_s at its focus. The perihelion and aphelion distances are r_2 and r_1 respectively. What will be the angular momentum of the planet relative to the sun?

Solution: When the planet is at A (Fig. 3.63), its total energy is given by

$$\frac{1}{2} \ m r_2^2 \ \omega_2^2 - \frac{G M_s m}{r_2}$$

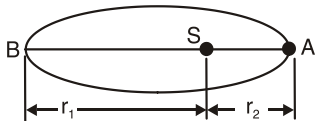


Fig. 3.63

Similarly, when the planet is at B, its total energy is given by

$\frac{1}{2} \ m r_1^2 \ \omega_1^2 - \frac{GM_s m}{r_1}$, here ω_1 and ω_2 are the angular velocities of the planet. In view of the conservation of energy we have

$$\frac{1}{2} \ m r_2^2 \ \omega_2^2 - \frac{G M_s m}{r_2} = \frac{1}{2} \ m\omega_1^2 \ r_1^2 - \frac{GM_s m}{r_1}$$

$$\text{or} \quad \frac{1}{2} \left(\frac{m^2 r_1^4 \omega_1^2}{m r_1^2} - \frac{m^2 r_2^4 \omega_2^2}{m r_2^2} \right) = GM_s m \quad \frac{1}{r_1} \quad \frac{1}{r_2}$$

$$\text{or} \quad \frac{1}{2} \left[\frac{(I_1 \omega_1)^2}{m r_1^2} - \frac{(I_2 \omega_2)^2}{m r_2^2} \right] = GM_s m \quad \frac{1}{r_1} \quad \frac{1}{r_2}$$

where I_1 and I_2 are the moments of inertia of the planet about an axis passing through sun. Further $I_1\omega_1 = I_2\omega_2 = J$.

$$\text{Therefore} \ \frac{J^2}{2m} \quad \frac{1}{r_1^2} \quad \frac{1}{r_2^2} \quad = GM_s m \quad \frac{1}{r_1} \quad \frac{1}{r_2}$$

$$\text{or} \quad J^2 \left(\frac{1}{r_1} - \frac{1}{r_2}\right) = 2GM_s m^2$$

$$\text{or} \quad J = \sqrt{\frac{2GM_s m^2 r_1 r_2}{r_1 - r_2}}$$

3.90. A spherical hollow is made in a solid sphere of radius R and mass M by taking out a mass from it. The surface of the hollow touches the surface of the sphere (see Fig. 3.64). A small solid sphere of mass m is placed at a distance x from the centre of the sphere. Find out the gravitational force of attraction between the two masses.

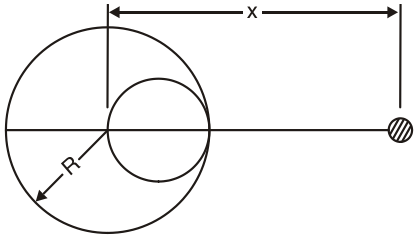


Fig. 3.64

Solution: Volume of the sphere = $\frac{4}{3}\pi R^3$

Mass per unit volume = $\frac{M}{\left(\frac{4}{3}\right)\pi R^3}$

Volume of the spherical hollow = $\frac{4}{3}\pi\left(\frac{R}{2}\right)^3$

Mass taken out = $\frac{M}{\left(\frac{4}{3}\right)\pi R^3} \times \frac{4\pi}{3}\left(\frac{R}{2}\right)^3 = \frac{M}{8}$

Net force of attraction = Force due to solid sphere of mass M – force due to solid sphere of mass $M/8$.

$$\text{or } F = \frac{GMm}{x^2} - \frac{GMm}{8(x - R/2)^2}$$

The centre of the hollow sphere will be at a distance $x - R/2$ from the mass m.

$$\therefore F = \frac{GMm}{x^2}\left[1 - \frac{1}{8[1-(R/2x)]^2}\right]$$

3.91. A double-star is a system of two stars moving around the centre of inertia of the system due to gravitation. Find the distance between the

components of the double star, if its total mass equals M and the period of revolution is T.

Solution: Fig. 3.65 shows a double-star system. The mass of the stars are m_1 and m_2 such that $m_1 + m_2 = M$. The gravitational force on star of mass m_1 must be equal to the centripetal force required for its motion in a circular orbit of radus r_1,

i.e., $\frac{G m_1 m_2}{r_1 \quad r_2^{\,2}} = m_1 \omega^2 r_1$

Further, about the centre of inertia, we have the identity

$$m_1 r_1 = m_2 r_2$$

or $$\frac{m_1}{m_2} = \frac{r_2}{r_1}$$

or $$\frac{m_1 \quad m_2}{m_2} = \frac{M}{m_2} = \frac{r_1 \quad r_2}{r_1}$$

or $$m_2 = \frac{M r_1}{r_1 \quad r_2}$$

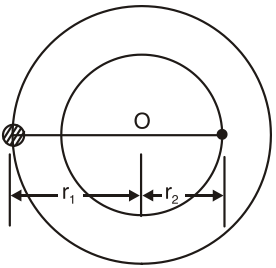


Fig. 3.65

On substituting the value of m_2, we have

$$\frac{G M r_1}{r_1 \quad r_2^{\,3}} = \omega^2 r_1$$

or $$\left(\frac{GMT^2}{4\pi^2}\right)^{1/3} = r_1 + r_2$$

3.92. Show that the total mechanical energy of a planet of mass m moving in an elliptical orbit with the sun at its focus depends only on the semi-major axis of the ellipse.

Solution: The motion of the planet of mass m around the sun is shown in Fig. 3.66.

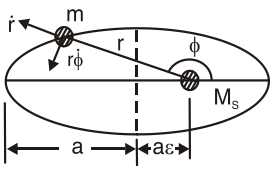


Fig. 3.66

Radial velocity of the planet = $\dot{r} = \frac{dr}{dt}$

Transverse velocity of the planet $r\dot{\phi} = r\frac{d\phi}{dt}$

Radial angular momentum $p_r = m\dot{r}$

Azimuthal angular momentum = $p_\phi = mr^2\,\dot{\phi}$

Total energy $W = \frac{1}{2}\, m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{GM_s m}{r}$

or $$W = \frac{1}{2m}\left(m^2\dot{r}^2 + \frac{m^2 r^4 \dot{\phi}^2}{r^2}\right) - \frac{GM_s m}{r}$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{GM_s m}{r}$$

As pointed out above, $p_r = m\dot{r} = m\,\frac{dr}{d\phi}\cdot\frac{d\phi}{dt} = \frac{mr^2\dot{\phi}}{r^2}\cdot\frac{dr}{d\phi} = \frac{p_\phi}{r^2}\,\frac{dr}{d\phi}$

Now $W = \frac{p_\phi^2}{2mr^2}\left[\left(\frac{1}{r}\frac{dr}{d\phi}\right)^2 + 1\right] - \frac{GM_s m}{r}$

Let the general equation of ellipse be

$$\frac{1}{r} = c_1 + c_2\cos\phi$$

At $r = a(1-\varepsilon)$, $\phi = 0$ and at $r = a(1+\varepsilon)$, $\phi = \pi$; a = semi-major axis.

On substituting these two conditions c_1 and c_2 can be evaluated. With these values of c_1 and c_2 the most general equation of ellipse is

$$\frac{1}{r} = \frac{1+\varepsilon\cos\phi}{a\left(1-\varepsilon^2\right)}$$

On substituting the value of r in the expression of energy, we have

$$W = \frac{p_\phi^2\left(1+\varepsilon\cos\phi\right)^2}{2ma^2\left(1-\varepsilon^2\right)^2}\left[\frac{1+\varepsilon^2+2\varepsilon\cos\phi}{\left(1+\varepsilon\cos\phi\right)^2}\right] - \frac{GM_s m(1+\varepsilon\cos\phi)}{a\left(1-\varepsilon^2\right)}$$

$$= \frac{p_\phi^2}{2ma^2\left(1-\varepsilon^2\right)^2}\left[1+\varepsilon^2+2\varepsilon\cos\phi\right] - \frac{GM_s m\left(1+\varepsilon\cos\phi\right)}{a\left(1-\varepsilon^2\right)}$$

Since the total energy is independent of ϕ, on equating the coefficients of $\varepsilon\cos\phi = 0$, we have

$$\frac{p_\phi^2}{2ma^2\left(1-\varepsilon^2\right)^2} = \frac{GM_s m}{2a\left(1-\varepsilon^2\right)}$$

The expression for energy is

$$W = \frac{GM_s m}{a\left(1-\varepsilon^2\right)}\cdot\left[\frac{1+\varepsilon^2}{2} - 1\right] = -\frac{GM_s m}{2a}$$

3.93. A satellite revolves from east to west in a circular equatorial orbit of radius R around the earth. Find the velocity and the acceleration of the satellite in the reference frame fixed to the earth.

Solution: The velocity of the satellite, $v' = v_1 + v_2$

where $v_1 = \sqrt{\frac{GM}{R}}$ and $v_2 = \frac{2\pi}{T}R$

M = mass of the earth

and T = period of revolution of earth about its own axis

$$\therefore\ v' = \sqrt{\frac{GM}{R}} + \frac{2\pi}{T}R$$

The acceleration of the satellite, $a' = a_1 + a_2$

where $a_1 = \frac{GM}{R^2}$ and $a_2 = \frac{2\pi}{T}\sqrt{\frac{GM}{R}}$

$$\therefore a' = \frac{GM}{R^2} + \frac{2\pi}{T}\sqrt{\frac{GM}{R}}$$

$$= \frac{GM}{R^2}\left[1 + \frac{2\pi}{T}R\frac{R}{GM}\cdot\sqrt{\frac{GM}{R}}\right]$$

$$= \frac{GM}{R^2}\left[1 + \frac{2\pi R}{T}\sqrt{\frac{GM}{R}}\right]$$

3.94. Two particles of mass M_1 and M_2 are at rest an infinite distance apart. Find the relative velocity of their approach.

Solution: The force exerted by M_1 on M_2 is given by

$$F_{M_2M_1} = \frac{GM_2M_1}{r^2} \text{ (see Fig. 3.67)}$$

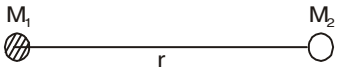


Fig. 3.67

Acceleration a_2 of mass $M_2 = \frac{GM_1}{r^2}$

Force exerted by mass M_2 on mass M_1 is given by

$$F_{M_1M_2} = -\frac{GM_2M_1}{r^2}$$

Acceleration a_1 of mass $M_1 = \frac{GM_2}{r^2}$

Relative acceleration $a = a_2 - a_1 = \frac{G}{r^2}(M_1 + M_2)$

We know that $a = \frac{dv}{dt} = \frac{dv}{dr}\cdot\frac{dr}{dt} = -\frac{dv}{dr}v$

On the basis of the above, we have

$$-\frac{G\ M_1\ \ M_2}{r^2} = v\frac{dv}{dr}$$

or $\int_0^V v\,dv = -\int_{\infty}^{x} G\ M_1\ \ M_2\ \frac{dr}{r^2}$ (x is the separation between the two particles)

or $V^2 = \frac{2G\ M_1\ \ M_2}{x}$

or $V = \sqrt{\frac{2G\ M_1\ \ M_2}{x}}$

3.95. A tunnel is dug through the centre of the earth as shown in Fig. 3.68. A particle of mass m is dropped through it. Find out its velocity at a depth of 1 km. Radius of earth $R_e = 6400$ km, $g = 9.8$ m/s^2.

Solution: Gravitational potential at A

$$= -\frac{GM}{R_e}$$

Gravitational potential at B

$$= -\frac{GM}{2R_e^3}\left(3R_e^2 - x^2\right)$$

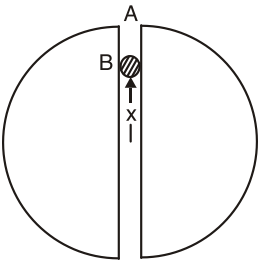


Fig. 3.68

Gravitational potential energy at $A = -\dfrac{GMm}{R_e}$

Gravitational potential energy at $B = -GMm\dfrac{\left(3R_e^2 - x^2\right)}{2R_e^3}$

Change in energy $= \dfrac{1}{2}mv_B^2$

$$= -\frac{GMm}{R_e} + \frac{3}{2}\frac{GMm}{R_e} - \frac{GMmx^2}{2R_e^3}$$

$$= \frac{1}{2}\frac{GMm}{R_e} - \frac{GMmx^2}{2R_e^3}$$

$$\therefore v_B = \sqrt{\frac{GM}{R_e}\left(1 - \frac{x^2}{R_e^2}\right)} = \sqrt{\frac{GM}{R_e}\frac{\left(R_e + x\right)\left(R_e - x\right)}{R_e^2}}$$

$$= \sqrt{\frac{g\left(R_e + x\right)\left(R_e - x\right)}{R_e}}\ ;\ GM = gR_e^2$$

Here x = 6399 km. On substituting the value of g and R_e we have

$$v_B = \sqrt{\frac{9.8\ (12{,}799)\ 10^3\ 10^3}{6400\ 10^3}} = \frac{1}{8}\sqrt{98\ (12{,}799)} = 140 \text{ m/s}$$

3.96. Two satellites S_1 and S_2 revolved round a planet in coplanar circular orbits in the same sense. Their periods of revolution are 1 hour and 8 hours respectively. The radius of the orbit of S_1 is 10^4 km. When S_2 is closest to S_1, find (i) the speed of S_2 relative to S_1, (ii) the angular speed of S_2 as actually observed by an astronaut in S_1. (IIT 1986)

Solution: Let R_1 and R_2 be the radii of the two circular orbits in which the two satellites S_1 and S_2 are revolving with periods of revolution T_1 and T_2.

Now according to Kepler's law of planetary motion, we have

$T_1^2 = kR_1^3$ and $T_2^2 = kR_2^3$

or $\left(\frac{T_1}{T_2}\right)^{2/3} = \frac{R_1}{R_2}$

$R_2 = R_1 \left(\frac{T_2}{T_1}\right)^{2/3} = 10^4 \times 8^{2/3} = 4 \times 10^4$ km

$\therefore R_2 = 4R_1$

Now speed of $S_2 = v_2 = \frac{2\pi R_2}{T_2} = \frac{2 \quad 3.14 \quad 4 \quad 10^4}{8} = 3.14 \times 10^4$ km/h

Speed of $S_1 = v_1 = \frac{2\pi R_1}{T_1} = 2 \times 3.14 \times 10^4 = 6.28 \times 10^4$ km/h

Speed of S_2 relative to $S_1 = |v_2 - v_1| = 3.14 \times 10^4$ km/h

Angular speed of S_2 as observed from $S_1 = \frac{|v_2 \quad v_1|}{R_2 \quad R_1}$

$= \frac{3.14 \quad 10^4}{3 \quad 10^4} = \frac{\pi}{3}$ rad/h

3.97. A satellite is revolving in a circular orbit of radius R around the earth. By external efforts its kinetic energy is suddenly increased by 50%. What maximum distance measured from the centre of the earth will be attained by the satellite in its new orbit?

Solution: Let the initial kinetic energy be $\frac{1}{2} m v_0^2$.

The kinetic energy of the satellite after its energy has been enhanced by 50% is given by

$\frac{1}{2} mv^2 = \frac{1}{2} m v_0^2 + \frac{1}{2} \cdot \frac{1}{2} m v_0^2 = \frac{3}{4} m v_0^2$

$\therefore v = \sqrt{\frac{3}{2}} \cdot v_0$

According to conservation of angular momentum

$mv_0 \sqrt{\frac{3}{2}} \cdot R = mv'r$; v' is the velocity in the new orbit of radius r.

$\therefore v' = v_0 \frac{R}{r} \cdot \sqrt{\frac{3}{2}}$

According to conservation of energy we have

$\frac{3}{4} m v_0^2 - \frac{GM_e m}{R} = \frac{1}{2} \frac{m \cdot v_0^2 \cdot 3R^2}{2r^2} - \frac{GM_e m}{r}$

$\frac{3}{4} v_0^2 \quad 1 \quad \frac{R^2}{r^2} = GM_e \quad \frac{1}{R} \quad \frac{1}{r}$; $\frac{GM_e}{R} = v_0^2$

$$\text{or} \quad \frac{3}{4} v_0^2 \quad 1 \quad \frac{R^2}{r^2} \quad = v_0^2 \quad 1 \quad \frac{R}{r}$$

$$\text{or} \quad \frac{3}{4} \quad 1 \quad \frac{R}{r} \quad = 1$$

$$r = 3R$$

3.98. At what angle θ with the horizontal should a missile be fired from the surface of earth with an initial velocity v in order to attain a height equal to the radius of the earth?

Solution: According to conservation of angular momentum

$$mv \cos\theta \cdot R_e = mV \cos\phi \cdot 2R_e$$

$$\therefore V \cos\phi = \frac{v \cos\theta}{2}$$

In view of conservation of energy we have

$$\frac{1}{2} mv^2 - \frac{GM_e m}{R_e} = - \frac{GM_e m}{2R_e} + \frac{1}{2} \frac{v^2 \cos^2\theta}{4}$$

$$\frac{1}{2} v^2 - gR_e = -g \frac{R_e}{2} + \frac{v^2}{8} \cos^2\theta$$

$$v^2 \left(1 - \frac{\cos^2\theta}{4}\right) = gR_e$$

$$1 - \frac{\cos^2\theta}{4} = \frac{2gR_e}{2v^2} = \frac{1}{2} \frac{v_{ese}}{v}^2$$

$$\cos\theta = 2 \quad 1 \quad \frac{1}{2} \frac{v_{ese}}{v}^{2 \ \frac{1}{2}}$$

3.99. Calculate the parking height of a geostationary satellite using $R_e = 6400$ km and $g = 9.8$ m/sec^2.

Solution: The equation of motion is given by

$$\frac{GM_e m}{r^2} = mr\omega^2 \text{ ; } r = \text{radius of the path measured from the centre of the earth.}$$

$$\text{or} \quad \frac{GM_e}{\omega^2} = r^3$$

$$\text{or} \quad \left(\frac{GM_e}{\omega^2}\right)^{\frac{1}{3}} = r$$

$$\text{or} \quad \left(\frac{gR_e^2 T^2}{4\pi^2}\right)^{\frac{1}{3}} = r$$

On substituting the values we have

$$r = \left[\frac{9.8 \times 6.4 \times 10^{12} \times (24 \times 60 \times 60)^2}{4 \times 3.14^2}\right]^{\frac{1}{3}}$$

$= 42.32 \times 10^6$ m

Parking height $h = (42.32 - 6.4) \times 10^6$ m $= 35.92 \times 10^6$ m

$= 35920$ km

3.100. The largest and the shortest distance of earth from the sun are R_1 and R_2. Find its distance from the sun when it is perpendicular to the major axis from the orbit drawn from the sun.

Solution: For the motion of earth, let us start with the general equation of ellipse $\frac{1}{r} = C_1 + C_2 \cos\theta$

When $\theta = 0$, $r = R_2$ and when $\theta = \pi$, $r = R_1$

Using these conditions we have

$$\frac{1}{R_1} = C_1 - C_2$$

and $\frac{1}{R_2} = C_1 + C_2$

on adding $2C_1 = \frac{1}{R_1} + \frac{1}{R_2}$

$$\therefore C_1 = \frac{R_1 \quad R_2}{2R_1R_2}$$

Let $r = R$ when $\theta = 90$

$$\therefore \frac{1}{R} = C_1 = \frac{R_1 \quad R_2}{2R_1R_2}$$

$$\therefore R = \frac{2R_1R_2}{R_1 \quad R_2}$$

3.101. A satellite is moving in a certain orbit of radius $R = 2 \times 10^4$ km from west to east around earth of mass $M = 6 \times 10^{24}$ kg. Calculate the time at which it will pass over a point at the equator.

Solution: When the satellite is moving downwards east its angular velocity increases hence.

$$\frac{GMm}{R^2} = \frac{m \; v_1 \quad v_2 \;^2}{R}$$

or $$\frac{GM}{R} = R^2\left(\frac{2\pi}{T} + \frac{2\pi}{t}\right)^2$$

or $$\sqrt{\frac{GM}{R^3}} - \frac{2\pi}{T} = \frac{2\pi}{t}$$

On substituting the values we have

$$\frac{2\pi}{t} = \sqrt{\frac{6.67 \quad 10^{11} \quad 6 \quad 10^{24}}{4 \quad 10^{14} \quad 2 \quad 10^{7}}} - \frac{2\pi}{24\times 60\times 60}$$

$$= 2.2366 \times 10^{-4} - 0.726 \times 10^{-4}$$

$$t = \frac{2 \quad 3.14 \quad 10^{4}}{1.5106 \quad 60 \quad 60}$$

$$= 11.548 \text{ hrs.}$$

3.102. A satellite moving around the earth at a height $H = R_e$ above the surface of earth ejects a packet (see Fig. 3.69). The speed of the packet is reduced to 1830 m/sec by using external methods. Determine (i) the speed with which the packet will reach the surface of earth (ii) the angle ϕ with the vertical at which it will strike surface of earth. $R_e = 6400$ km.

Solution: According to conservation of energy we have

$$-\frac{GM_e m}{2R_e} + \frac{1}{2} m v_0^2 = -\frac{GM_e m}{R_e} + \frac{1}{2} m v^2$$

or $\sqrt{gR_e \quad v_0^2} = v$

On substituting the values we have

$$v = \sqrt{9.8 \quad 6400 \quad 10^3 \quad 1830^2}$$

$$= 8.128 \times 10^3 \text{ m/sec.}$$

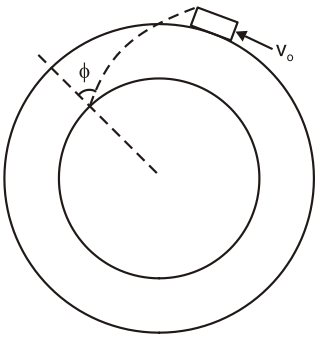


Fig. 3.69

According to conservation of angular momentum we have

$$m \cdot 1830 \times 2R_e = m \times 8.128 \times 10^3 \sin\phi \cdot R_e$$

$$\sin\phi = \frac{3660}{0.128 \quad 10^3} = 0.4502$$

$$\phi = 26.8°$$

3.103. An artificial satellite of moon is revolving in a circular orbit of radius $\eta\, r_m$ where r_m = radius of moon. In the state of motion it experiences a resistive force $F = \alpha v^2$. How long it will stay in the orbit until it falls on the surface of moon?

Solution: The equation of motion of satellite is given by

$$\frac{G \cdot M_m \cdot m}{\eta^2 r_m^2} = \frac{mv^2}{\eta r_m}$$

Kinetic energy of satellite $= \frac{1}{2} mv^2 = \frac{1}{2} m \frac{G \cdot M_m}{\eta r_m}$

Potential energy of satellite $= -\frac{GM_m \cdot m}{\eta r_m}$

Total energy $E = -\dfrac{GM_m \cdot m}{2\eta r_m} = -\dfrac{g_m \cdot r_m^2 \cdot m}{2\eta r_m} = -\dfrac{g_m \cdot r_m \cdot m}{2\eta}$

Now $dE = F \cdot dx = F \cdot v \cdot dt$

$$\frac{dE}{dt} = F \cdot v = \alpha v^2 \cdot v = \alpha v^3 = \alpha \frac{g_m^{\frac{3}{2}} \cdot r_m^{\frac{3}{2}}}{\eta^{\frac{3}{2}}};\ v = \left(\frac{g_m \cdot r_m}{\eta}\right)^{\frac{1}{2}}$$

$$-\frac{g_m \cdot m}{2\eta} \cdot \frac{dr_m}{dt} = \alpha \cdot \frac{g_m^{\frac{3}{2}} \cdot r_m^{\frac{3}{2}}}{\eta^{\frac{3}{2}}}$$

$$\frac{m \cdot \sqrt{\eta}}{2\alpha\sqrt{g_m}} \int_{\eta r_m}^{r_m} r_m^{-\frac{3}{2}} dr_m = \int_o^T \cdot dt$$

$$T = \frac{m\sqrt{\eta}}{\alpha\sqrt{g_m}} \left[\frac{1}{\sqrt{r_m}} - \frac{1}{\sqrt{\eta r_m}}\right] = \frac{m \cdot \left(\sqrt{\eta} - 1\right)}{\alpha\sqrt{g_m \cdot r_m}}$$

3.104. An engine burn out a satellite which has reached an altitude h and has velocity v_0 forming an angle ϕ with the vertical as shown in Fig. 3.70. Calculate the maximum and minimum height attained by the satellite.

Solution: In view of conservation of energy we have

$$\frac{1}{2} m v_0^2 - \frac{GM_e m}{r_0} = \frac{1}{2} mv^2 - \frac{GM_e m}{r};\ r_0 = R_e + h$$

r = height attained

$$v_0^2 - \frac{2gR_e^2}{r_0} = v^2 - 2g\frac{R_e^2}{r}$$

According to conservation of angular momentum

$$mr_0 v_0 \sin\phi = mrv$$

$$\therefore v = \frac{v_0 r_0 \sin\phi}{r}$$

$$v_0^2 - \frac{2gR_e^2}{r_0} = \frac{v_0^2 r_0^2}{r^2} \sin^2\phi - \frac{2gR_e^2}{r}$$

$$1 - \frac{2gR_e^2}{v_0^2 \cdot r_0} = \frac{r_0^2 \sin^2\phi}{r^2} - \frac{2gR_e^2}{v_0^2 \cdot r}$$

Let us put $\dfrac{r_0 v_0^2}{gR_e^2} = \eta$

$$\left(1 - \frac{2}{\eta}\right) = \frac{r_0^2 \sin^2\phi}{r^2} - \frac{2r_0}{\eta r}$$

$$(2 - \eta)\, r^2 - 2r_0 r + \eta r_0^2 \sin^2\phi = 0$$

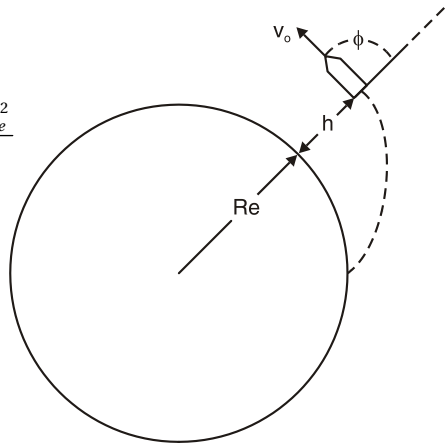


Fig. 3.70

$$r = \frac{2r_0 \pm \sqrt{4r_0^2 - 4(2-\eta)\eta r_0^2 \sin^2\phi}}{2(2-\eta)}$$

$$= \frac{r_0}{2-\eta}\left[1 \pm \sqrt{1-(2-\eta)\eta\sin^2\phi}\right]$$

$$\text{Maximum height} = \frac{r_0}{2-\eta}\left[1+\sqrt{1-(2-\eta)\eta\sin^2\phi}\right] - R_e$$

$$\text{Minimum height} = \frac{r_0}{2-\eta}\left[1-\sqrt{1-(2-\eta)\eta\sin^2\phi}\right] - R_e$$

3.105. A satellite is launched in a direction parallel to the surface of earth with a velocity v_0 = 10 km/sec from an altitude of $x = 0.6 \times 10^6$ m (see Fig. 3.71). Calculate the maximum height attained by it. Radius of earth = 6.4×10^6 m.

Solution: According to conservation of angular momentum

$$mr_0v_0 = mr_1v_1$$

$$\therefore v_1 = \frac{r_0v_0}{r_1}; \quad r_0 = (6.4 + 0.6) \times 10^6 \text{ m} = 7 \times 10^6 \text{ m}$$

$$v_0 = 10^4 \text{ m/sec.}$$

According to conservation of energy we have

$$\frac{1}{2}mv_0^2 - \frac{GM_em}{r_0} = \frac{1}{2}mv_1^2 - \frac{GM_em}{r_1}$$

$$\text{or } \frac{1}{2}v_0^2 - \frac{gR_e^2}{r_0} = \frac{1}{2}\frac{r_0^2v_0^2}{r_1^2} - \frac{gR_e^2}{r_1}$$

$$\text{or } \frac{1}{2}v_0^2 \quad 1 \quad \frac{r_0^2}{r_1^2} = gR_e^2 \quad \frac{1}{r_0} \quad \frac{1}{r_1}$$

$$\text{or } \frac{1}{2}v_0^2\left[1+\frac{r_0}{r_1}\right] = \frac{gR_e^2}{r_0}$$

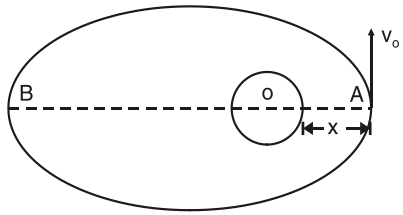


Fig. 3.71

$$1 + \frac{r_0}{r_1} = \frac{2gR_e^2}{r_0 \cdot v_0^2}$$

On putting the value we have

$$1 + \frac{r_0}{r_1} = \frac{2 \quad 9.8 \quad (6.4 \quad 10^6)^2}{7 \quad 10^6 \quad 10^8}$$

$$1 + \frac{r_0}{r_1} = 1.14688$$

$$r_1 = \frac{r_0}{0.14688} = \frac{7 \quad 10^6}{0.14688} = 47.65 \times 10^6$$

$\therefore$ Maximum height = $(47.65 - 6.4) \times 10^6 = 41.25 \times 10^6$ m

3.106. A satellite of mass m is revolving in a circular orbit of radius r_0 with speed v_0 around earth (see Fig. 3.72). At a certain point in its orbit the direction of motion of the satellite is changed by turning its velocity vector through an angle θ keeping its magnitude constant. As a result of it the satellite moves into an elliptical orbit. Calculate the distances and speeds of satellite both at perigee and apogee of the orbit. Given $\cos\theta = \frac{3}{5}$.

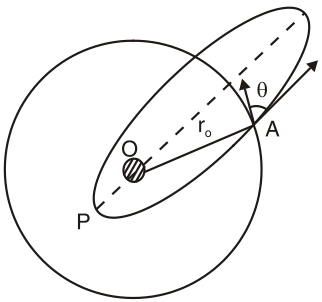


Fig. 3.72

Solution: While moving in the circular orbit the angular momentum is given by $L_0 = mv_0r_0$.

Equation of motion is given by

$$\frac{GM_em}{r_0^2} = \frac{mv_0^2}{r_0}$$

$$K.E. = \frac{GM_em}{2r_0} \text{ and } P.E. = -\frac{GMm}{r_0}$$

When the satellite is at A then angular momentum about O is given by $mr_0v_0\cos\theta = \frac{3}{5}\,mv_0r_0$

According to conservation of angular momentum

$$mv_pr_p = \frac{3}{5}\,mv_0r_0$$

$$\text{Energy at } P = E_\rho = \frac{1}{2}\,mv_\rho^2 - \frac{GM_em}{r_\rho}$$

$$= \frac{1}{2}\,m\cdot\frac{9}{25}\,\frac{v_0^2r_0^2}{r_\rho^2} - \frac{mr_0v_0^2}{r_\rho}$$

$$= \left(\frac{9}{50}\cdot x^2 - x\right) mv_0^2 \text{; here } x = \frac{r_0}{r_\rho}$$

According to conservation of energy

$$\frac{9}{50}x^2 \quad x \quad mv_0^2 = -\frac{1}{2}\,mv_0^2$$

or $9x^2 - 50x + 25 = 0$

$$x = 5 \text{ or } \frac{5}{9}$$

with $x = 5\ r_0 = 5r_p$

$$r_{\min} = \text{perigee} = r_p = \frac{r_0}{5}$$

$$r_{\max} = \text{apogee} = \frac{9r_0}{5}$$

v_{max} at perigee $= 3v_0$

v_{min} at apogee $= \frac{v_0}{3}$

3.107. Two spaceships move around earth in the same elliptical path of eccentricity e. At perigee the two ships are separated by a small distance d. Calculate the separation between them at apogee.

Solution: According to Kepler's law the areal speed is constant.

$$\therefore r_{min} \cdot d = r_{max} \cdot x$$

$$\therefore x = \frac{r_{min}}{r_{max}} \cdot d = \frac{1 \quad e}{1 \quad e} \quad d$$

3.108. A satellite while moving around earth has maximum and minimum orbital speed as v_{max} and v_{min} respectively. Calculate the eccentricity of the orbit.

Solution: According to conservation of angular momentum

$$m \cdot v_{max} \cdot r_{min} = m\, v_{min} \cdot r_{max}$$

The general equation of conic is

$$\frac{l}{r} = 1 + e\cos\theta$$

From this $\frac{l}{r_{max}} = 1 - e$ and $\frac{l}{r_{min}} = 1 + e$

Now $m \cdot v_{max} \cdot \frac{l}{1 \quad e} = m \cdot v_{min} \frac{l}{1 \quad e}$

$$\frac{v_{max}}{1 \quad e} = \frac{v_{min}}{1-e}$$

$$\frac{v_{max}}{v_{min}} - 1 = \frac{1 \quad e}{1 \quad e} - 1 = \frac{2e}{1 \quad e}$$

$$\frac{v_{max}}{v_{min}} + 1 = \frac{1 \quad e}{1 \quad e} + 1 = \frac{2}{1-e}$$

$$\therefore \frac{v_{max} \quad v_{min}}{v_{max} \quad v_{min}} = e$$

3.109. Two equal spheres of radius r and density equal to mean density of earth are placed at a very large distance. Find the common speed with which they will collide each other due to their mutual force of attraction.

Solution:

$$\frac{1}{2} mv^2 + \frac{1}{2} mv^2 = \frac{G \cdot m^2}{2r}$$

$$v^2 = \frac{G \cdot m}{2r} = \frac{G \cdot \frac{4}{3}\pi r^3 \rho}{2r}$$

$$= \frac{4}{3}\pi G \cdot \rho \cdot \frac{r^2}{2}$$

$$GM_e = g R_e^2$$

$$G \cdot \frac{4}{3}\pi R_e^3 \cdot \rho = g R_e^2$$

$$\therefore \frac{4}{3}\pi G \cdot \rho = \frac{g}{R_e}$$

$$\therefore v = r \cdot \sqrt{\frac{g}{2R_e}}$$

3.110. A particle is in a circular orbit of height h above the surface of earth. Find the value of h if the work done to take the particle to the orbit is equal to its kinetic energy in the orbit.

Solution:

$$\text{Work done} = \int_{R_e}^{R_e+h} \frac{GM_e m \cdot dx}{x^2} = \frac{GM_e m \cdot h}{R_e[R_e + h]}$$

$$\text{As given } \frac{GM_e m \cdot h}{R_e[R_e + h]} = \frac{1}{2} mv^2; \quad \frac{mv^2}{R_e \quad h} = \frac{GM_e m}{R_e \quad h^2}$$

$$\frac{G \cdot M_e m h}{R_e[R_e + h]} = \frac{1}{2} \cdot \frac{GM_e m}{R_e \quad h}$$

$$h = \frac{R_e}{2}$$

3.111. A star with a mass of more than three times that of the sun contracts so much upon cooling that it is not able to radiate; neither material particles or light are able to overcome its gravitational field. Such objects are known as black hole. Find its radius.

Solution: The star will retain light and material particles if its escape velocity is equal to velocity of light.

$$\frac{G \cdot M_{star} \cdot m}{R_{star}} = \frac{1}{2} mc^2$$

$$\therefore \quad R_{star} = \frac{2G \cdot M_{star}}{C^2}$$

$$= \frac{2G \cdot 3M_S}{C^2}$$

$$= \frac{6G \cdot M_S}{C^2}$$

On putting the values we have

$$R_{star} = \frac{6 \quad 6.67 \quad 10^{11} \quad 1.98 \quad 10^{30}}{9 \quad 10^{16}}$$
$$= 8.8 \text{ km}$$

3.112. A cosmic body moves towards sun with velocity v_0 from a very large distance. The arm of vector v_0 is at a distance l from the centre of the sun (see Fig. 3.73). Find the minimum distance $r_{\min}$ by which the body will get to the sun.

Solution: According to conservation of angular momentum we have

$$mv_0 l = mr_{\min} \cdot v$$

and according to conservation of energy we have

$$\frac{1}{2} m v_0^2 = -\frac{GM_s \cdot m}{r_{\min}} + \frac{1}{2} mv^2$$

$$v_0^2 = -\frac{2GM_s}{r_{\min}} + \frac{v_0^2 l^2}{r_{\min}^2}$$

$$r_{\min}^2 + \frac{2GM_s}{v_0^2} r_{\min} - l^2 = 0$$

$$r_{\min} = -\frac{2GM_s}{v_0^2} \pm \sqrt{\frac{4G^2M_s^2}{v_0^4} \quad 4l^2} \Big/ 2$$

$$= \left[\left(\frac{GM_S}{v_0^2}\right)^2 + l^2\right]^{\frac{1}{2}} - \frac{GM_S}{v_0^2}$$

$$r_{\min} = \frac{GM_s}{v_0^2} \quad \sqrt{1 \quad \frac{l v_0^2}{GM_S}^2} \quad 1$$

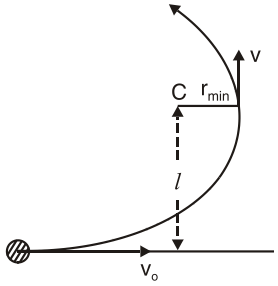


Fig. 3.73

3.113. Calculate the amount of work performed in transferring a particle of mass m from the centre of hemisphere of mass M and radius R to infinity.

Solution: Work done = Mass of the particle × potential at the centre of hemisphere.

Consider an element at P having coordinates r, θ and ϕ (see Fig. 3.74).

Volume of the element

$= r^2 \sin\theta \cdot dr \cdot d\theta \cdot d\phi$

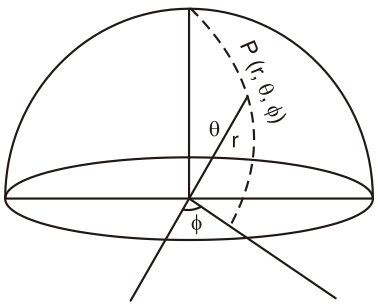


Fig. 3.74

Mass of the element $= r^2 \sin\theta \cdot dr \cdot d\theta \cdot d\phi \cdot \dfrac{3M}{2\pi R^3}$

Potential at the centre of the hemisphere due to the element

$$= -G \cdot \frac{3M}{2\pi R^3} \cdot \sin\theta \; \frac{d\theta \cdot r^2 dr \cdot d\phi}{r}$$

Net potential at the centre $= -\dfrac{G \cdot 3M}{2\pi R^3} \int_0^{\pi/2} \sin\theta d\theta \int_0^R r dr \cdot \int_0^{2\pi} d\theta$

$$= -\frac{G \cdot 3M}{2\pi R^3} \times 1 \times \frac{R^2}{2} \times 2\pi$$

$$= -\frac{3GM}{2R}$$

Work done $= -\dfrac{3GMm}{2R}$

3.114. Earth is moving around sun in a circular orbit. Determine the path of earth if its velocity at any point of its orbit is increased by 41.4%.

Solution: Equation of motion of earth is given by the relation

$$\frac{GM_S M_e}{R^2} = \frac{M_e v_0^2}{R}$$

$$v_0 = \sqrt{\frac{GM_S}{R}}$$

After increasing the velocity by 41.4% its new velocity $v = \sqrt{2}\; v_0$

or $v^2 = 2v_0^2$

Under this condition earth will describe a parabolic path with sun at its focus.

3.115. A plant is moving around the sun in a circular orbit with a period of revolution T. Calculate the time in which it will fall into the sun when it were suddenly stopped into its orbit.

Solution: Let the planet be suddenly stopped at P. Its velocity is zero. It will move towards the sun along the line PS with an acceleration $\dfrac{GM_s}{r^2}$ (see Fig. 3.75).

$$\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} = v \cdot \frac{dv}{dr} = -\frac{GM_s}{r^2} \text{ (Since } r \text{ is decreasing)}$$

$$v dv = -GM_s \cdot \int \frac{dr}{r^2}$$

$$\frac{v^2}{2} = \frac{GM_s}{r} + A$$

at $r = R$, $v = 0$ $\therefore A = -\dfrac{GM_s}{R}$

$$v = \sqrt{2GMs\left(\frac{1}{r} - \frac{1}{R}\right)}$$

Now $\frac{dr}{dt} = -\sqrt{\frac{2GM_s}{R}} \cdot \sqrt{\frac{R\ r}{r}}$ here –ve sign has been introduced since r is decreasing

$$\int_R^0 -dr\ \sqrt{\frac{r}{R\ r}} = \sqrt{\frac{2GM_s}{R}}\ \int_0^t dt$$

Let us substitute $r = R\cos^2\theta$

$$\int_0^{\frac{\pi}{2}} \frac{2R\cdot\cos\theta\cdot\sin\theta d\theta\cdot\sqrt{R}\cdot\cos\theta}{\sqrt{R}\sin\theta} = \sqrt{\frac{2GM_s}{R}}\cdot t$$

$$\int_0^{\frac{\pi}{2}} R(1+\cos 2\theta)\, d\theta = \sqrt{\frac{2GM_s}{R}}\cdot t$$

$$R\cdot\frac{\pi}{2} = \sqrt{\frac{2GM_s}{R}}\cdot t$$

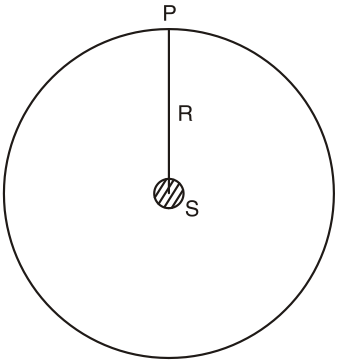


Fig. 3.75

$$t = \frac{\pi R^{\frac{3}{2}}}{2\cdot\sqrt{2GM_s}}$$

$$\frac{GM_s}{R^3} = \omega^2 = \frac{4\pi^2}{T^2}$$

$$\therefore\ T = \frac{2\pi R^{3/2}}{\sqrt{GM_s}}$$

In view of this

$$t = \frac{1}{2\sqrt{2}}\cdot\frac{T}{2}$$

$$= \frac{\sqrt{2}T}{8}$$

3.116. Two particles of masses m_1 & m_2 moving in the parabolic path in the same plane around sun collides at right angles and coalesce (see Fig. 3.76). At this instant their common distance from sun is R. Determine the substiquent path of the combined particle.

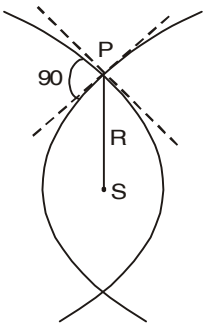


Fig. 3.76

Solution: Let V_1 and V_2 be the velocities of the two particles at P then we have

$$V_1^2 = V_2^2 = \frac{2GM_s}{R}$$

The above identity shows that the two velocities are equal.

Now $(m_1 + m_2)^2 V^2 = m_1^2 V_1^2 + m_2^2 V_2^2$; V = velocity of the combined mass

$$V^2 = \frac{m_1^2 + m_2^2}{(m_1 + m_2)^2} V_1^2$$

From the above relation it is observed that

$$V^2 < V_1^2 < \frac{2GM_S}{R}$$

Therefore the path of the combined mass is an ellipse.

For point P

$V^2 = GM_S \left(\frac{2}{R} - \frac{1}{a}\right)$; a = semi-major axis of the ellipse.

$$\frac{m_1^2 + m_2^2}{(m_1 + m_2)^2} V_1^2 = \frac{m_1^2 + m_2^2}{(m_1 + m_2)^2} \frac{2GM_s}{R} = GM_s \left(\frac{2}{R} - \frac{1}{a}\right)$$

$$\frac{1}{a} = \frac{2}{R}\left(1 - \frac{m_1^2 + m_2^2}{(m_1 + m_2)^2}\right) = \frac{4m_1 m_2}{R(m_1 + m_2)^2}$$

$$\therefore 2a = \frac{R(m_1 + m_2)^2}{2m_1 m_2}$$

This is the length of the semi-major axis of the ellipse.

3.117. Two satellites A and B of same mass m moves in equatorial plane of earth close to its surface. A moves in direction of earth's rotation whereas B against it. Find the ratio of kinetic energy of B to that of A in the reference frame fixed to the earth. $g = 10$ m/sec^2 $R_e = 6400$ km.

Solution:

$$\frac{GM_e m}{R_e^2} = \frac{m(v_1 + \omega R_e)^2}{R_e} \quad \text{For } A$$

$$\left(\sqrt{gR_e} - \omega R_e\right) = v_1$$

$$\frac{GM_e m}{R_e^2} = \frac{m(v_2 - \omega R_e)^2}{R_e} \quad \text{For } B$$

$$\left(\sqrt{gR_e} + \omega R_e\right) = v_2$$

$$\therefore \text{Ratio of kinetic energy } K = \frac{\frac{1}{2}m\left[\sqrt{gR_e} + \omega R_e\right]^2}{\frac{1}{2}m\left[\sqrt{gR_e} - \omega R_e\right]^2}$$

On putting the values we have

$$K = \frac{\sqrt{10 \quad 6400 \quad 10^3} \quad \left(\frac{2 \quad 3.14 \quad 6400 \quad 10^3}{24 \quad 60 \quad 60}\right)^2}{\sqrt{10 \quad 6400 \quad 10^3} \quad \left(\frac{2 \quad 3.14 \quad 6400 \quad 10^3}{24 \quad 60 \quad 60}\right)^2}$$

$$= 8 \quad \left(\frac{4 \quad 3.14}{27}\right)^2 \Big/ 8 \quad \left(\frac{4 \quad 3.14}{27}\right)^2$$

$$= \left(\frac{228.56}{203.4}\right)^2$$

$$= 1.263$$

3.118. A satellite of radius r_1 moves around a planet of radius r_3 in a circular orbit with a period T. The distance between their surfaces is r_2. Find the mass of the planet.

Solution: The equation of motion of the satellite is given by

$$\frac{G \cdot M_p \cdot M_S}{(r_1 + r_2 + r_3)^2} = M_S (r_1 + r_2 + r_3) \cdot \omega^2 ; \quad M_p = \text{mass of planet}$$

M_s = mass of satellite

$$\text{or} \quad \frac{G \cdot M_p}{(r_1 + r_2 + r_3)^3} = \frac{4\pi^2}{T^2}$$

$$\therefore M_p = \frac{4\pi^2 \cdot (r_1 + r_2 + r_3)^2}{G \cdot T^2}$$

3.119. The escape velocity on the surface of earth is 11.2 km/s. A small body is projected out with thrice this speed. What will be the speed of the body far away from the earth ? Radius of earth = 6400 km, $g = 9.8$ m/s^2.

Solution: According to the law of conservation of energy, we have

$$-\frac{GM_e m}{R_e} + \frac{1}{2} m (3v_e)^2 = \frac{1}{2} mv^2$$

M_e = mass of earth, v_e = escape velocity and R_e = radius of the earth

$$\text{or } v^2 = 9 \; v_e^2 - \frac{2GM_e}{R_e}$$

$$= 9 \; v_e^2 - 2g R_e ; \; GM_e = g R_e^2$$

$$v = \sqrt{9v_e^2 \quad 2g R_e}$$

On substituting the values of v_e, g and R_e, we have

$v = \sqrt{9 \quad 11.2 \quad 11.2 \quad 2 \quad 9.8 \quad 10^{3} \quad 6400}$

$= (9 \times 11.2 \times 11.2 - 2 \times 9.8 \times 6.4)^{1/2}$ km/s

$= 31.66$ km/s

3.120. What is the angular deflection of a light beam which passes by the sun at its edge.

Solution: Let the mass of the light beam be m_l. The gravitational force

$$F = \frac{GM_s m_l}{l^2 \quad y^2}$$

Component of the force acting along the x-axis

$$F_x = F \sin \phi = - \frac{GM_s m_l \cdot l}{\left(l^2 + y^2\right)^{3/2}}$$ (see Fig. 3.77); M_S = mass of sun)

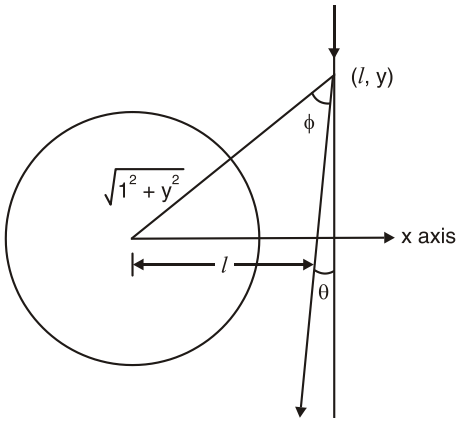


Fig. 3.77

Now $m_l \cdot \frac{d}{dt} (v_x) = F_x$ where v_x is the velocity along x-axis.

$$m_l v_x = \quad F_x dt \quad = \quad F_x \frac{dy}{v_y} \quad = \frac{1}{c} \quad F_x dy \; ; \; c = \text{velocity of light}$$

$$v_x \cong - \frac{2GM_s l}{c} \int_0^{\infty} \frac{dy}{\left(l^2 + y^2\right)^{3/2}} \cong - \frac{2GM_s}{cl}$$

when $l = R_S$ (radius of sun), then $v_x \cong \frac{2GM_s}{cR_s}$

$\therefore$ θ = angular deflection = $\frac{|v_x|}{c} = \frac{2GM_s}{c^2 R_s}$

4

COLLISION AND RIGID BODY ROTATION

4.1. Six particles of mass 1, 2, 3, 4, 5 and 6 kg are placed at the corners of a hexagon of side L. Find out the distance between the centre of the hexagon and the centre of gravity.

Solution: In view of the geometry of the hexagon, the coordinates of points A, B, C, D, E and F are shown in Fig. 4.1.

Now

$$x = -\frac{\frac{L}{2}\times 1+\frac{L}{2}\times 2+L\times 3+\frac{L}{2}\times 4-\frac{L}{2}\times 5-L\times 6}{21} = -\frac{L}{7}$$

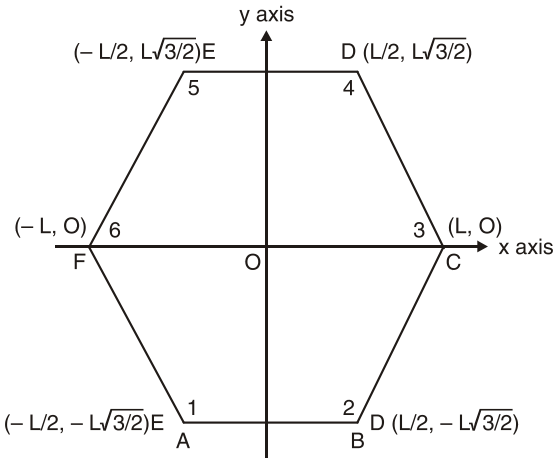


Fig. 4.1

$$y = -\frac{\frac{L\sqrt{3}}{2}\times 1-\frac{L\sqrt{3}}{2}\times 2+3\times 0+\frac{L}{2}\sqrt{3}\times 4+\frac{L}{2}\sqrt{3}\times 5+0\times 6}{21} = \frac{L}{7}\sqrt{3}$$

Distance between the centre of hexagon and centre of gravity

$$= \sqrt{\frac{L^2}{49}\ \frac{3L^2}{49}} = \frac{2L}{7}$$

4.2. A square hole is cut out from a circular uniform disc of radius R (see Fig. 4.2). The diagonal of the hole is equal to R. Show that the centre of mass of the remaining portion of the disc is at a distance $\frac{R}{2\ 2\ \ 1}$ from the centre of the disc.

Solution: Let $OD = L$. Then we have $L^2 + L^2 = R^2$

or $L = \frac{R}{\sqrt{2}}$

Area of the square = $\frac{R^2}{2}$

Area of the disc = πR^2

Area of the remaining portion

$= \frac{R^2}{2}\ (2\pi - 1)$

Taking the moment about the point O we have

$\rho\ \frac{R^2}{2}\ (2\pi - 1) \times OG = \rho\ \frac{R^2}{2} \times \frac{R}{2}$

(here ρ is mass per unit area)

$\therefore\ OG = \frac{R}{2\,(2\ \ \ 1)}$

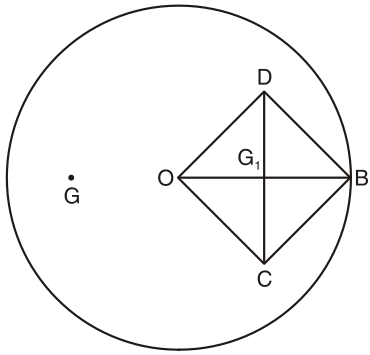


Fig. 4.2

4.3. A circular plate of uniform thickness has a diameter of 56 cm. A circular portion of diameter 42 cm is removed from one edge of the plate as shown in Fig. 4.3. Find the centre of mass of the remaining portion. (IIT 1980)

Solution: $OA = 28$ cm.

$O'A = 21$ cm

$\therefore\ OO' = 7$ cm

Mass of the disc of radius 28 cm

$= \pi 28^2 \cdot \rho$ (here ρ is mass per unit area)

Mass of circular hole of radius 21 cm

$= \pi \cdot 21^2\ \rho$

Mass of the remaining portion

$= \pi\ (28^2 - 21^2)\ \rho = \pi \times 49 \times 7 \times \rho$

Taking moment about O, we have

$\pi \times 49 \times 7\rho \times OG = \pi \times 21^2\ \rho \times OO'$

$\therefore\ OG = \frac{21^2\ \ 7}{49\ \ 7} = 9$ cm

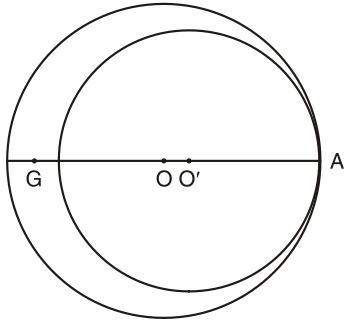


Fig. 4.3

4.4. A hemisphere of radius R is made up of a nonuniform material. If the density ρ at any point at a distance x from the centre is $\rho = \rho_0 x^2$, find the centre of mass of the hemisphere.

Solution: Consider a section of the hemisphere. The section is a circular disc of radius y and thickness dx (see Fig. 4.4).

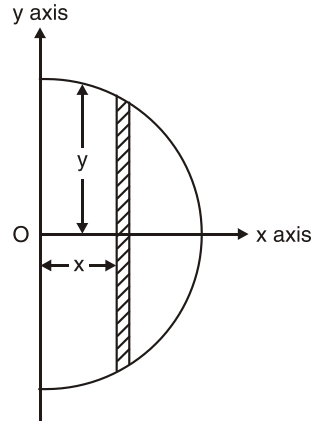


Fig. 4.4

Mass of the section

$\pi (R^2 - x^2)\, dx\, \rho_0 x^2$

If X is the distance of centre of mass from O, then

$$X = \frac{\int_0^R x\,dm}{\int_0^R dm} = \frac{\pi\rho_0\left[R^2\int_0^R x^3dx - \int_0^R x^5dx\right]}{\pi\rho_0\left[R^2\int_0^R x^2dx - \int_0^R x^4dx\right]}$$

$$= \frac{R^6/4 \quad R^6/6}{R^5/3 \quad R^5/5} = \frac{5}{8}R$$

4.5. Two identical spheres of mass m are placed at two of the vertices of an equilateral triangle. A sphere of mass $2m$ is placed at the other vertex. Find the centre of the mass of the system formed.

Solution: Let L be the length of the triangle. The coordinates of the three masses are shown in Fig. 4.5.

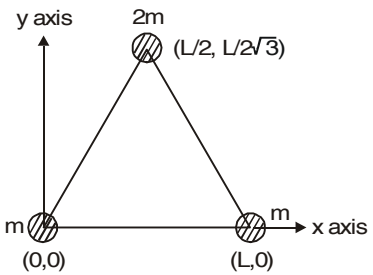


Fig. 4.5

Now

$$X = \frac{m\ 0\ m\ L\ 2m\ \frac{L}{2}}{4m} = \frac{L}{2}$$

$$Y = \frac{m\ 0\ m\ 0\ 2m\ L\frac{\sqrt{3}}{2}}{4m} = \frac{L\sqrt{3}}{4}$$

4.6. A thin bar of length L is placed along x axis as in Fig. 4.6. The mass per unit length of the bar varies with x via relation $\lambda = ax$ where a is a positive constant. Find the x coordinate of centre of mass.

Solution: x component of centre of mass is given by

$$x_c = \frac{\int_0^L x\,dm}{\int dm} = \frac{\int_0^L x\cdot\lambda\,dx}{\int_0^L \lambda\,dx}$$

$$= \frac{a\int_0^L x^2dx}{a\int_0^L x\,dx} = \frac{\frac{L^3}{3}}{\frac{L^2}{2}} = \frac{2}{3}L$$

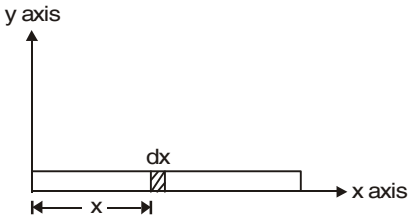


Fig. 4.6

4.7. A wooden board is of mass M is in the form of right angle triangle of base L_1 and height L_2 as shown in the Fig. 4.7. The mass per unit area of the board is constant. Find the y component of centre of mass.

Solution: $$y_c = \frac{y\,dm}{dm} = \frac{\int_0^{L_2} y \cdot \frac{2M}{L_1 L_2} \cdot (L_1 - x)dy}{\int_0^{L_2} \frac{2M}{L_1 L_2} \cdot (L_1 - x)dy}$$

$$= \frac{\int_0^{L_2} y(L_1 - x)dy}{\int_0^{L_2} (L_1 - x)dy}$$

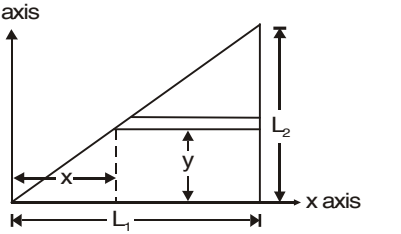


Fig. 4.7

Further $\frac{L_2}{L_1} = \frac{y}{x}$

$\therefore x = \frac{L_1 y}{L_2}$

On substituting the value of x we have

$$y_c = \frac{\int_0^{L_2} y(L_2 - y)dy}{\int_0^{L_2} (L_2 - y)dy}$$

$$= \frac{\frac{L_2^3}{2} \quad \frac{L_2^3}{3}}{L_2^2 \quad \frac{L_2^2}{2}} = \frac{L_2}{3}$$

4.8. Find x and y component of centre of mass of a quadrant of a disc of radius R.

Solution: Consider two concentric circles of radius r and $r + dr$ and two lines at angles θ and $\theta + d\theta$.

Area of the element = $rd\theta \cdot dr$ see Fig. 4.8.

$$x_c = \frac{\int \frac{4M}{\pi R^2} \cdot x \cdot rd\theta \cdot dr}{\int \frac{4M}{\pi R^2} \cdot rd\theta \cdot dr}$$; M = mass of the disc

$$= \frac{\int r\cos\theta \cdot rd\theta \cdot dr}{\int rd\theta \cdot dr}$$

$$x_c = \frac{\int_0^{\frac{\pi}{2}} \cos\theta \cdot d\theta \int_0^R r^2 dr}{\int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^R r\,dr}$$

$$= \frac{\frac{R^3}{3}}{\frac{}{2} \cdot \frac{R^2}{2}} = \frac{4R}{3}$$

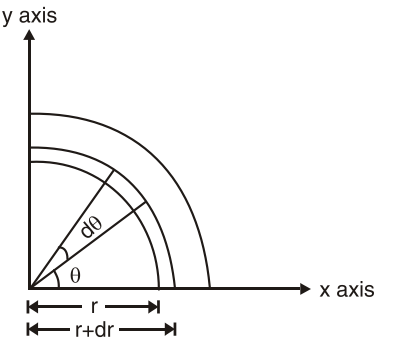


Fig. 4.8

$$y_c = \frac{\int \frac{4M}{\pi R^2} \cdot yr\, d\theta \cdot dr}{\int \frac{4M}{\pi R^2} \cdot r\, d\theta \cdot dr}$$

$$= \frac{\int r \sin\theta \cdot r\, d\theta \cdot dr}{\int r\, d\theta \cdot dr}$$

$$= \frac{\int_0^{\frac{\pi}{2}} \sin\theta d\theta \int_0^R r^2 dr}{\int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^R r\, dr}$$

$$= \frac{4R}{3}$$

4.9. Find the centre of mass of a right circular cone of height h and semi-vertical angle θ.

Solution: Consider a disc perpendicular to the axis of the cone at a depth y from the vertex.

Radius of the disc = $y \tan\theta$

Volume of the disc = $\pi y^2 \tan^2\theta \cdot dy$

$$\text{Mass of the disc} = \frac{M}{\frac{1}{3} h^2 \tan^2 \cdot h} \cdot \pi y^2 \tan^2\theta \cdot dy$$

$$= \frac{3M}{h^3} y^2 \cdot dy$$

$$y_c = \frac{\int_0^h \frac{3M}{h^3} \cdot y^3 dy}{\int_0^h \frac{3M}{h^3} \cdot y^2 dy} = \frac{\frac{h^4}{4}}{\frac{h^3}{3}}$$

$$= \frac{3}{4} h$$

4.10. Two uniform spheres composed of the same material and having radii 3 cm and 6 cm respectively are united as shown in the Fig. 4.9. Find the centre of mass from the centre of the sphere of radius 6 cm.

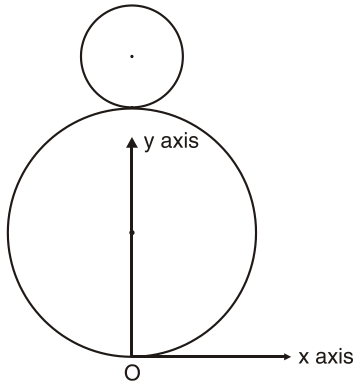


Fig. 4.9

Solution: $y_c = \dfrac{\frac{4}{3}\pi 3^3 \cdot \rho \times 15 + \frac{4}{3}\pi 6^3 \times \rho \times 6}{\frac{4}{3}\pi 3^3 \rho + \frac{4}{3}\pi 6^3 \rho}$

$$= \frac{27 \quad 15 \quad 36 \quad 36}{27 \quad 36 \quad 6} = \frac{45 \quad 144}{3 \quad 24}$$

$= 7$ cm.

Thus the centre of mass is 7 – 6 = 1 cm above the centre of the larger sphere.

4.11. Determine the coordinates of the centroid of the area of one quadrant of an ellipse with semi-major and semi-minor axes a and b respectively.

Solution: Consider an element of thickness dx as shown in Fig. 4.10.

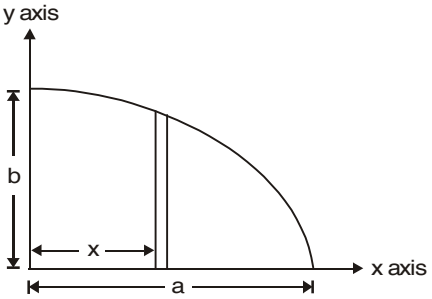


Fig. 4.10

$$x_c = \frac{\int x dA}{\int dA}$$

$$= \frac{\int x \cdot y \cdot dx}{\int y \cdot dx}$$

For an ellipse we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$= \frac{a^2 - x^2}{a^2}$$

$$y = \frac{b}{a}\sqrt{a^2 \quad x^2}$$

On putting the value of y we have

$$x_c = \frac{\frac{b}{a}\int x\sqrt{a^2 - x^2} \cdot dx}{\frac{b}{a}\int \sqrt{a^2 - x^2} \cdot dx}$$

$$= \frac{\int_0^a x\sqrt{a^2 - x^2} \cdot dx}{\int_0^a \sqrt{a^2 - x^2} \cdot dx}$$

$$= -\frac{\left|\frac{\left(a^2 - x^2\right)^{\frac{3}{2}}}{3}\right|_0^a}{\left|\frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right)\right|_0^a} = \frac{\frac{a^3}{3}}{\frac{a^2}{4}} = \frac{4a}{3}$$

On similar lines $y_c = \frac{4b}{3}$

4.12. Find the centroid of the area of the circular section *AOB* of radius *r* subtending an angle ϕ at the centre.

Solution: Consider an element subtending an angle $d\theta$. see Fig. 4.11.

$$x_c = \frac{x\,dA}{dA}$$

$dA = r^2 \frac{d}{2}$ and $x = x$ component of the centroid of the element $= \frac{2}{3} r \cos\theta$

$$\therefore x_c = \frac{2\int_0^{\frac{\phi}{2}} \frac{2}{3} r\cos\theta \cdot \frac{r^2 d\theta}{2}}{2\int_0^{\phi} \frac{r^2 d\theta}{2}}$$

$$= \frac{4r}{3} \sin\frac{}{2}$$

From symmetry $y_c = 0$

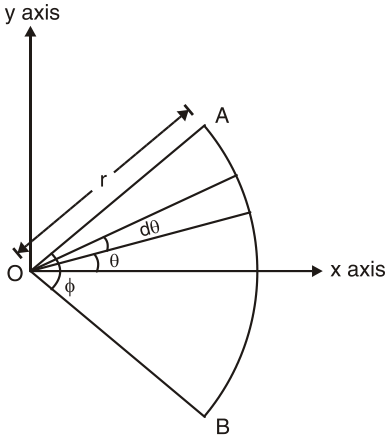


Fig. 4.11

4.13. Find the coordinates of the centroid of the area *OAB* (see Fig. 4.12) in which part *OA* is a portion of parabola represented by $y = \frac{b}{a^2} x^2$.

Solution: Consider an element of width *dx* and height *y* then

$$x_c = \frac{x\,dA}{dA} = \frac{\int x \cdot y \cdot dx}{\int y\,dx}$$

$$= \frac{\int_0^a x \cdot \frac{b}{a^2} \cdot x^2 dx}{\int_0^a \frac{b}{a^2} \cdot x^2 dx} = \frac{3}{4} a$$

$$y_c = \frac{\frac{y}{2} dA}{dA} = \frac{\int \frac{y}{2} \cdot y dx}{\int y dx}$$

$$= \frac{b^2}{2a^4} \cdot \frac{\int_0^a x^2 dx}{\frac{b}{a^2}\int_0^a x^2 dx}$$

$$= \frac{3}{10} b$$

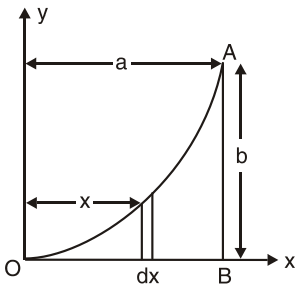


Fig. 4.12

4.14. Determine *x* component of the centroid of the area between the *x* axis and the half sine wave *OAB* as shown in the Fig. 4.13.

Solution: The equation of the sine wave is given by $y = a \sin \frac{x}{b}$

Consider an element of width dx and height y.

$$x_c = \frac{\int x \cdot y dx}{\int y dx}$$

$$= \frac{a\int_0^b x \cdot \sin\left(\frac{\pi x}{b}\right) dx}{a\int_0^b \sin\left(\frac{\pi x}{b}\right) dx}$$

$$= \frac{b}{2}$$

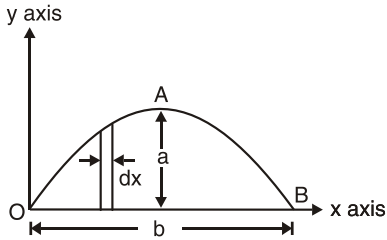


Fig. 4.13

4.15. Determine the centroid of the length of the arc AB (see Fig. 4.14) of radius r forming an angle ϕ at the centre.

Solution: The x component of the centroid is given by

$$x_c = \frac{xdL}{dL}$$

Now $dL^2 = dx^2 + dy^2$

and $x^2 + y^2 = r^2$

$$xdx + ydy = 0$$

$$\therefore \quad dL = \frac{r}{y} \cdot dx$$

$$\therefore \quad x_c = \frac{\int \frac{xr \cdot dx}{y}}{\int dL}$$

$$= \frac{\int \frac{rx \cdot dx}{\sqrt{r^2 - x^2}}}{\int dL}$$

$$x = r\cos\theta$$

$$x_c = \frac{2\int_0^{\frac{\phi}{2}} \frac{r \cdot r\cos\theta \cdot r\sin\theta d\theta}{r\sin\theta}}{2\frac{r\phi}{2}}$$

$$= \frac{2r}{\phi} \sin\frac{\phi}{2}$$

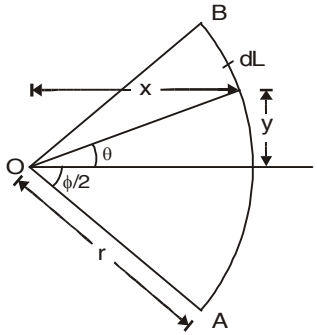


Fig. 4.14

On symmetry considerations $y_c = 0$

4.16. A 4 kg mass moving with an initial speed of 5 m/sec collides and sticks to a 6 kg mass initially at rest. The combined mass then moves to collide with a 2 kg mass and stick to it. Prior to collision the 2 kg mass was at rest. Find the speed of the mass formed and the amount of energy lost in the process.

Solution: According to conservation of momentum we have

$$4 \times 5 = 10 \times v$$

$$\therefore v = 2 \text{ m/sec}$$

$$10 \times 2 = 12 \times v'$$

$$\therefore v' = \frac{10 \quad 2}{12} = \frac{5}{3} \text{ m/sec}$$

Amount of energy lost is given by

$$\frac{1}{2} \times 4 \times 25 - \frac{1}{2} \times 12 \times \frac{25}{9} = 33.33 \text{ Joule}$$

4.17. A 0.2 kg ball is attached to one end of a 2 m long string. The other end of the string is pivoted at *O* (see Fig. 4.15). The ball is released from its rest position keeping the string horizontal. The ball swings down and collides with a 0.4 kg block resting on a frictionless surface elastically. Calculate the speed of the ball and block after collision.

Solution: In view of conservation of energy the velocity of the ball before collision is given by

$$0.2 \times 9.8 \times 2 = \frac{1}{2} \times 0.2 v_1^2$$

$$\therefore v_1 = \sqrt{9.8 \quad 4}$$

According to conservation of momentum we have

$0.2 v_1 + 0.4 \times 0 = 0.2 v'_1 + 0.4 v'_2$; v'_1 = velocity of the ball after collision

$v_1 = v'_1 + 2v'_2$ v'_2 = velocity of the block after collision.

According to conservation of energy

$$\frac{1}{2} \times 0.2 v_1^2 = \frac{1}{2} \times 0.2 \; v_1{}^2 + \frac{1}{2} \; 0.4 \; v_2{}^2$$

$$v_1^2 = v_1'^2 + 2v_2'^2$$

$$v_1^2 = v_1{}^2 + 2 \; \frac{v_1 \quad v_1{}^2}{4}$$

$$4 v_1^2 = 4 v_1{}^2 + 2 v_1{}^2 + 2 v_1^2 - 4 v_1 v'_1$$

$$6 v_1'^2 - 4 v_1 v'_1 - 2 v_1^2$$

$$v'_1 = \frac{4 v_1 \quad \sqrt{16 v_1^2 \quad 48 v_1^2}}{12}$$

$$= -\frac{8 v_1 + 4 v_1}{12} = -\frac{v_1}{3} = -\sqrt{\frac{9.8 \times 4}{3}}$$

$$= -2.086 \text{ m/sec.}$$

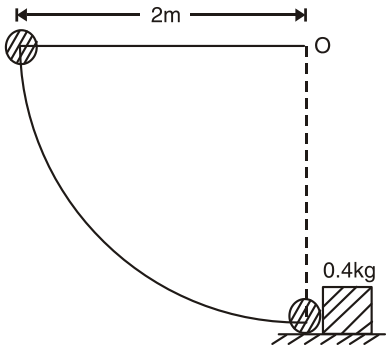


Fig. 4.15

$$2v'_2 = v_1 + \frac{v_1}{3}$$

$$v'_2 = \frac{2v_1}{3} = \frac{2 \quad 2.086}{3} = 1.39 \text{ m/sec}$$

4.18. Three particles *A*, *B* and *C* of equal masses move with equal speed of *v* along the meridians of an equilateral triangle as shown in the Fig. 4.16. They collide at the centroid *G* of the triangle. After collision *A* comes to rest, while *B* retraces its path with the speed *v*. What is the velocity of *C*.

Solution: Before collision the momentum along *x* axis is given by

$mv \cos 30 - mv \cos 30 = 0$

Before collision the momentum along *y* axis is given by

$mv \sin 30 + mv \sin 30 - mv = 0$

After collision. Let *V* be the velocity of *C* in a direction as shown in the figure then along *x* and *y* axes we have

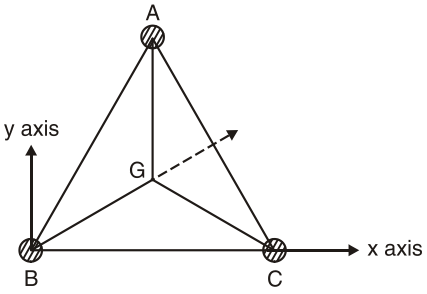


Fig. 4.16

$$mv \cos 30 = mV \cos 30$$

and $mv \sin 30 = mV \sin 30$

$$\therefore \quad v = V$$

i.e. *C* will move with the velocity of *B* in the direction *BG*.

4.19. A moving particle of mass *m* makes a head on elastic collision with a particle of mass $2m$ which is initially at rest. Show that the colliding particle looses $\frac{8}{9}$ th of its energy after collision. (Roorkee 1982)

Solution: Let *u* be the initial velocity of mass *m* and v_1 and v_2 be the velocities of the two masses *m* and $2m$ after collision then according to conservation of momentum we have

$$mu = mv_1 + 2mv_2$$

$$u = v_1 + 2v_2$$

$$\therefore \quad v_2 = \frac{u \quad v_1}{2}$$

According to conservation of energy we have

$$\frac{1}{2} mu^2 = \frac{1}{2} m v_1^2 + \frac{1}{2} \cdot 2m \cdot v_2^2$$

or $$u^2 = v_1^2 + 2 v_2^2$$

or $$u^2 = v_1^2 + 2 \frac{u \quad v_1}{4}^2$$

or $$2 \quad u^2 \quad v_1^2 = u \quad v_1^2$$

or $2(u + v_1) = u - v_1$

$$v_1 = -\frac{u}{3}$$

$\therefore$ Final kinetic energy of particle of mass $m = \frac{1}{2} m \frac{u^2}{9}$

$\therefore$ loss in K.E. of the particle $= \dfrac{\frac{1}{2}mu^2 \quad \frac{1}{2}\frac{mu^2}{9}}{\frac{1}{2}mu^2}$

$$= \frac{8}{9}$$

4.20. A ball of mass m moving with a velocity v collides with another ball of same mass. Prior to collision the second ball is at rest. If the coefficient of restitution be e then find the ratio of velocities of the two balls after collision.

Solution: e is defined as

$e = \dfrac{v_2 \quad v_1}{u_1 \quad u_2}$; where u_1 and u_2 are the velocities of the two balls before collision and v_1 and v_2 are the velocities after collision.

Now $e = \dfrac{v_2 \quad v_1}{v}$

or $ev = v_2 - v_1$

According to conservation of momentum we have

$$mv = mv_1 + mv_2$$

$$\therefore v = v_1 + v_2$$

or $e(v_1 + v_2) = v_2 - v_1$

$$(e + 1) v_1 = (1 - e) v_2$$

$$\therefore \frac{v_2}{v_1} = \frac{1 \quad e}{1 \quad e}$$

4.21. A golf ball dropped from rest on a surface from height h. It rebounds to eight tenths of the height h. Find the coefficient of restitution.

Solution: The velocity with which the ball approaches the surface is given by

$u = \sqrt{2gh}$

The velocity with which the ball recedes from the surface is given by

$v = \sqrt{\dfrac{8}{10} \quad 2gh}$

$e = \dfrac{v}{u} = \sqrt{0.8} = 0.89$

4.22. A ball of mass m moves with a velocity v and hits a stationary ball of mass $3m$. Assuming the coefficient of restitution $e = 0.5$, find the velocities of the two balls after collision.

Solution: $e = \frac{1}{2} = \frac{v_2 \quad v_1}{v}$; v_1 is the velocity of the ball of mass m after impact

or $v = 2v_2 - 2v_1$ v_2 is the velocity of the ball of mass $3m$ after impact.

According to conservation of momentum we have

$$\begin{aligned} mv &= mv_1 + 3mv_2 \\ v &= v_1 + 3v_2 \\ v_1 + 3v_2 &= 2v_2 - 2v_1 \\ 3v_1 &= -v_2 \\ v &= v_1 - 9v_1 \\ \therefore\ v_1 &= -\frac{v}{8} \\ v_2 &= \frac{3v}{8} \end{aligned}$$

4.23. Consider a one dimensional elastic collision between a given incoming body A and body B, initially at rest. How would you choose the mass of B in comparison to the mass of A in order that B should recoil with (a) greatest speed (b) greatest momentum and (c) greatest kinetic energy. (Roorkee 1983)

Solution: Let m_A and m_B be the mass of the two bodies.

According to conservation of momentum we have

$m_A v = m_A v_1 + m_B v_2$; v_1 is the velocity of A after collision v_2 is the velocity of B after collision

$$\begin{aligned} v &= v_1 + \frac{m_B}{m_A} v_2 \\ &= v_1 + K v_2 \,;\, K = \frac{m_B}{m_A} \end{aligned}$$

According to conservation of energy we have

$$\frac{1}{2} m_A v^2 = \frac{1}{2} m_A v_1^2 + \frac{1}{2} m_B v_2^2$$

or $$v^2 = v_1^2 + K v_2^2$$

$$v^2 = (v - K v_2)^2 + K v_2^2$$

$$\begin{aligned} (K + K^2)\, v_2^2 - 2vKv_2 &= 0 \\ (K+1)\, v_2 &= 2v \\ \therefore\ v_2 &= \frac{2v}{K \quad 1} \end{aligned}$$

(a) v_2 will be maximum when K is min.
or $m_A >> m_B$

(b) momentum of $B = p_B = m_B v_2$

$$= \frac{2m_B v}{1+K} = \frac{2Km_A v}{1+K} = \frac{2m_A v}{\frac{1}{K}+1}$$

For momentum to be maximum $\frac{1}{K}$ should be min or K should be maximum, hence

$m_B >> m_A$

(c) Kinetic energy of B is given by

$$E_2 = \frac{1}{2} m_B v_2^2$$

$$= \frac{1}{2} K \cdot m_A \left(\frac{2v}{K \quad 1}\right)^2$$

$$E_2 = \frac{4K E_1}{1 \quad K^2 \quad 4K}; \; E_1 = \text{kinetic energy of } A$$

Now E_2 is maximum when $1 - K$ is minimum and equal to zero
Hence $K = 1$
or $m_A = m_B$

4.24. A steel ball of mass m falls from a height h on a floor. The ball bounces for quite large number of times. Calculate the cumulative momentum imparted to the floor if at each collision the velocity of the ball decreases η times.

Solution: Velocity of the ball when it just strikes the floor = $\sqrt{2gh}$

Momentum before impact = $m\sqrt{2gh}$

Momentum after impact = $\frac{m}{\quad}\sqrt{2gh}$

$\therefore$ Change of momentum of the ball after first impact

$$= m\sqrt{2gh} + \frac{m}{\quad}\sqrt{2gh}$$

Similarly change of momentum = $\frac{m}{\quad}\sqrt{2gh} + \frac{m}{2}\sqrt{2gh}$ of the ball after second impact.

The change of momentum of the ball at each impact is equal to momentum transferred to the floor.

$\therefore$ Net cumulated momentum of the floor is equal to

$$P = m\sqrt{2gh} + \frac{m}{\quad}\sqrt{2gh} + \frac{m}{\quad}\sqrt{2gh} + \frac{m}{2}\sqrt{2gh} + \ldots\ldots$$

$$= m\sqrt{2gh}\left[\left(1+\frac{1}{\eta}+\frac{1}{\eta^2}+\ldots..\right)+\frac{1}{\eta}\left(1+\frac{1}{\eta}+\frac{1}{\eta^2}+\ldots..\right)\right]$$

$$= m\sqrt{2gh}\quad \frac{1}{1-\frac{1}{\eta}}\quad \frac{1}{\eta}\cdot\frac{1}{1-\frac{1}{\eta}}$$

$$= m\sqrt{2gh}\quad \frac{}{1}\quad \frac{1}{1}$$

$$= m\sqrt{2gh}\quad \frac{1}{1}$$

4.25. Two identical particles A and B of mass m, A moving with velocity $v = 10$ m/s suffers a head-on collision with B. Prior to collision B was at rest. The collision results in a decrease in kinetic energy of the system by $\eta = 2\%$. What will be the velocity of particle A after collision.

Solution: Let V_{Ai} and V_{Af} represent the velocities of the partice A before and after collision and V_{Bf}, the final velocity of particle B after collision.

According to the law of conservation of momentum we have

$m V_{Ai} + 0 = m V_{Af} + m V_{Bf}$

$\therefore V_{Ai} = V_{Af} + V_{Bf}$

Change in kinetic energy as a result of collision

$$= \frac{1}{2}\ mV_{Ai}^2 - \ \frac{1}{2}mV_{Af}^2\quad \frac{1}{2}mV_{Bf}^2$$

Now we have $\dfrac{\frac{1}{2}mV_{Ai}^2\quad \frac{1}{2}mV_{Af}^2\quad \frac{1}{2}mV_{Bf}^2}{\frac{1}{2}mV_{Ai}^2} = \eta$

or $V_{Ai}^2 - V_{Af}^2\quad V_{Bf}^2 = \eta V_{Ai}^2$

or $V_{Ai}^2\ (1-\eta) = V_{Af}^2 + (V_{Ai} - V_{Af})^2$ (after setting $V_{Bf} = V_{Ai} - V_{Af}$)

or $V_{Ai}^2\ (1-\eta) = V_{Af}^2 + V_{Ai}^2 - 2\ V_{Ai}\ V_{Af} + V_{Af}^2$

or $2\ V_{Af}^2 - 2V_{Ai}\ V_{Af} + \eta\ V_{Ai}^2 = 0$

or $2\ V_{Af}^2 - 2v\ V_{Af} + \eta v^2 = 0;\ V_{Ai} = v$

$$\therefore V_{Af} = \frac{2v \pm \sqrt{4v^2 - 8\eta v^2}}{4} = \frac{v \pm v\sqrt{1-2\eta}}{2}$$

Now V_{Af} cannot be greater than v, therefore V_{Af} will have the value $V_{Af} = \dfrac{v}{2}\left[1-\sqrt{1-2\eta}\right]$ and will be in the same direction as v.

$$V_{Af} = \frac{v}{2}\left[1-(1-2\eta)^{1/2}\right] \cong \frac{v}{2}(1-1+\eta) = \frac{\eta v}{2}$$

On substituting the values of η and v, we have

$$V_{Af} = \frac{2}{100} \times \frac{1000}{2} = 10 \text{ cm/s}$$

4.26. Particle 1 of mass m moving with velocity v collides with particle 2 of mass M. Prior to collision particle 2 is at rest. As a result of collision particle 1 deviates through an angle 90° whereas particle 2 recoils at an angle $\theta = 22°$ with respect to initial direction of motion of particle 1. Assuming a mass ratio $M/m = 4$, find out the percentage change in the kinetic energy of the system.

Solution: v is the velocity with which particle 1 of mass m moves prior to colliding with particle 2 of mass M. After collision let v_1 and v_2 be the velocities of the two particles of mass m and M respectively (see Fig. 4.17).

In view of the law of conservation of momentum, we have

$$mv = M v_2 \cos\theta$$

and $$mv_1 = M v_2 \sin\theta$$

$$\therefore \tan\theta = \frac{v_1}{v}$$

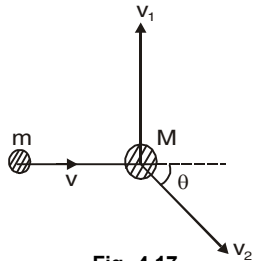


Fig. 4.17

Now the ratio of change of kinetic energy after collision and kinetic energy before collision is given by

$$\frac{T}{T} = \frac{\frac{1}{2}mv_1^2 \quad \frac{1}{2}Mv_2^2 \quad \frac{1}{2}mv^2}{\frac{1}{2}mv^2} \quad (T \text{ stands for kinetic energy})$$

$$= \frac{v_1}{v}^2 + \frac{M}{m}\frac{v_2}{v}^2 - 1$$

$$= \tan^2\theta + \frac{M}{m}\frac{v_2}{v_1}\frac{v_1}{v}^2 - 1 = \tan^2\theta + \frac{m}{M}\sec^2\theta - 1$$

$$= 1 \quad \frac{m}{M} \quad \tan^2\theta + \frac{m}{M} - 1$$

On substituting the values of the various terms we have

$$\frac{T}{T} = (1 + 0.25) \times 0.16 + 0.25 - 1;\ \tan 22° = 0.4$$

$$= 0.2 + 0.25 - 1 = -0.55 = -55\%$$

4.27. A disc of radius r_1 and mass m_1 moving at a speed v undergoes an elastic collision with a stationary disc of radius r_2 and mass m_2 (see Fig. 4.18a). After collision the disc of radii r_1 and r_2 moves at angles 60° and 30° respectively with respect to the initial direction of motion. Find

out the velocities of each of the discs after collision and the impact parameter d.

Solution: Let v_1 and v_2 be the velocities of the two discs after impacts (see Fig. 4.18b).

On resolving the momentum along x and y axes we have

$m_1 v_1 \sin 60 + m_2 v_2 \cos 30 = m_1 v$ and $m_1 v_1 \sin 60 = m_2 v_2 \sin 30$

or $v_2 = \frac{m_1}{m_2} v_1 \frac{\sin 60}{\sin 30} = \frac{m_1}{m_2} \sqrt{3}\, v_1$

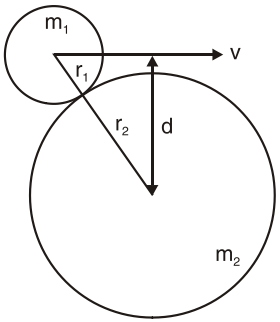


Fig. 4.18a

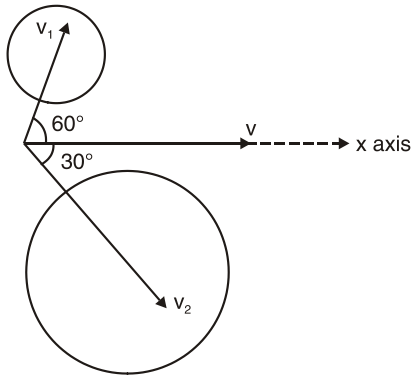


Fig. 4.18b

On substituting the value of v_2 in the above equation, we have

$\frac{1}{2} m_1 v_1 + m_2 \frac{\sqrt{3}}{2} \times \frac{m_1}{m_2} \sqrt{3}\, v_1 = m_1 v$ or $\frac{m_1 v_1}{2} + \frac{3}{2} m_1 v_1 = m_1 v$

or $\therefore v_1 = \frac{v}{2}$, $v_2 = \frac{m_1}{m_2} \frac{\sqrt{3}}{2} v$

Further $\sin 30 = \frac{d}{r_1 \quad r_2}$

Impact parameter $d = \frac{r_1 \quad r_2}{2}$

4.28. A wooden cylinder of mass M is suspended horizontally with the help of two identical threads of length l (see Fig. 4.19). A bullet of mass m ($m < M$) moving parallel to the axis of the cylinder gets stuck to it. The two strings swing through an angle θ. Calculate (i) the initial velocity of the bullet, and (ii) the fraction of kinetic energy of the bullet converted into heat.

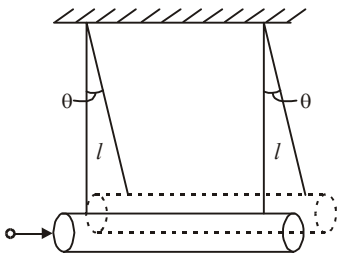


Fig. 4.19

Solution: (i) Let v' be the velocity of cylinder of mass M after collision and v the initial velocity of the bullet. In view of the law of conservation of momentum, we have

$$mv = Mv'$$

or $$v' = \frac{m}{M} v$$

Loss in kinetic energy $= \frac{1}{2} mv^2 - \frac{1}{2} M \cdot \frac{m^2}{M^2} v^2$

$$= \frac{1}{2} mv^2 \left(1 - \frac{m}{M}\right)$$

Gain in potential energy $= \frac{1}{2} mv^2 - \frac{1}{2} mv^2 \left(1 - \frac{m}{M}\right) = Mgl(1 - \cos\theta)$

or $$\frac{1}{2} \frac{m^2 v^2}{M} = Mgl(1 - \cos\theta) = 2Mgl \sin^2 \frac{\theta}{2}$$

$$\therefore v = \frac{2M}{m} \sqrt{gl} \sin \frac{\theta}{2}$$

(ii) Fraction of energy which will turn into heat

$$= \frac{\text{loss in kinetic energy}}{\text{initial kinetic energy of bullet}}$$

$$= \frac{\frac{1}{2} mv^2 \left(1 - \frac{m}{M}\right)}{\frac{1}{2} mv^2}$$

$$= 1 - \frac{m}{M}$$

4.29. A small sphere of mass m resting at a height h slides down a frictionless hill and reaches a long flat pieces of wood of mass M at the base of the hill. Due to large fricton between the sphere and the wood, the sphere slows down and the two masses move as a single unit. Calculate the amount of work done by the force of friction in this process.

Solution: The velocity of the sphere at the base of the hill is given by

$$v = \sqrt{2gh}$$

Let v' be the velocity with which the two masses are moving. Then according to the law of conservation of momentum, we have

$$mv = (m + M)v'$$

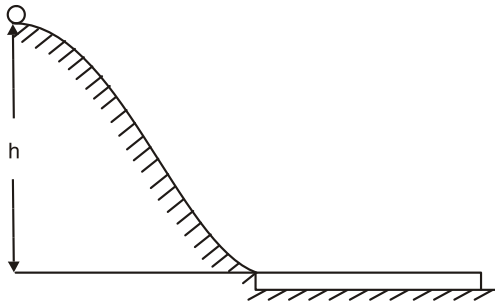


Fig. 4.20

Work done by the frictional forces $= \frac{1}{2}(M+m)\,v'^2 - \frac{1}{2}mv^2$

$$= \frac{1}{2}(M+m) \times \left(\frac{mv}{M\ \ m}\right)^2 - \frac{1}{2}mv^2$$

$$= \frac{1}{2}mv^2\left(\frac{m}{M\ \ m}\ \ 1\right) = -\frac{1}{2}\frac{mMv^2}{M\ \ m}$$

$$= \frac{-mMgh}{M+m};\ v = \sqrt{2gh}$$

4.30. Three identical particles *P*, *Q* and *R* rest on a smooth horizontal table. Particle *P* starts moving with a velocity v and undergoes an elastic collision with particle *Q* and *R* simultaneously, as shown in Fig. 4.21. The distance between the centres of *Q* and *R* before collision is x times as much as the diameter of each particle. Obtain the velocity of particle *P* after the collision. For what value of x will the particle *P* (i) recoil (ii) stop and (iii) move on, after collision ?

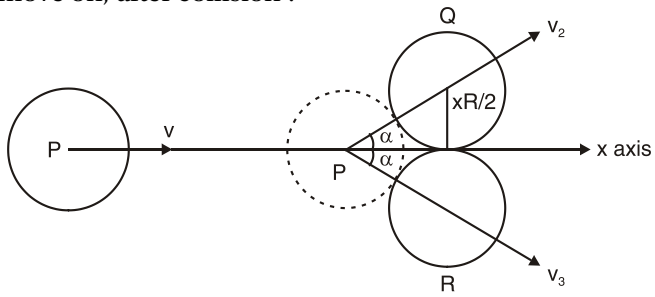


Fig. 4.21

Solution: Let us assume that after collision *P* moves in the same direction with velocity v_1, and that *Q* and *R* move at an angle α with respect to the initial direction with velocities v_2 and v_3.

On resolving the momentum along x and y axes, we have

$mv = mv_1 + mv_2 \cos\alpha + mv_3 \cos\alpha$; m is the mass of each particle

and $v_2 \sin\alpha = v_3 \sin\alpha$.

In the light of this the above equation is now transformed into the following form $v - v_1 = 2\,v_2 \cos\alpha$.

Further,

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}mv_3^2$$

or $\quad v^2 = v_1^2 + 2v_2^2$

or $\quad v^2 - v_1^2 = 2\left|\frac{v\ \ v_1}{2\cos}\right|^2$

or $v + v_1 = \dfrac{(v - v_1)}{2\cos^2 \alpha}$

or $2v \cos^2 \alpha + 2v_1 \cos^2 \alpha = v - v_1$

or $v_1 = \dfrac{v \quad 2\cos^2 \quad 1}{1 \quad 2\cos^2}$

Using R as the diameter of each particle and on substituting $\cos \alpha = \dfrac{\sqrt{4R^2 \quad x^2 R^2}}{2R}$ we have

$$v_1 = - \frac{v \quad \dfrac{4R^2 \quad x^2 R^2}{2R^2} \quad 1}{1 \quad \dfrac{4R^2 \quad x^2 R^2}{2R^2}} = - v \frac{2 \quad x^2}{6 \quad x^2}$$

P will recoil with smaller value of x (v_1 will be negative). When $x = \sqrt{2}$, it will stop ($v_1 = 0$).

When $x > \sqrt{2}$, it will move on (v_1 will be positive).

4.31. The centres of the three spheres A, B and C lie on a horizontal line. The spheres A and C of mass m_1 and m_3 respectively are initially at rest. Sphere A strikes sphere B elastically with an initial velocity v_1 (see Fig. 4.22). As a result of this collision sphere B acquires a velocity v_2 and hits the sphere C elastically. For what value of mass of sphere B with sphere C gain maximum velocity?

Solution: Let m_2 be the mass of sphere B. In view of the law of conservation of momentum, we have

$m_1 v_1 = m_1 v_f + m_2 v_2$ (v_f is the final velocity acquired by sphere A)

or $v_1 = v_f + \dfrac{m_2}{m_1} v_2$ (v_2 is the final velocity acquired by sphere B)

From the law of conservation of energy,

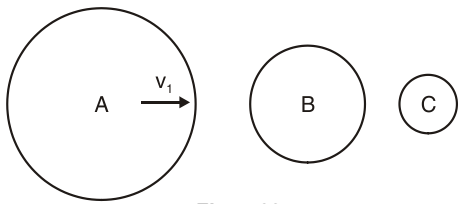


Fig. 4.22

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_f^2 + \frac{1}{2} m_2 v_2^2$$

or $v_1^2 = v_f^2 + \dfrac{m_2}{m_1} v_2^2$

^{2}

or $v_1^2 - \dfrac{m_2}{m_1} v_2^2 = v_1 \quad \dfrac{m_2}{m_1} v_2$

or $v_1^2 - \dfrac{m_2}{m_1} v_2^2 = v_1^2 - 2v_1 v_2 \dfrac{m_2}{m_1} + \dfrac{m_2^2}{m_1^2} v_2^2$

or $- v_2^2 = -2v_1 v_2 + \dfrac{m_2}{m_1} v_2^2$

or $v_2 = \dfrac{2m_1}{m_1 \quad m_2} v_1$

Similarly the velocity v_3 acquired by the sphere

$$C = \frac{4m_1 m_2 v_1}{m_1 \quad m_2 \quad m_2 \quad m_3}$$

Now v_3 is function of mass m_2. For v_3 to be maximum find out the value of $\dfrac{d}{dm_2}(v_3)$ and equate it to zero.

$$\therefore \frac{d}{dm_2}(v_3) = 4m_1 v_1 \frac{m_2}{m_1 \quad m_2 \quad m_2 \quad m_3^{\,2}} \frac{1}{m_1 \quad m_2 \quad m_2 \quad m_3} \frac{m_2}{m_2 \quad m_3 \quad m_1 \quad m_2^{\,2}}$$

$$= \frac{4m_1 v_1 \quad m_1 m_3 \quad m_2^2}{m_2 \quad m_2^{\,2} \quad m_2 \quad m_3^{\,2}}$$

This will be zero when $m_2 = \sqrt{m_1 m_3}$

Hence $m_2 = \sqrt{m_1 m_3}$

4.32. A molecule collides with another stationary molecule of the same mass m. Show that the angle of divergence is 90° when the collision is ideally elastic.

Solution: Let u_1 be the initial velocity of the incident molecule. After collision the two molecules are moving at angles θ and ϕ with velocities v_1 and v_2 (see Fig. 4.23). On resolving the momentum along the x and y axes, we have

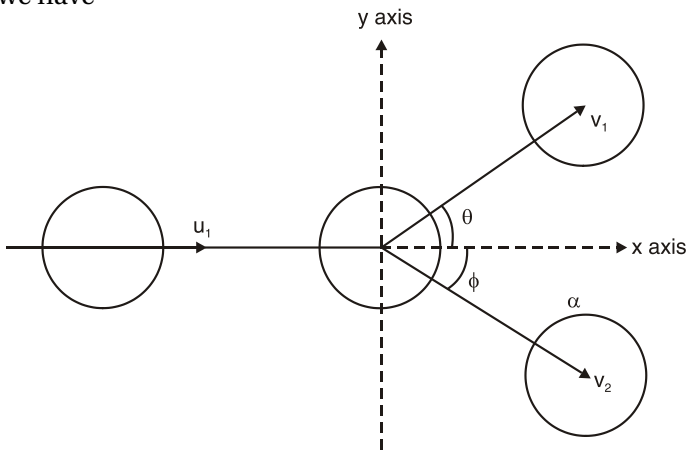


Fig. 4.23

$$mu_1 = mv_1 \cos\theta + mv_2 \cos\phi \quad \text{(i)}$$

and $$mv_1 \sin\theta = mv_2 \sin\phi \quad \text{(ii)}$$

Further $$\frac{1}{2} m u_1^2 = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 \quad \text{(iii)}$$

or $$u_1^2 = v_1^2 + v_2^2$$

$$= v_1^2 + \left(\frac{v_1 \sin}{\sin}\right)^2$$

or $$u_1^2 = v_1^2 \frac{\sin^2 \quad \sin^2}{\sin^2} \quad \text{(iv)}$$

Now $$u_1 = v_1 \cos\theta + v_1 \frac{\sin}{\sin} \cos\phi$$

or $$u_1 \sin\phi = v_1 \sin(\theta + \phi) \quad \text{(v)}$$

On substituting the value of $u_1 \sin\phi$ as $v_1 \sin(\theta + \phi)$, in equation (iv), we have

$\sin^2(\theta + \phi) = \sin^2\theta + \sin^2\phi$

or $\sin^2\theta \cos^2\phi + \cos^2\theta \sin^2\phi + 2 \sin\theta \cos\theta \sin\phi \cos\phi = \sin^2\theta + \sin^2\phi$

or $2 \sin\theta \cos\theta \sin\phi \cos\phi = \sin^2\theta (1 - \cos^2\phi) + \sin^2\phi (1 - \cos^2\theta)$

$$= 2 \sin^2\theta \sin^2\phi$$

or $$\frac{\cos \quad \cos}{\sin \quad \sin} = 1$$

or $$\cot\theta = \tan\phi = \cot(90 - \phi)$$

or $$\theta = 90 - \phi$$

or $$\theta + \phi = 90°$$

4.33. A small particle A of mass 20 g moving with a velocity $\mathbf{u}_1 = (\mathbf{i} + \mathbf{j})$ collides with another particle B of mass 10 g and velocity $\mathbf{u}_2 = \left(\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 6\hat{\mathbf{k}}\right)$ inelastically, and the two coalesce. Find the velocity of the formed particle AB.

Solution: When two particles of mass m_1 and m_2 moving with velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ collide inelastically and form a new particle moving with velocity $\mathbf{u}$ we have $m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = (m_1 + m_2)\,\mathbf{u}$

On substituting the value of m_1, m_2, $\mathbf{u}_1$ and $\mathbf{u}_2$, we have

$$2\left(\hat{\mathbf{i}} + \hat{\mathbf{j}}\right) + 1\left(\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 6\hat{\mathbf{k}}\right) = 3\mathbf{u}$$

$$3\hat{\mathbf{u}} = 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$$

or $$\mathbf{u} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$$u = \sqrt{1 \quad 4 \quad 4} = 3 \text{ m/s}$$

4.34. A particle of mass m_1 moving with velocity v collides elastically with a stationary particles of mass m_2 ($m_1 > m_2$). Through what maximum angle the particle of mass m_1 will deviate from its initial path ?

Solution: Let us assume that the two particles are moving at angles θ and ϕ with velocities v_1 and v_2 respectively (Fig. 4.24).

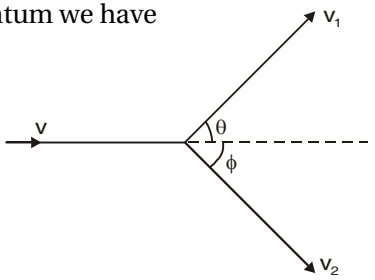


Fig. 4.24

According to conservation of momentum we have

$$m_1 v = m_1 v_1 \cos\theta + m_2 v_2 \cos\phi$$

and $m_1 v_1 \sin\theta = m_2 v_2 \sin\phi$

or $m_1 [v - v_1 \cos\theta] = m_2 v_2 \cos\phi$

$$= \cos\phi \cdot m_1 v_1 \frac{\sin}{\sin}$$

or $(v - v_1 \cos\theta) = v_1 \dfrac{\cos\phi \cdot \sin\theta}{\sin\phi}$

or $v \sin\phi - v_1 \cos\theta \cdot \sin\phi = v_1 \cos\phi \sin\theta$

or $v_1 = \dfrac{v \sin\phi}{\sin(\theta + \phi)}$

According to conservation of energy we have

$$\frac{1}{2} m_1 v^2 - \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_2 v_2^2$$

or $m_1 \quad v^2 \quad \dfrac{v^2 \sin^2}{\sin^2} = m_2 \cdot \dfrac{m_1^2 \sin^2}{m_2^2 \sin^2} \cdot \dfrac{v^2 \sin^2}{\sin^2}$

or $\sin^2(\theta + \phi) - \sin^2\phi = \dfrac{m_1}{m_2} \cdot \sin^2\theta$

or $(\sin\theta \cdot \cos\phi + \cos\theta \sin\phi)^2 - \sin^2\phi = \dfrac{m_1}{m_2} \cdot \sin^2\theta$

or $\sin^2\theta \cdot \cos^2\phi + \cos^2\theta \sin^2\phi$

$$+ 2 \sin\theta \cos\theta \cdot \sin\phi \cos\phi - \sin^2\phi - \frac{m_1}{m_2} \cdot \sin^2\theta = 0$$

or $\sin^2\theta \cdot \cos^2\phi + (1 - \sin^2\theta) \sin^2\phi$

$$+ 2 \sin\theta \cos\theta \cdot \sin\phi \cos\phi - \sin^2\phi - \frac{m_1}{m_2} \sin^2\theta = 0$$

or $\sin^2\theta \cdot \cos 2\phi - \dfrac{m_1}{m_2} \sin^2\theta + \sin\theta \sqrt{1 \quad \sin^2} \cdot \sin 2\phi = 0$

On differentiating the above equation w.r.t. ϕ and equating it to zero we have

$$-\sin^2\theta \cdot \sin 2\phi \cdot 2 + \sin\theta \sqrt{1 \quad \sin^2} \cdot \cos 2\phi \cdot 2 = 0$$

$$\tan 2\phi = \frac{\sqrt{1 \quad \sin^2}}{\sin}$$

On substituting the values of $\sin 2\phi$ and $\cos 2\phi$ we have

$$\sin^2\theta \cdot \sin\theta - \frac{m_1}{m_2}\sin^2\theta + \sin\theta \cdot \sqrt{1 \quad \sin^2} \cdot \sqrt{1 \quad \sin^2} = 0$$

$$\sin^3\theta - \frac{m_1}{m_2} \cdot \sin^2\theta + \sin\theta - \sin^3\theta = 0$$

or $\sin\theta = \dfrac{m_2}{m_1}$

4.35. In a closed system two particles of masses m_1 and m_2 moving with velocities v_1 and v_2 collide inelastically. Find the increment in the kinetic energy of the system.

Solution: According to conservation of momentum we have

$m_1 v_1 + m_2 v_2 = (m_1 + m_2)V$; V is the final velocity of the mass $(m_1 + m_2)$

$$\therefore V = \frac{m_1 v_1 \quad m_2 v_2}{m_1 \quad m_2}$$

Increase in kinetic energy

$$= \Delta E = \frac{1}{2}(m_1 + m_2) \cdot V^2 - \frac{1}{2} m_1 v_1^2 - \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2}(m_1 + m_2) \frac{m_1 v_1 \quad m_2 v_2}{m_1 \quad m_2}^2 - \frac{1}{2} m_1 v_1^2 - \frac{1}{2} m_2 v_2^2$$

$$= -\frac{m_1 m_2}{2 \; m_2 \quad m_2} \cdot \quad v_1^2 \quad v_2^2 \quad 2v_1 v_2$$

$$= -\frac{1}{2} \mu (v_1 - v_2)^2 \; ; \; \mu = \frac{m_1 m_2}{m_1 \quad m_2}$$

4.36. A particle of mass m_A had a perfectly elastic collision with a particle of mass m_B. Prior to collision the particle of mass m_B is at rest. Calculate the fraction of energy lost by the particle of mass m_A when (i) it recoils at right angles to direction in which it was incident (ii) the collision is head on.

Solution: Let v be the velocity of the particle of mass m_A before collision and v_1 and v_2 be the velocities of the two respective masses m_A and m_B after collision.

In view of conservation of energy and momentum we have

$$\frac{1}{2} m_A v^2 = \frac{1}{2} m_A \cdot v_1^2 + \frac{1}{2} m_B \cdot v_2^2$$

and $m_A v = m_B v_2 \cos\phi$ (see Fig. 4.25a)

$$m_A v_1 = m_B v_2 \sin\phi$$

$$m_A v = m_B \cdot v_2 \cdot \sqrt{1 - \left(\frac{m_A v_1}{m_B v_2}\right)^2}$$

or $m_A^2 v^2 = m_B^2 v_2^2 - m_A^2 v_1^2$

$$\frac{m_A^2}{m_B^2} \quad v^2 \quad v_1^2 \quad = v_2^2$$

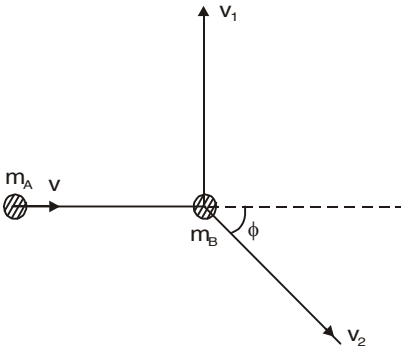


Fig. 4.25a

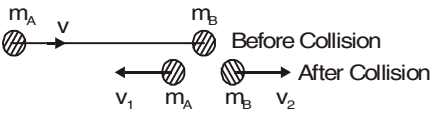


Fig. 4.25b

Now,

$$m_A v^2 = m_A v_1^2 + m_B \cdot \frac{m_A^2}{m_B^2} \quad v^2 \quad v_1^2$$

$$v^2 = v_1^2 + \frac{m_A}{m_B} \quad v^2 \quad v_1^2$$

$$\frac{m_B \quad m_A}{m_B} \quad v^2 = v_1^2 \quad \frac{m_A \quad m_B}{m_B}$$

or $$\frac{m_B \quad m_A}{m_B \quad m_A} = \frac{v_1^2}{v^2}$$

$$\text{Fractional change in energy} = \frac{\frac{1}{2} m_A v^2 \quad \frac{1}{2} m_A v_1^2}{\frac{1}{2} m_A v^2}$$

$$= 1 - \quad \frac{v_1}{v}^2$$

$$= 1 - \quad \frac{m_B \quad m_A}{m_B \quad m_A}$$

$$= \frac{2 m_A}{m_B \quad m_A}$$

(ii) In view of figure 4.25b and according to conservation of momentum, we have

$$m_A v = m_B v_2 - m_A v_1$$

or $\frac{m_A}{m_B}(v + v_1) = v_2$

According to conservation of energy we have

$$\frac{1}{2} m_A v^2 = \frac{1}{2} m_A v_1^2 + \frac{1}{2} m_B v_2^2$$

After substituting the value of v_2, we have

$$m_A v^2 = m_A v_1^2 + m_B \cdot \frac{m_A^2}{m_B^2} (v_1 + v)^2$$

or $v^2 = v_1^2 + \frac{m_A}{m_B}(v_1 + v)^2$

or $(v - v_1) = \frac{m_A}{m_B}(v_1 + v)$

or $v \frac{m_B - m_A}{m_B} = v_1 \left(\frac{m_A + m_B}{m_B} \right)$

or $\frac{v_1}{v} = \frac{m_B - m_A}{m_A - m_B}$

Fractional change in energy $= 1 - \left(\frac{v_1}{v}\right)^2$

$$= 1 - \left(\frac{m_B - m_A}{m_A - m_B}\right)^2$$

$$= \frac{4 m_A m_B}{(m_A - m_B)^2}$$

4.37. A particle A of mass m_A experiences a perfectly elastic collision with another stationary particle B of mass m_B. Find the ratio $\frac{m_A}{m_B}$ when (i) the two particles moves in the opposite direction with equal velocity after a head on collision (ii) the two particles move symmetrically with respect to the initial direction of motion of A with a divergence angle θ.

Solution:

(i) According to conservation of momentum we have

$$m_A v = (m_B - m_A) v_1$$

According to conservation of energy we have

$$\frac{1}{2} m_A v^2 = \frac{1}{2} m_A v_1^2 + \frac{1}{2} m_B v_1^2$$

or $m_A v^2 = (m_A + m_B) v_1^2$

$$= \frac{m_A m_B m_A^2 v^2}{(m_B - m_A)^2}$$

or $(m_B - m_A)^2 = m_A (m_A + m_B)$

or $m_B^2 - 2m_A m_B + m_A^2 = m_A^2 + m_A m_B$

or $\frac{m_A}{m_B} = \frac{1}{3}$

(ii) In view of conservation of momentum in the initial direction of motion we have

$m_A v = (m_A v_1 + m_B v_2) \cos \frac{\theta}{2}$ (see Fig. 4.26)

According to conservation of momentum in a direction perpendicular to the initial direction of motion we have

$m_A v_1 = m_B v_2$

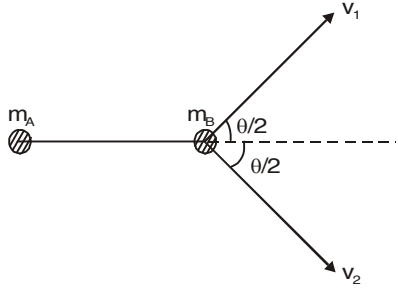


Fig. 4.26

hence $m_A v = 2m_A v_1 \cos \frac{\theta}{2}$

or $v_1 = \frac{v}{2 \cos \frac{\theta}{2}}$

Now $v_2 = \frac{m_A \cdot v}{2 m_B \cos \frac{\theta}{2}}$

According to conservation of energy we have

$$\frac{1}{2} m_A v^2 = \frac{1}{2} m_A v_1^2 + \frac{1}{2} m_B v_2^2$$

or $m_A v^2 = \frac{m_A \cdot v^2}{4 \cos^2 \frac{\theta}{2}} + m_B \cdot \frac{m_A^2 v^2}{4 m_B^2 \cos^2 \frac{\theta}{2}}$

or $1 = \frac{1}{4 \cos^2 \frac{\theta}{2}} \quad 1 \quad \frac{m_A}{m_B}$

or $4 \cos^2 \frac{\theta}{2} - 1 = \frac{m_A}{m_B}$

or $2 \cos \theta + 1 = \frac{m_A}{m_B}$

4.38. Two bodies A and B of masses m and $2m$ respectively are placed on a smooth floor. They are connected by a spring. A third body C of mass m moves with velocity v_0 along the line joining A and B and collides elastically with A. At a certain instant of time t_0 after collision it is found that the instantaneous velocities of A and B are same. Further at this instant the

compression of the spring is x_0. Determine (a) the common velocity of A and B at time t_o (b) the spring constant. (IIT 1984)

Solution: Consider the elastic collision between C and A as shown in Fig. 4.27.

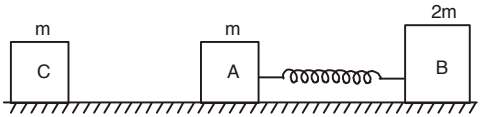


Fig. 4.27

According to conservation of momentum we have

$mv_0 = mv_A + mv_C$; v_A and v_C are the velocities of A and C after collision

or $v_0 = v_A + v_C$

According to conser-vation of energy we have

$$\frac{1}{2} m v_0^2 = \frac{1}{2} m v_A^2 + \frac{1}{2} m v_C^2$$

or $v_0^2 = v_A^2 + v_C^2$

or $(v_A + v_C)^2 = v_A^2 + v_C^2$

or $v_A \cdot v_C = 0$

When $v_A = 0$ that is the case in which there is no interaction between the blocks A and C. Thus the only possibility is $v_C = 0$.

$\therefore v_A = v_0$

(a) Applying the principles of conservation of momentum between masses A and B we have

$mv_0 = (m + 2m)\, V$

or $V = \dfrac{v_0}{3}$

(b) According to conservation of energy we have

$$\frac{1}{2} m v_0^2 = \frac{1}{2} (m + 2m)\, V^2 + \frac{1}{2} K x_0^2$$

$$m v_0^2 - 3m \frac{v_0^2}{9} = K x_0^2$$

$$\therefore K = \frac{2 m v_0^2}{3 x_0^2}$$

4.39. A ball of mass m executing translational motion collides elastically with a stationary ball of same mass. At the time of impact the line passing through the centres of the two balls makes an angle θ with the initial direction of motion (see Fig. 4.28). Assuming the balls to be smooth, find the fraction of the kinetic energy of the incident ball that turns into potential at the instant of maximum deformation.

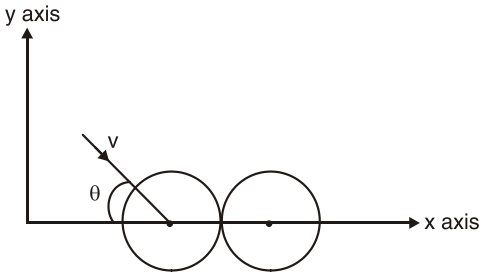


Fig. 4.28

Solution: Applying the principles of conservations of momentum along x axis we have

$mv\cos\theta = mv_1 + mv_2$; v is the velocity of the striking ball before collision and v_1 and v_2 are the velocities of the two balls after collision.

For maximum deformation

$$v_1 = v_2$$

$$\therefore \frac{v\cos}{2} = v_1$$

Initial kinetic energy $= \frac{1}{2} mv^2$

Final kinetic energy $= 2 \times \frac{1}{2} m v_1^2$

$$= \frac{mv^2\cos^2\theta}{4}$$

Fraction of kinetic energy $= \frac{mv^2\cos^2}{4} \Big/ \frac{mv^2}{2} = \frac{1}{2}\cos^2\theta$

4.40. A bullet of mass m moving with a horizontal velocity v strikes a stationary block of mass M suspended by a string of length L. The bullet gets embedded in the block. What is the maximum angle made by the string after impact. (Roorkee 1988)

Solution: According to conservation of momentum we have

$mv = (M + m)\ V$; v = velocity of bullet

V = velocity of block and bullet

$$\therefore v = \frac{M \quad m}{m} \quad V$$

According to conservation of energy we have

$\frac{1}{2}\ (m + M)\ V^2 = (M + m)\ gh$ (see Fig. 4.29)

$\therefore V = \sqrt{2gh}$

$$v = \frac{M \quad m}{m} \sqrt{2gh}$$

or $$h = \frac{v^2}{2g} \quad \frac{m}{M \quad m}^2$$

In view of figure, $\cos\theta = \frac{L \quad h}{L}$

or $$\cos\theta = 1 - \frac{h}{L}$$

$$\cos\theta = 1 - \frac{v^2}{2gL} \quad \frac{m}{M \quad m}^2$$

or $$\theta = \cos^{-1} \quad 1 \quad \frac{v^2}{2gL} \quad \frac{m}{M \quad m}^2$$

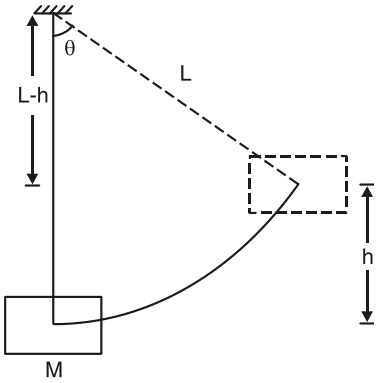


Fig. 4.29

4.41. A simple pendulum is suspended from a peg on a vertical wall. The pendulum is pulled away from the wall to a horizontal position and released. The ball hits the wall, the coefficient of restitution being $\frac{2}{\sqrt{5}}$. What is the minimum number of collisions after which the amplitude of oscillation becomes less than 60°? (IIT 1987)

Solution: When the pendulum is released it will strike the wall with a velocity v given by

$v = \sqrt{2gL}$; L = length of the pendulum

Speed of the pendulum after first collision is given by

$v_1 = ev = e\sqrt{2gL}$

After second collision its velocity is equal to

$v_2 = e^2 \sqrt{2gL}$

Velocity after n oscillation will be

$v_n = e^n \sqrt{2gL}$

According to conservation of energy

$\frac{1}{2} m v_n^2 = mgh$ (see Fig. 4.30)

$= \frac{1}{2} m \cdot e^{2n} \cdot 2gL = mgh$

or $e^{2n} = \frac{h}{L} = 1 - \cos\theta$ (see Fig.)

$\frac{2}{\sqrt{5}}^{2n} = 1 - \cos\theta$

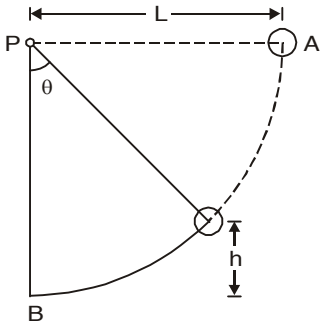


Fig. 4.30

$$\left(\frac{4}{5}\right)^n = 1 - \cos\theta$$

θ should be less than 60° hence $\cos\theta > \frac{1}{2}$

or $\left(\frac{4}{5}\right)^n < \frac{1}{2}$

or $\left(\frac{5}{4}\right)^n > 2$

$n\,[\log 5 - \log 4] > \log 2$

$$n > \frac{0.301}{0.097}$$

or $n > 3.1$

The number of oscillations must be whole member hence $n = 4$.

4.42. A ball of mass m falls from a height h. Assuming the coefficient of restitution as e find the distance travelled by it before it stops.

Solution: The velocity of the ball when it is going to strike the ground is given by

$v = \sqrt{2gh}$

Velocity at the first rebound = ev

The height attained by it = $\frac{(ev)^2}{2g} = e^2h$

The distance travelled by the ball before second rebound = $2e^2h$

Similarly the distance travelled by the before third round = $2e^4h$

$\therefore$ Total distance travelled $S = h + 2e^2h + 2e^4h + \ldots\ldots$

$$= h + \frac{2e^2h}{1 \quad e^2} = h\,\frac{(1+e^2)}{1-e^2}$$

4.43. A bullet of mass m is fired at an angle $\theta = 60°$ with the horizontal with a velocity v on a wooden rod of mas M and length L. The rod is hinged at its top O (see Fig. 4.31a). Prior to collision the rod is at rest. The bullet gets embedded into the rod in a short interval of time Δt. Calculate (i) the angular velocity of the rod (ii) the reaction at the hinge.

Solution:

(i) Taking the moment about O we have

$$mv \cdot \cos\theta \cdot \frac{L}{2} = I\omega + MV \cdot \frac{L}{2}\ ;\ V = \text{linear velocity of the rod along } x \text{ axis}$$

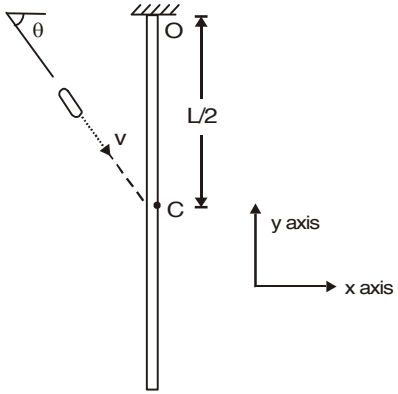


Fig. 4.31a

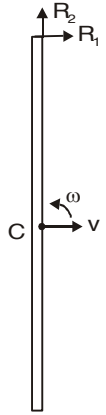


Fig. 4.31b

Resolving the momentum along x and y axes (see Fig. 4.31b) we have

$mv\cos\theta + R_1 \cdot \Delta t = MV$; R_1 is reaction at the hinge along x axis and $-mv\sin\theta + R_2 \cdot \Delta t = 0$; R_2 is reaction at the hinge along y axis

$$mv \cdot \cos\theta \cdot \frac{L}{2} = \frac{ML^2}{12}\ \frac{ML^2}{4}\ \omega;\ V = \frac{L}{2} \cdot \omega$$

$$\text{or}\quad mv\cos\theta \cdot \frac{L}{2} = \frac{ML^2}{3} \cdot \omega$$

$$\text{or}\quad \omega = \frac{3mv\cos\theta}{2ML} = \frac{3mv}{4ML}$$

(ii) $R_1 \Delta t = MV - mv\cos\theta$

$$= M \cdot \frac{L}{2} \cdot \omega - mv\cos\theta$$

$$= \frac{ML}{2} \cdot \frac{3mv\cos}{2ML} - mv\cos\theta$$

$$= -\frac{mv\cos\theta}{4}$$

$$\therefore R_1 = -\frac{mv\cos\theta}{4 \cdot \Delta t} = -\frac{mv}{8\Delta t}$$

$$R_2 = \frac{mv\sin}{t} = \frac{mv \cdot \sqrt{3}}{2\Delta t}$$

4.44. A wooden rod of mass M and length l pivoted at O is capable of rotating about it. A bullet of mass m ($m << M$) moving horizontally with a velocity v_0 strikes at the lower end of the rod and gets embedded into it (see Fig. 4.32). As a result of this impact the rod moves through an angle θ. Calculate (i) the velocity of the bullet (ii) the change in the momentum of the system.

Solution:

(i) According to the conservation of angular momentum we have

$$mv_0 l = \frac{1}{3} Ml^2\omega$$

or $\omega = \dfrac{3mv_0}{Ml}$

According to conservation of energy we have

$$\frac{1}{2} \cdot \frac{1}{3} Ml^2 \cdot \frac{9m^2v_0^2}{M^2l^2} = Mg\frac{l}{2}(1 - \cos\theta)$$

$$= Mg\frac{l}{2} \cdot 2\sin^2\frac{\theta}{2}$$

or $v_0^2 = \dfrac{M^2}{m^2} \cdot \dfrac{2gl}{3} \sin^2\dfrac{\theta}{2}$

or $v_0 = \dfrac{M}{m}\sqrt{\dfrac{2gl}{3}} \cdot \sin\dfrac{\theta}{2}$

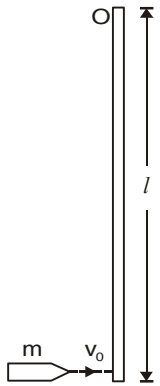


Fig. 4.32

(ii) Linear velocity of the bullet rod system = $v = \dfrac{l}{2} \cdot \omega = \dfrac{3mv_0}{2M}$

Final momentum of the bullet rod system = $Mv = \dfrac{3mv_0}{2}$

Δp = change in momentum = $\dfrac{3mv_0}{2} - mv_0 = \dfrac{mv_0}{2}$

$$\Delta p = \frac{mv_0}{2} = \frac{M}{2} \cdot \sqrt{\frac{2gl}{3}} \cdot \sin\frac{\theta}{2}$$

4.45. A rod of mass M and length l is hinged at O. The rod is initially at rest. A sphere of mass m moving horizontally with a velocity v_0 strikes at the lower end A of the rod (see Fig. 4.33a). Taking the coefficient of restitution as e determine (i) the angular velocity of the rod (ii) the velocity of the sphere after the impact.

Solution: (i) According to conservation of angular momentum, we have

$$mv_0 l = mv_1 l + M \cdot V_c \cdot \frac{l}{2} + I\omega\,;\ V_c = \frac{l}{2}\omega \text{ (see Fig. 4.33b)}$$

$$mv_0 l = mv_1 l + \frac{Ml^2}{4} \quad \frac{Ml^2}{12}$$

$$mv_0 = mv_1 + \frac{Ml}{3} \cdot \omega$$

$$V'_A - v_1 = e(v_0 - V_A); \quad V_A = \text{velocity of the end } A \text{ of the rod before impact}$$

$$= ev_0; \qquad V'_A = \text{velocity of the end } A \text{ of the rod after impact}$$

$$l\omega - v_1 = ev_0; V'_A = l\omega$$

$$\therefore v_1 = (l\omega - ev_0)$$

Now $$mv_0 = m(l\omega - ev_0) + \frac{Ml}{3} \cdot \omega$$

$$mv_0(1+e) = l \quad m \quad \frac{M}{3} \quad \omega$$

or $$\omega = \frac{mv_0(1 \quad e)}{l \quad m \quad \frac{M}{3}}$$

(ii) $$v_1 = l\omega - ev_0$$

$$= \frac{mv_0 \quad 1 \quad e}{m \quad \frac{M}{3}} - ev_0$$

$$= \frac{v_0 \quad m \quad \frac{eM}{3}}{m \quad \frac{M}{3}}$$

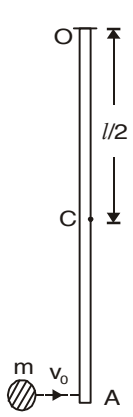


Fig. 4.33a

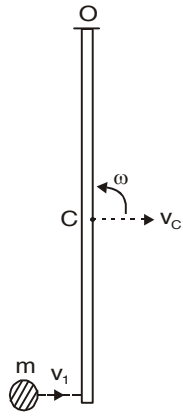


Fig. 4.33b

4.46. A thin uniform bar lies on a frictionless horizontal surface and is free to move in anyway on the surface. Its mass is 0.16 kg and length is $\sqrt{3}$ metres. Two particles each of mass 0.08 kg are moving on the same surface and towards the bar in a direction perpendicular to the bar, one with a velocity of 10 m/sec and other with 6 m/sec as shown in Fig. 4.34. The first particle strikes the bar at a point A and the other at point B. Points A and B are at a distance of 0.5 m from the centre of the bar. The particle strike the bar at the same instant of time and stick to the bar on collision. Calculate the loss of kinetic energy of the system in the above collision process. (IIT 1989)

Solution: The centre of mass of the system coincides with the mid-point of the rod. In the process of collision the forces developed will cause the rod to execute translational as well as rotational motion. Therefore, the linear momentum as well as the angular momentum both will be conserved.

According to conservation of linear momentum we have

$mu_1 + mu_2 = (M + 2m)\, V$; u_1 and u_2 are the velocities of the two particles

V = velocity of the centre of mass.

On putting the values of various terms we have

$0.08 \times 10 + 0.08 \times 6 = (0.16 + 2 \times 0.08)\ V$

$V = 4$ m/sec

Moment of inertia of the system about O is given by

$$I = \left[\frac{Ml^2}{12} + 2my^2\right]; y = OA = OB$$

On substituting the values of various terms we have

$$I = \frac{0.16 \quad \sqrt{3}}{12}^{\ 2} \quad 2 \quad 0.08 \quad (0.5)^2$$

$= 0.08$ kg m^2

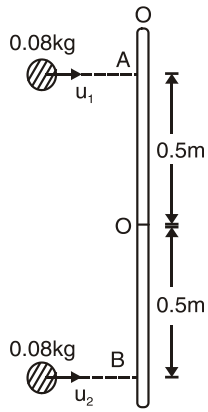


Fig. 4.34

According to the conservation of angular momentum we have

$mu_1 y - mu_2 y = I\omega$

$$\omega = \frac{mu_1 y \quad mu_2 y}{I}$$

On substituting the values we have

$$\omega = \frac{0.08 \quad 10 \quad 0.5 \quad 0.08 \quad 6 \quad 0.5}{0.08}$$

$= 2$ rad/sec

$$\text{Loss in energy} = \frac{1}{2} m u_1^2 + \frac{1}{2} m u_2^2 - \frac{1}{2} (M + 2m)\ V^2 + \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} \times 0.08 \times 10^2 + \frac{1}{2} \times 0.08 \times 6^2 - \frac{1}{2} (0.16 + 2 \times 0.08) \times 4^2 + \frac{1}{2} (0.08 \times 2^2)$$

$= 2.72$ Joule.

4.47. A block A of mass $2m$ is placed on another block B of mass $4m$ which in turn is placed on a fixed table. The two blocks have the same length $4d$ and they are placed as shown in Fig. 4.35. The coefficient of friction (both static and kinetic) between the block B and the table is μ. There is no friction between the two blocks. A small object of mass m moving horizontally along a line passing through the centre of mass (see figure) of the block B and perpendicular to its face with a speed v collides elastically with the block B at a height d above the table.

(a) What is the minimum value of v (call it v_0) required to make the block A topple ?

(b) If $v = 2v_0$, find the distance (from the point P in the Fig. 4.35) at which the mass m falls on the table after collision (Ignore the role of friction during the collision). (IIT 1991)

Solution: When the object of mass m collides with the block B it moves forward and as a result the block A moves backward. The friction between the two block is zero therefore velocity of A is zero.

For toppling the block A, the minimum distance traversed by block B must be $2d$.

Force of friction between block B and the table $= \mu \cdot 6mg$

Work done in displacing B through a distance $2d$ is given by

$W = \mu \cdot 6mg \times 2d = 12\mu\, mgd$

On applying the law of conservation of momentum in the collision of the object of mass m and block B of mass $4m$, we have

$mv = mv_1 + 4mv_2$; v_1 is the velocity of object of mass m after collision

v_2 is the velocity of block B after collision.

$v = v_1 + 4v_2$

According to conservation of energy, we have

$$\frac{1}{2} mv^2 = \frac{1}{2} m v_1^2 + \frac{1}{2} \cdot 4m v_2^2$$

or $v^2 = v_1^2 + 4v_2^2$

on solving $v_1 = -\frac{3}{5} v$ and $v_2 = \frac{2}{5} v$

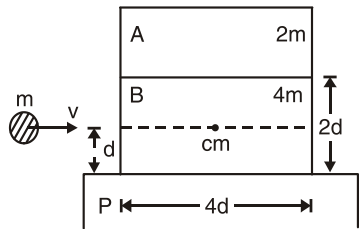


Fig. 4.35

Kinetic energy of block $B = \frac{1}{2} \times 4m \cdot v_2^2$

$$= 2m \times \left(\frac{2}{5} \cdot v_0\right)^2 = \frac{8m}{25} v_0^2$$

Now $\frac{8m}{25} v_0^2 = 12\,\mu\, mgd$

$$v_0 = \frac{5}{2} \sqrt{6\,\mu gd}$$

(b) When $v = 2v_0$ (given)

$$v_1 = -\frac{3}{5} \times 2v_0 = -\frac{6}{5} v_0$$

Time taken by the mass m to fall through a distance d is equal to

$d = \frac{1}{2} gt^2$ or $t = \sqrt{\frac{2d}{g}}$

Distance from P at which it will fall is equal to

$$x = -\frac{6v_0}{5} \cdot \sqrt{\frac{2d}{g}} = -\frac{6}{5} \cdot \frac{5}{2} \left(6\,\mu gd \times \frac{2d}{g}\right)^{\frac{1}{2}}$$

$$= -6d \sqrt{3}$$

4.48. A string with one end fixed on a rigid wall, passing over a fixed frictionless pulley at a distance of 2 m from the wall, has a point mass $M = 2$ kg attached to it at a distance of 1 m from the wall. A mass $m = 0.5$ kg attached at the free end is held at rest so that the string is horizontal between the wall and the pulley and vertical beyond the pulley. What is the speed with which the mass M will hit the wall when the mass m is released? (IIT 1985)

Solution: When the mass m is released then the block C of mass M moves in an arc CC' with A as centre (Fig. 4.36b).

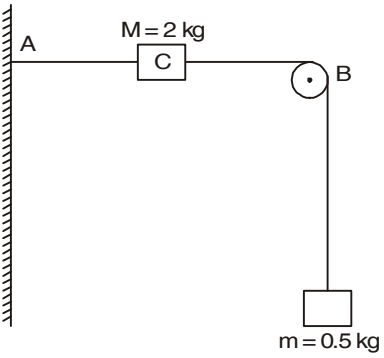


Fig. 4.36a

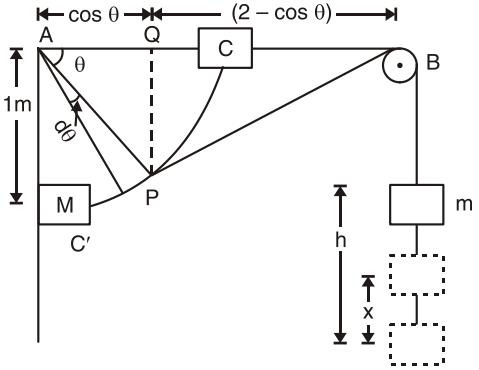


Fig. 4.36b

As the mass M descends mass m moves upward through a distance h given by

$$h = BC' - BC = \sqrt{AB^2 \quad AC^2} - BC$$

$$= \sqrt{5} \quad 1$$

Let P be the position of mass M at an instant t then

$$\omega = \frac{d}{dt}$$

Speed of mass M at point $P = v = r\omega = \frac{d}{dt}$; $r = 1$ m

Now $x = BP - BC = \sqrt{(BQ)^2 \quad (PQ)^2} - BC$

$$= \left[(2 - \cos\theta)^2 + \sin^2\theta\right]^{\frac{1}{2}} - 1$$

$$= \left[5 - 4\cos\theta\right]^{\frac{1}{2}} - 1$$

The velocity of mass m at this instant is given by

$$v_1 = \frac{dx}{dt} = \frac{1}{2} \cdot \frac{4\sin}{\sqrt{5 \quad 4\cos}} \cdot \frac{d}{dt}$$

$$= \frac{2\sin\theta \cdot v}{\sqrt{5 - 4\cos\theta}}$$

Let V be the speed of mass M at C'

$V = v$; here $\theta = 90°$

$$v_1 = \frac{2}{\sqrt{5}} \cdot V$$

According to conservation of energy, we have

Loss in P.E. of mass M = gain in K.E. of M + gain in P.E. of m + gain in K.E. of m

$$Mg = \frac{1}{2} MV^2 + mg \quad \sqrt{5} \quad 1 \quad + \frac{1}{2} m v_1^2$$

On putting the values of various terms we have

$$2 \times 9.8 = \frac{1}{2} \times 2 \cdot V^2 + 0.5 \times 9.8 \quad \sqrt{5} \quad 1 \quad + \frac{1}{2} \times 0.5 \times \quad \frac{2}{\sqrt{5}} V \quad ^2$$

On solving $V = 3.36$ m/sec.

4.49. A neutron of kinetic energy 65 eV collides in elastically with a singly ionised helium atom at rest. It is scattered at an angle of 90° with respect to its original direction. (a) Find the allowed values of the energy of the neutron and that of the atom after collision (b) If atom gets de-excited subsequently be emitting radiation, find the frequencies of the emitted radiation [Given mass of the atom = 4 × mass of neutron, ionisation energy of H-atom = 13.6 eV] (IIT 1993)

Solution: According to conservation of momentum (see Fig. 4.37) we have

Along x axis $mu = 4mv_2 \cos\theta$; m = mass of neutron

Along y axis $mv_1 = 4mv_2 \sin\theta$; u = velocity of neutron before collision

v_1 = velocity of neutron after collision

v_2 = velocity of helium ion after collision

Now $u^2 + v_1^2 = 16 v_2^2$

or $\frac{1}{2} mu^2 + \frac{1}{2} m v_1^2 = \frac{1}{2} m \cdot 16 v_2^2$

$65 + E_1 = 4E_2$ E_1 = K.E. of neutron after collision

E_2 = K.E. of helium ion after collision

Now $65 = E_1 + E_2 + \Delta x$; $\Delta x = 13.6 \times 2^2 \quad 1 \quad \frac{1}{n^2}$

or $65 = E_1 + E_2 + 54.4 \quad 1 \quad \frac{1}{n^2}$

or $E_1 + E_2 = 10.6 + \frac{54.4}{n^2}$

On solving the values of E_1 and E_2 we have

$$E_1 = \frac{43.52}{n^2} \quad 4.52 \quad eV$$

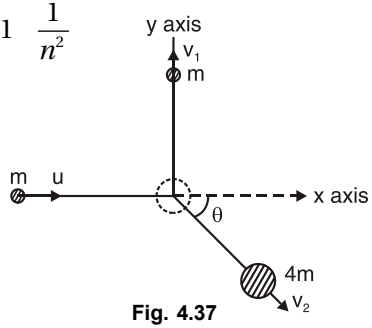


Fig. 4.37

$$E_2 = 15.12 \quad \frac{10.88}{n^2} \quad eV$$

E_1 is positive which is possible when $\frac{43.52}{n^2} > 4.52$

or $n < 3.1$

Now when $n = 2$, $E_1 = 6.36\ eV$ and $E_2 = 17.84\ eV$

when $n = 3$ $E_1 = 0.32\ eV$ and $E_2 = 16.33\ eV$

The frequency of radiation emitted as a result of transitions is given by

$$\nu = \frac{13.6 \quad 4 \quad 1.6 \quad 10^{-19}}{6.63 \quad 10^{-34}} \quad \frac{1}{n_1^2} \quad \frac{1}{n_2^2}$$

When $n_2 = 3$ and $n_1 = 1$ $\nu = 11.67 \times 10^{15}$ Hz

When $n_2 = 2$ and $n_1 = 1$ $\nu = 9.84 \times 10^{15}$ Hz

When $n_2 = 3$ and $n_1 = 2$ $\nu = 1.83 \times 10^{15}$ Hz

4.50. Find the moment of inertia of a solid sphere of mass M and radius R about an axis passing through its diameter.

Solution: Mass per unit volume of the sphere = $\frac{M}{\frac{4}{3} \quad R^3}$

Consider a circular disc of thickness dx at a distance x from the centre of the sphere (Fig. 4.38).

Volume of the disc = $\pi (R^2 - x^2) \cdot dx$

Mass of the disc

$$= \frac{3M}{4\pi R^3} \cdot \pi (R^2 - x^2)\, dx$$

$$= \frac{3M}{4R^3} (R^2 - x^2)\, dx$$

Moment of inertia of the disc about AB

$$= \frac{1}{2} \cdot \frac{3M}{4R^3} (R^2 - x^2)\, dx \cdot (R^2 - x^2)$$

$$= \frac{3M}{8R^3} (R^2 - x^2)^2\, dx$$

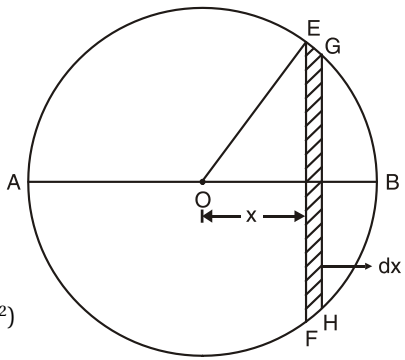


Fig. 4.38

Moment of inertia of the solid sphere about the diameter AB is given by

$$I = 2\int_0^R \frac{3M}{8R^3}\left(R^2 - x^2\right)^2 dx$$

$$= \frac{3M}{4R^3}\left[\int_0^R R^4 dx - 2R^2\int_0^R x^2 dx + \int_0^R x^4 dx\right]$$

$$= \frac{3M}{4R^3} \quad R^5 \quad \frac{2}{3}R^5 \quad \frac{R^5}{5} = \frac{3M}{4R^3}\left[\frac{15R^5 - 10R^5 + 3R^5}{15}\right] = \frac{2}{5} MR^2$$

4.51. Find out the moment of inertia of a spherical shell of mass M and radius R about an axis passing through its diameter.

Solution: Mass per unit area of the shell = $\frac{M}{4\ R^2}$

Consider a thin slice of thickness dx at a distance x from the centre O. The slice under consideration is a ring of radius PE and width dx.

Mass of the ring $= 2\pi \cdot PE \cdot EG \cdot \frac{M}{4\ R^2}$ (see Fig. 4.39)

$$= 2\pi \cdot PE \cdot Rd\theta \cdot \frac{M}{4\ R^2}$$

$$= \frac{M}{2R} \cdot dx\ ;\ PE \cdot d\theta = dx$$

Moment of inertia of the ring about $AB = \frac{M}{2R} \cdot dx \cdot (R^2 - x^2)$

Moment of inertia of the spherical shell is given by

$$I = 2\int_0^R \frac{M}{2R} \cdot \left(R^2 - x^2\right) dx$$

$$= \frac{M}{R}\left[R^2 \int_0^R dx - \int_0^R x^2 dx\right]$$

$$= \frac{M}{R}\ R^3\ \frac{R^3}{3}$$

$$= \frac{2}{3}\ MR^2$$

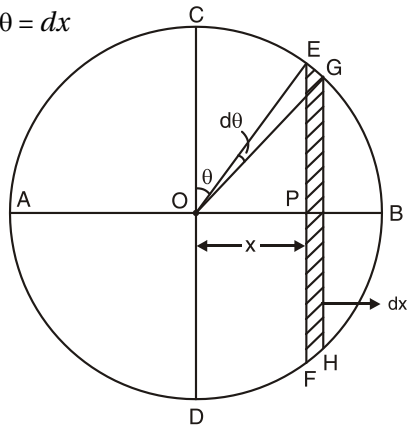


Fig. 4.39

4.52. A uniform disc of radius R has a round cut of diameter R as shown in the Fig. 4.40. The mass of the portion left is equal to m. Find the moment of inertia of the disc about an axis passing through the centre of inertia and perpendicular to the plane of the disc.

Solution: Area of the portion left $= \pi r^2 - \pi\frac{R^2}{4}$

$$= \frac{3\ R^2}{4}$$

Mass per unit area $= \frac{4m}{3\ R^2}$

Mass of the portion left $= \frac{4m}{3\ R^2} \cdot \frac{R^2}{4}$

$$= \frac{m}{3}$$

Total mass of the disc $= \frac{4m}{3}$

G is the centre of inertia.

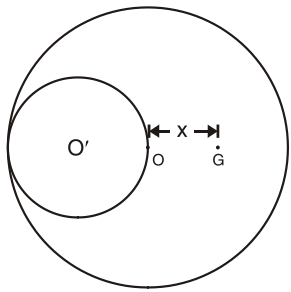


Fig. 4.40

Now, $\frac{4m}{3} \cdot OG = \frac{m}{3} \cdot O'G$

$$4x = \frac{R}{2} + x$$

or $$x = \frac{R}{6}$$

Let I_1 be the moment of inertia of the whole disc about G then

$$I_1 = \frac{1}{2} \cdot \frac{4m}{3} \cdot R^2 + \frac{4m}{3} \cdot \left(\frac{R}{6}\right)^2$$

$$= \frac{2}{3} mR^2 + \frac{mR^2}{27}$$

Let I_2 be the moment of inertia of the disc to be taken out about G then

$$I_2 = \frac{1}{2} \cdot \frac{m}{3} \cdot \left(\frac{R}{2}\right)^2 + \frac{m}{3} \cdot \frac{R}{2} \cdot \left(\frac{R}{6}\right)^2$$

$$= \frac{mR^2}{24} + \frac{m}{3} \cdot \frac{4R^2}{9}$$

Moment of inertia of the portion left about G is equal to

$$I_G = I_1 - I_2 = \frac{2}{3} mR^2 + \frac{mR^2}{27} - \frac{mR^2}{24} - \frac{m}{3} \cdot \frac{4R^2}{9}$$

$$= \frac{37}{72} mR^2$$

4.53. Fig. 4.41 shows a circular disc of mass M and radius R having the centre at O. From this disc a small circular portion of radius r is removed. Find out the moment of inertia of the remaining portion about an axis passing through O and perpendicular to the plane of the disc.

Solution: Mass per unit area of the disc = $\frac{M}{R^2}$

Mass of the area removed = $\frac{M}{R^2} \cdot \pi r^2$

$$= \frac{Mr^2}{R^2}$$

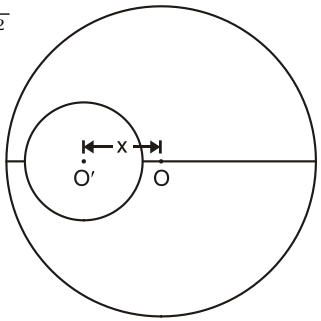


Fig. 4.41

Moment of inertia of the removed portion of mass about its centre O'

$$= \frac{1}{2} \frac{Mr^2}{R^2} \cdot r^2$$

$$= \frac{1}{2} \frac{Mr^4}{R^2}$$

Moment of inertia of the removed portion about O

$$= \frac{1}{2} \frac{Mr^4}{R^2} + \frac{Mr^2}{R^2} \cdot x^2$$

Moment of inertia of the portion of the disc left about an axis passing through O is equal to

$$I = \frac{1}{2} MR^2 - \frac{1}{2} \frac{Mr^4}{R^2} - \frac{Mr^2x^2}{R^2}$$

$$= \frac{M}{2R^2} \left[R^4 - r^4 - 2r^2x^2\right]$$

4.54. Find out the moment of inertia of a solid cone of mass M and base radius R about its vertical axis.

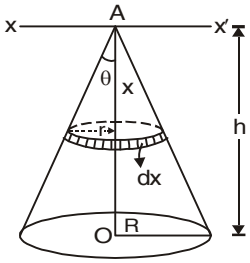


Fig. 4.42

Solution: Volume of the cone = $\frac{1}{3} \pi R^2 h$

Mass per unit volume of the cone = $\frac{3M}{R^2h}$

Consider a thin disc of radius r and thickness dx lying at a distance x below the vertex A (Fig. 4.42).

Volume of the disc = $\pi x^2 dx$

Mass of the disc = $\frac{3M}{R^2h} \cdot \pi r^2 dx$

$$= \frac{3M}{R^2h} r^2 dx$$

Moment of inertia of the disc about OA is given by

$$\frac{1}{2} \cdot \frac{3Mr^2dx}{R^2h} \cdot r^2 = \frac{3}{2} \frac{M}{R^2h} \cdot \tan^4\theta \cdot x^4 \cdot dx; \; r = x \tan\theta$$

Moment of inertia of the cone about its vertical axis is given by

$$I = \frac{3}{2} \frac{M}{R^2h} \tan^4\theta \int_0^h x^4 dx$$

$$= \frac{3M}{2R^2} \tan^4\theta \frac{h^4}{5}$$

$$= \frac{3M}{10R^2} \frac{R^4}{h^4} h^4$$

$$= \frac{3MR^2}{10}$$

4.55. Find the moment of inertia of an elliptical disc of mass M about its minor axis. Take the length of semi-major and semi-minor axes as a and b respectively.

Solution: Consider a strip of width dx parallel to minor axis and at a distance x from it.

Area of the strip = $2y \cdot dx$ (see Fig. 4.43)

Mass of the strip = $2ydx \cdot \sigma$; σ = mass per unit area

Moment of inertia of the strip = $2ydx\,\sigma \cdot x^2$

Moment of inertia of the whole elliptical disc about yy' is given by

$$I_y = 2\int_0^a 2y\,dx\,\sigma\cdot x^2$$

$$= 4\,\sigma\int_0^a y\,x^2\cdot dx$$

The equation of ellipse is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

or $y=\dfrac{b}{a}\sqrt{a^2 \quad x^2}$

$$\therefore I_y=\frac{4}{a}\frac{b}{}\cdot\int_0^a\sqrt{a^2-x^2}\cdot x^2dx$$

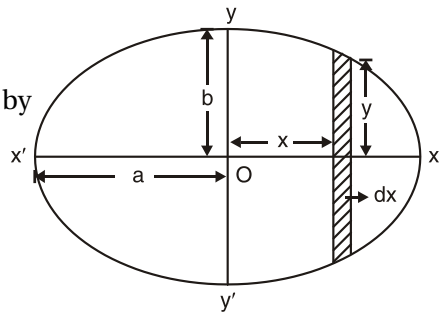


Fig. 4.43

Let us substitute $x = a\sin\theta$, $dx = a\cos\theta\cdot d\theta$

$$I_y=\frac{4}{a}\frac{b}{}\int_0^{\frac{\pi}{2}}a\cos\theta\cdot a^2\sin^2\theta\cdot a\cos\theta\cdot d\theta$$

$$=4\sigma a^3b\int_0^{\frac{\pi}{2}}(\cos\theta\cdot\sin\theta)^2\cdot d\theta$$

$$=\sigma a^3b\int_0^{\frac{\pi}{2}}\sin^2 2\theta\cdot d\theta$$

$$=\frac{a^3b}{2}\left[\int_0^{\frac{\pi}{2}}d\theta-\int_0^{\frac{\pi}{2}}\cos 4\theta\cdot d\theta\right]$$

$$=\frac{a^3b}{4}=\frac{Ma^2}{4};\ \sigma\,\pi\,a\,b=M$$

4.56. A point moves along on arc of a circle of radius r. Its velocity is no longer constant but varies with the distance covered s via relation $v=a\sqrt{s}$ where a is a positive constant. Calculate the angle α between the vectors of the total acceleration and velocity in terms of the distance covered.

Solution: $v=a\sqrt{s}$

$v = r\omega = a\sqrt{s}$

On differentiation we have

$$\frac{rd}{dt}=a\cdot\frac{1}{2}\ s^{\frac{1}{2}}\cdot v$$

a_τ = tangential acceleration

$$=\frac{rd}{dt}=\frac{a}{2\sqrt{s}}\cdot v=\frac{a}{2\sqrt{s}}\cdot a\sqrt{s}=\frac{a^2}{2}$$

$$\omega^2=\frac{a^2s}{r^2}$$

a_r = radial acceleration = $\omega^2 r=\dfrac{a^2s}{r}$

$$\tan\alpha=\frac{a_r}{a}=\frac{a^2s}{r\cdot a^2}\cdot 2=\frac{2s}{r}$$ (see Fig. 4.44)

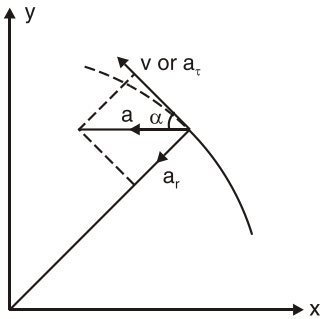


Fig. 4.44

4.57. A solid body rotates about a given axis with an angular acceleration which varies linearly with time via relation $\beta = at$ where a is a positive constant and equal to $\sqrt{3} \times 10^{-2}$ rad/s^3. Calculate the time from the beginning during which the angle α between the vectors of the total acceleration and velocity is equal to 60°.

Solution: $\dfrac{d}{dt} = \beta = at$

$d\omega = at \cdot dt$

or $\omega = \dfrac{at^2}{2}$

a_r = radial acceleration = $\left(\dfrac{at^2}{2}\right)^2 \cdot r$

a_τ = tangential acceleration = $r \cdot \dfrac{d}{dt} = at \cdot r$

$\tan\alpha = \dfrac{a_r}{a} = \dfrac{a^2t^4 \cdot r}{4 \cdot at \cdot r}$

$= \dfrac{at^3}{4}$

$\therefore\ t = \left(\dfrac{4}{3}\tan\alpha\right)^{\frac{1}{3}}$

On substituting the values we have

$t = \left(\dfrac{4 \cdot \sqrt{3}}{\sqrt{3} \times 10^{-2}}\right)^{\frac{1}{3}}$

$= (400)^{\frac{1}{3}}$

$= 7.37$ sec.

4.58. A wheel with the initial angular velocity ω_0 rotates about a given axis with an angular retardation $\beta = \alpha\sqrt{\ }$ where α is constant and ω is angular velocity. Find the mean value of its angular velocity averaged over the total period of rotation.

Solution:

$$\beta = -\alpha\sqrt{\ }$$

or $$\frac{d}{dt} = -\alpha\sqrt{\ }$$

or $$\int_{\omega_0}^{0} \omega^{-\frac{1}{2}} d\omega = -\alpha \int_0^T dt$$

$$T = \frac{2\omega_0^{\frac{1}{2}}}{\alpha}$$

Now $2\omega^{\frac{1}{2}} = \alpha t$ is the most general equation

$$4\omega = \alpha^2 t^2$$

$$4\frac{d}{dt} = \alpha^2 t^2$$

$$d = \frac{2}{4} t^2 dt$$

$$\phi = \frac{2}{12} T^3$$

Average angular velocity $= \frac{}{T} = \frac{^2T^3}{12\ T} = \frac{^2T^2}{12} = \frac{2}{12}\ \frac{4}{2}$

$$= \frac{}{3}$$

4.59. A flywheel rotates about a given axis in a way in which the angle described varies with the square of the time taken via relation $\theta = at^2$ where a is a positive constant. In the state of rotation the linear velocity at a point on the rim at an instant t is v. Calculate the total acceleration at that point at that instant.

Solution: $\theta = at^2$

$$\omega = \frac{d}{dt} = 2at$$

$$\frac{d}{dt} = 2a$$

Tangential acceleration $a_T = \frac{dv}{dt} = r \cdot \frac{d}{dt}$

$$= \frac{v}{} \cdot 2a$$

$$= \frac{v \cdot 2a}{2at} = \frac{v}{t}$$

Radial acceleration $a_r = \omega^2 r = \omega^2 \cdot \frac{v}{}$

$$= v\omega = v \cdot 2at$$

Total acceleration $a = \sqrt{a_T^2 \quad a_r^2} = \sqrt{\frac{v^2}{t^2} \quad 4v^2a^2t^2}$

4.60. A body rotates about an axis according to the law $\phi = at - bt^3$ where a and b are constants. Find (i) the mean value of angular velocity and angular acceleration averaged over the total time during which the body rotates. (ii) the angular acceleration at the instant when the body stops.

Solution: (i) $\phi = at - bt^3$

$$\frac{d}{dt} = a - 3bt^2$$

at $t = 0$, $\phi = 0$; at $t = T = \sqrt{\frac{a}{3b}}$, $\frac{d\phi}{dt} = 0$ i.e. the body stops

Total time during which the body rotates = $T = \sqrt{\frac{a}{3b}}$

Average angular velocity $= \frac{}{T} = \frac{aT \quad bT^3}{T}$

$= (a - bT^2) = a - b \cdot \frac{a}{3b} = \frac{2a}{3}$

at $t = 0$, $\frac{d}{dt} = \omega_1 = a$

at $t = \sqrt{\frac{a}{3b}}$, $\frac{d}{dt} = \omega_2 = 0$

$\therefore \beta = \frac{^2 \quad ^1}{T} = \frac{a\sqrt{3b}}{\sqrt{a}} = \sqrt{3ab}$

(ii) $\frac{d^2}{dt^2} = -6bt$

$= -6b \sqrt{\frac{a}{3b}}$

$= -2 \sqrt{3ab}$

4.61. The angular velocity of a solid body rotating about an axis varies with the angle of rotation via relation $\omega = \omega_0 - a\phi$ where ω_0 and a are positive constants. Assuming the angle of rotation $\phi = 0$ at $t = 0$ find the expression for the angle of rotation and angular velocity as a function of time.

Solution: $\omega = \omega_0 - a\,\phi$

or $\frac{d}{_0 \quad a} = dt$

$-\frac{1}{a} \log (\omega_0 - a\,\phi) = t + A'$

or $\log (\omega_0 - a\,\phi) = -at + A$

At $t = 0$, $\phi = 0$, $\therefore \quad A = \log \omega_0$

$\therefore \log \frac{_0 \quad a}{_0} = -at$

$1 - \frac{a}{_0} = e^{-at}$

or $\phi = \frac{^0}{a} \quad 1 \quad e^{\ at}$

$\frac{d}{dt} = \omega_0 e^{-at}$

4.62. The angular acceleration of a body rotating with an initial angular acceleration α_0 varies with the angle of rotation ϕ via relation $\alpha = \alpha_0 \cos \phi$. Find the angular velocity of the body as a function of the angle of rotation.

Solution:

$$\alpha = \alpha_0 \cos \phi$$

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\phi} \cdot \frac{d\phi}{dt} = \alpha_0 \cos \phi$$

$$\omega \cdot d\omega = \alpha_0 \cos \phi \cdot d\phi$$

$$\omega = \pm \sqrt{2 \quad _0 \sin}$$

4.63. A particle moves on the circumference of a circle of radius r with an angular velocity ω. Its displacement from the initial position measured along the arc is given by $l = b \sin \omega t$. Where b is a positive constant. Find (i) the total acceleration of the particle at $l = 0$ and $l = \pm b$ (ii) the minimum value of total acceleration $A_{\min}$ and the corresponding displacement $l_{\min}$.

Solution:

(i) $l = b \sin \omega t$

$$\frac{dl}{dt} = b\omega \cos \omega t = v = r\omega'$$

Tangential acceleration $a_T = r \cdot \frac{d}{dt} = -b\omega^2 \sin \omega t$

$$= -\omega^2 l$$

Radial acceleration $a_r = \frac{v^2}{r} = \frac{b^2 \quad ^2}{r} \cos^2 \omega t$

At $l = 0$, $a_T = 0$ and $a_r = \frac{b^2 \quad ^2}{r}$

$\therefore a = \frac{b^2 \quad ^2}{r}$

At $l = \pm b$; $\omega t = \frac{}{2}$

$a_T = b\omega^2$ and $a_r = 0$

$\therefore a = b\omega^2$

(ii) Total acceleration $A = \sqrt{b^2\omega^4 \cdot \sin^2 \omega t + \frac{b^2\omega^4}{r^2} \cdot \cos^4 \omega t}$

$$= b\omega^2 \sqrt{\frac{b^2 \cos^4 \omega t}{r^2} + \sin^2 \omega t}$$

Now $\frac{dA}{dt} = \frac{1}{2} b\omega^2 \frac{\left[-\frac{b^2}{r^2} \cdot 4\cos^3 \omega t \cdot \sin \omega t \cdot \omega + 2 \sin \omega t \cdot \cos \omega t \cdot \omega \right]}{\sqrt{\frac{b^2}{r^2} \cos^4 \omega t + \sin^2 \omega t}} = 0$

$$\cos^2 \omega t = \frac{r^2}{2b^2}, \sin^2 \omega t = 1 - \frac{r^2}{2b^2}$$

$$\therefore A_{min} = \sqrt{b^2\omega^4 \cdot \left(1 - \frac{r^2}{2b^2}\right) + \frac{b^4\omega^4}{r^2} \cdot \frac{r^4}{4b^4}}$$

$$= \sqrt{b^2 \ ^4 \quad \frac{r^2 \ ^4}{2} \quad \frac{r^2 \ ^4}{4}}$$

$$= \sqrt{b^2 \ ^4 \quad \frac{r^2 \ ^4}{4}}$$

$$A_{min} = b\omega^2 \sqrt{1 - \left(\frac{r}{2b}\right)^2}$$

$$\cos^2 \omega t = \frac{r^2}{2b^2}, \sin^2 \omega t = 1 - \frac{r^2}{2b^2}$$

$$\sin \omega t = \pm \sqrt{1 \quad \frac{r^2}{2b^2}}$$

$$\therefore l_{min} = \pm b \sin \omega t = \pm b \sqrt{1 \quad \frac{r^2}{2b^2}}$$

4.64. A wheel of radius r rolls without slip along x axis with velocity v. A point A is located on the rim of the wheel. The wheel starts from the origin O'. Find (i) the acceleration of the point A (ii) the distance covered by the point A between two successive intervals at which it touches the surface.

Solution: (i) After a lapse of time t the centre O of the wheel will move through a distance vt (see Fig. 4.45). Length of the arc CA is also equal to vt. Angle $AOC = \frac{vt}{r}$. The x and y coordinates of point A are given by

$$x = vt - r \sin \frac{vt}{r}$$

$$y = r - r \cos \frac{vt}{r}$$

The path of the point A is cycloid as shown in the Fig. 4.45.

$$\frac{dx}{dt} = v - \frac{rv}{r} \cos \frac{vt}{r}$$

$$= v \quad 1 \quad \cos \frac{vt}{r}$$

$$\frac{d^2x}{dt^2} = \frac{v^2}{r} \sin \frac{vt}{r}$$

Similarly $\frac{dy}{dt} = \frac{rv}{r} \sin \frac{vt}{r}$

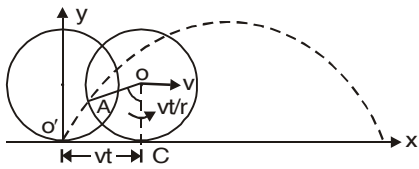


Fig. 4.45

$$\frac{d^2y}{dt^2} = \frac{v^2}{r} \cdot \cos\frac{vt}{r}$$

Resultant acceleration $a = \sqrt{\frac{d^2x}{dt^2} \quad \frac{d^2y}{dt^2}}$

$$= \frac{v^2}{r}$$

The acceleration of point A is constant in magnitude but it is always directed toward the centre of the wheel.

(ii) Result velocity $V = \sqrt{\frac{dx}{dt}^2 \quad \frac{dy}{dt}^2}$

$$= v\sqrt{1 \quad \cos\frac{vt}{r}^2 \quad \sin^2\frac{vt}{r}}$$

$$= 2v \cdot \sin\frac{vt}{2r}$$

$$S = \int_0^{\frac{2\pi r}{v}} V dt = 2v \int_0^{\frac{2\pi r}{v}} \sin\frac{vt}{2r} dt$$

$$= -2v \cdot \frac{2r}{v} \left| \cos\frac{vt}{2r} \right|_0^{\frac{2\pi r}{v}}$$

$$= 8r$$

4.65. A solid cylinder of radius r rolls without slipping on a horizontal surface with a velocity v. Find the radius of curvature of the path described by the point A and B marked on its surface.

Solution: The x and y coordinates of point A (see Fig. 4.46a) is given by

$$x = vt + r\sin\frac{vt}{r}$$

$$y = r - r\cos\frac{vt}{r}$$

$$\frac{dx}{dt} = \dot{x} = v + v\cos\frac{vt}{r}$$

$$\frac{d^2x}{dt^2} = \ddot{x} = -\frac{v^2}{r} \cdot \sin\frac{vt}{r}$$

$$\frac{dy}{dt} = \dot{y} = v\sin\frac{vt}{r}, \quad \frac{d^2x}{dt^2} = \ddot{y} = \frac{v^2}{r} \cdot \cos\frac{vt}{r}$$

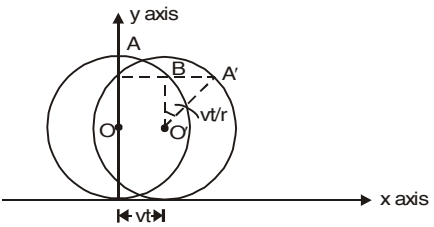


Fig. 4.46a

Radius of curvature $\rho = \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{\frac{3}{2}}}{\left(\dot{x}\ddot{y} - \ddot{x}\dot{y}\right)}$

On substituting the values we have

$$\rho = \frac{v^3\left[\left(1+\cos\frac{vt}{r}\right)^2+\sin^2\frac{vt}{r}\right]^{\frac{3}{2}}}{v\left(1+\cos\frac{vt}{r}\right)\frac{v^2}{r}\cdot\cos\frac{vt}{r}+\frac{v^2}{r}\sin\frac{vt}{r}\cdot v\sin\frac{vt}{r}}$$

$$= \frac{r\left[1+2\cos\frac{vt}{r}+1\right]^{\frac{3}{2}}}{\cos\frac{vt}{r}+\cos^2\frac{vt}{r}+\sin^2\frac{vt}{r}}$$

$$= \frac{r\left[2+2\cos\frac{vt}{r}\right]^{\frac{3}{2}}}{1+\cos\frac{vt}{r}}$$

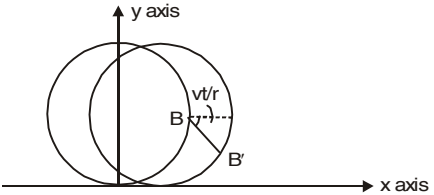


Fig. 4.46b

On putting $\frac{vt}{r} = 0$

$$\rho = \frac{r}{2}\ [4]^{\frac{3}{2}} = 4r$$

For point B the x and y coordinates (see Fig. 4.46b) are given by

$$x = vt - r\cos\frac{vt}{r}$$

$$y = r - r\sin\frac{vt}{r}$$

$$\dot{x} = v\left(1+\sin\frac{vt}{r}\right),\ \ddot{x} = \frac{v^2}{r}\cos\frac{vt}{r}$$

$$\dot{y} = -v\cos\frac{vt}{r},\ \ddot{y} = \frac{v^2}{r}\sin\frac{vt}{r}$$

$$\rho = \frac{\left(\dot{x}^2+\dot{y}^2\right)^{\frac{3}{2}}}{\left(\dot{x}\ddot{y}-\ddot{x}\dot{y}\right)}$$

On putting the values we have

$$\rho = \frac{v^3\left[1+2\sin\frac{vt}{r}+\sin^2\frac{vt}{r}+\cos^2\frac{vt}{r}\right]^{\frac{3}{2}}}{\frac{v^3}{r}\left[\sin\frac{vt}{r}+\sin^2\frac{vt}{r}+\cos^2\frac{vt}{r}\right]}$$

$$= \frac{r\left[2+2\sin\frac{vt}{r}\right]^{\frac{3}{2}}}{1+\sin\frac{vt}{r}}$$

On putting $\frac{vt}{r} = 0$ we have

$$\rho = 2\sqrt{2}\, r$$

4.66. A particle of mass m moves in xy plane. The motion is defined by the equation $x = a \cos \omega t$ and $y = b \sin \omega t$. Find the angular momentum of the particle with respect to the origin.

Solution: $x = a \cos \omega t$, $\dot{x} = -a\omega \sin \omega t$

$y = b \sin \omega t$, $\dot{y} = b\omega \cos \omega t$

Angular momentum $J = m \quad \dot{y}x \quad \dot{x}y$

On putting the values we have

$J = m\,[b\omega \cos \omega t \cdot a \cos \omega t + a\omega \sin \omega t\, b \sin \omega t]$

$= mab\omega$

4.67. A particle of mass m is projected into earth gravitational field with an initial velocity v_0 at an angle ϕ with the horizontal (see Fig. 4.47). Find the angular momentum about the origin when (i) the particle is at the highest point of its parabolic path (ii) the particle is just going to hit the ground.

Solution:

(i) The angular momentum is defined by

$J = m\,[x\dot{y} - y\dot{x}]$

At the highest point $\dot{y} = 0$,

$\dot{x} = v_0 \cos \phi$, and $y = \frac{v_0^2 \sin^2}{2g}$

$\therefore J = -m \cdot \frac{v_0^2 \sin^2}{2g} \cdot v_0 \cos \phi$

$= -\frac{mv_0^3}{2g} \sin^2 \phi \cos \phi$

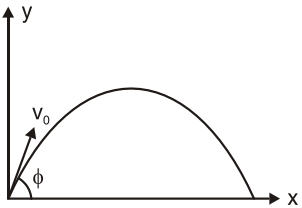


Fig. 4.47

(ii) In the second case $\dot{x} = 0$, $\dot{y} = -v_0 \sin \phi$, $x = \frac{2v_0^2}{g} \sin \phi \cos \phi$

$J = -m \cdot \frac{2v_0^2}{g} \sin \phi \cos \phi \cdot v_0 \sin \phi$

$= -\frac{2mv_0^3}{g} \sin^2 \phi \cos \phi$

4.68. At what height above the surface a ball of radius r be struck by a horizontal impact inorder to have no sliding at the point of contact O.

Solution: Angular momentum $L = m\, v \cdot x = \frac{2}{5}\, mr^2\omega$ (see Fig. 4.48)

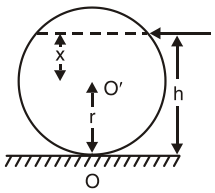


Fig. 4.48

$$mr\omega \cdot x = \frac{2}{5} mr^2\omega;\ v = r\omega$$

$$\therefore\ x = \frac{2}{5} r$$

or $$h = x + r = \frac{2}{5} r + r = \frac{7}{5} r$$

4.69. A ball of mass m is located at the origin O of a frame of reference at a height h above the earth surface. As the ball starts falling from the point O without any initial velocity the frame of reference starts moving horizontally with a constant velocity V. Calculate the increment in the angular momentum of the ball during the time of its free fall.

Solution: Let O' be the position of O after a lapse of time dt (see Fig. 4.49). Then $OO' = Vdt$. Let dy be the downward displacement of the ball after time dt. Then angular momentum about O' is given by

$$dJ = m \cdot \frac{dy}{dt} \cdot V \cdot dt$$

$$= mV \cdot dy$$

Net angular momentum is given by

$$J = \int dJ = \int_0^h mVdy = mVh$$

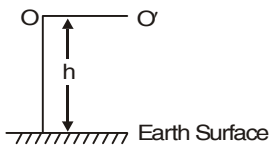


Fig. 4.49

4.70. A particle moves in a closed path under the action of a central field of force. The potential energy of the particle is Kr^2 where K is constant and r is the distance of the particle from the centre of the field. Find the mass of the particle if the minimum distance of the particle from the centre of the field is r_1 and its velocity at the farthest point from the centre of the field is v_2.

Solution: U = Potential energy = Kr^2

$$\text{Force} = -\frac{dU}{dr} = -2Kr$$

$$m \times \text{acceleration} = -2K \cdot r$$

$$\therefore\ \text{Acceleration} = -\frac{2K}{m} \cdot \text{displacement}$$

$$\therefore\ \omega = \sqrt{\frac{2K}{m}}$$

According to conservation of angular momentum we have

$mv_1r_1 = mv_2r_2$; r_1 and r_2 are the distance from the centre of the field where the velocities are v_1 and v_2 respectively.

Now $v_2 = r_2\omega$ and $v_1 = r_1\omega$

$$r_1^2\,\omega = v_2 \cdot \frac{v_2}{\omega}$$

or $$\omega^2 = \frac{v_2^2}{r_1^2} = \frac{2K}{m} \quad \text{or} \quad m = \frac{2Kr_1^2}{v_2^2}$$

4.71. A solid sphere of mass m and radius r starts rolling down an inclined plane at an angle θ with the horizontal. Find the angular momentum of the sphere relative to the point of contact at the initial moment.

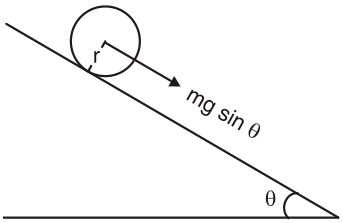


Fig. 4.50

Solution: Torque $= I\dfrac{d}{dt} = mgr\sin\theta$ (see Fig. 4.50)

$$Id\omega = mgr\sin\theta \cdot dt$$

or $$J = I\omega = mgr\sin\theta \cdot t$$

4.72. A smooth bar AB of mass M and length l rotates with an angular velocity ω_0 about a vertical axis passing through the end A. A sleeve of mass m initially at A starts sliding along the length of the bar. Find the linear velocity of the sleeve relative to the bar at the instant when it reaches the other end B.

Solution: According to conservation of energy we have

$$\frac{1}{2}\cdot\frac{1}{3}Ml^2\cdot\ {}^{2}_{0} = \frac{1}{2}\quad\frac{Ml^2}{3}\quad ml^2\quad\omega^2$$

$${}^{2}_{0} = 1\quad\frac{3m}{M}\quad\omega^2$$

or $$\omega = \frac{v}{l} = \frac{{}^{0}}{\sqrt{1\quad\dfrac{3m}{M}}}$$

or $$v = \frac{{}^{0}l}{\sqrt{1\quad\dfrac{3m}{M}}}$$

4.73. A solid sphere of mass M and radius R is pivoted at its centre O. A particle of mass m travelling with velocity v collides and adhere at the edge of the sphere. Prior to the collision the sphere was at rest. Find (i) the angular velocity of the system after collision (ii) the amount of energy lost in the collision.

Solution: (i) According to conservation of angular momentum we have

$$mvR = \frac{2}{5}MR^2\quad mR^2\quad\omega \text{ (see Fig. 4.51)}$$

$$\therefore\ \omega = \frac{mv}{\dfrac{2}{5}M\quad m\quad R}$$

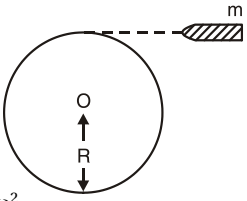


Fig. 4.51

(ii) Loss in energy $= \dfrac{1}{2}\ mv^2 - \dfrac{1}{2}\quad\dfrac{2}{5}M\quad m\quad R^2\cdot\omega^2$

$$= \frac{1}{2} mv^2 - \frac{1}{2} \left(\frac{2}{5}M + m\right) R^2 \cdot \frac{m^2v^2}{\left(\frac{2}{5}M + m\right)^2 \cdot R^2}$$

$$= \frac{1}{2} \left(m - \frac{m^2}{\frac{2}{5}M + m}\right) v^2$$

$$= \frac{Mmv^2}{2M + 5m}$$

4.74. A thin bar BB' of mass m and length l is falling in the vertical plane with velocity v_0. During motion the end B is connected the pivot O as shown in the Fig. 4.52. As a result of this the bar begin to rotate about B. Find the angular velocity of rotation.

Solution: Just before the connection at B with pivot O the bar has only translational motion as shown in Fig. 4.52. Its angular momentum with respect to $B = m v_0 \cdot \frac{l}{2}$

After being connected at B the new angular momentum is given by

$m\frac{l^2}{3} \cdot \omega$

According to conservation of angular momentum we have

$$mv_0 \cdot \frac{l}{2} = \frac{ml^2}{3} \cdot \omega$$

$$\therefore \omega = \frac{3v_0}{2l}$$

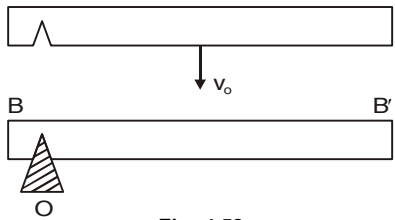


Fig. 4.52

4.75. One end of a thin rigid massless wire of length l is fixed to a wooden block of mass M as shown in the Fig. 4.53. A particle of mass m moving at right angles to the wire with a velocity v_0 strikes the block and adheres with it. Find (i) the angular momentum of the system (ii) the fraction of the particles energy lost in the collision.

Solution: (i) According to the conservation of momentum we have

$$mv_0 = (M + m)\, v$$

$$\therefore v = \frac{mv_0}{M + m}$$

Angular momentum

$$J = (M + m) \cdot \frac{mv_0}{M + m} l = mv_0 l$$

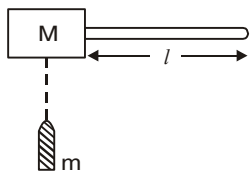


Fig. 4.53

(ii) Fraction of energy lost in the collision is given by

$$F = \frac{\frac{1}{2}mv_0^2 \quad \frac{1}{2}(M \quad m)\frac{m^2v_0^2}{M \quad m^{\ 2}}}{\frac{1}{2}mv_0^2} = \frac{M}{M \quad m}$$

4.76. One end of a long string of length L and negligible radius is fixed at the top of a pole of radius r. The other end of the string carries a small ball of mass m as shown in the Fig. 4.54. On imparting the ball an initial velocity v_0 in the horizontal plane the string starts winding itself around the pole and as a consequence the ball moves in a spiral of decreasing radius. Calculate the angular velocity of the ball after it has completed ten complete revolution around the pole.

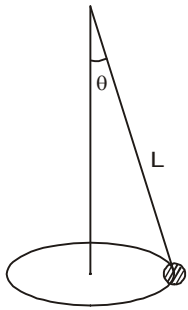


Fig. 4.54

Solution: According to conservation of energy we have

$$\frac{1}{2}\,m \cdot L^2 \sin^2\theta \cdot \ {}^{2}_{0} = \frac{1}{2}\,m\,(L - 20\pi r)^2 \sin^2\theta\ \omega^2\ ;\ \omega_0 = \text{initial angular velocity}$$

or $\quad L\omega_0 = v_0 = (L - 20\pi r)\omega$

or $\quad \omega = \dfrac{v_0}{L \quad 20 \quad r}$

4.77. A rod of length L and mass M lies on a smooth horizontal surface, where it can freely move in any manner. A ball of mass m moving with a velocity v_0 collides the rod at a point lying at a depth d below the centre of the rod (see Fig. 4.55) elastically. What should be the mass of the ball so that it remains at rest immediately after collision?

Solution: According to conservation of momentum we have

$mv_0 = m \times 0 + MV$

According to conservation of angular momentum we have

$mv_0 d = I\omega$; I is the moment of inertia of the rod about an axis passing through its centre.

According to conservation of energy we have

$$\frac{1}{2}\,mv_0^2 = \frac{1}{2}\,MV^2 + \frac{1}{2}\,I\omega^2$$

Now $\dfrac{1}{2}\,mv_0^2 = \dfrac{1}{2}\,MV^2 + \dfrac{1}{2}\,I\ \left(\dfrac{mv_0 d}{I}\right)^2$

or $\quad mv_0^2 = MV^2 + \dfrac{12m^2v_0^2d^2}{ML^2}$

or $\quad mv_0^2 = M \cdot \left(\dfrac{mv_0}{M}\right)^2 + \dfrac{12m^2v_0^2d^2}{ML^2}$

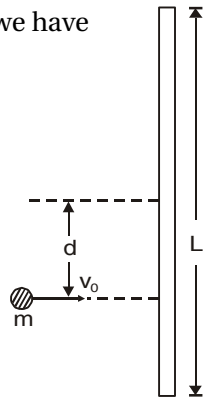


Fig. 4.55

or $1 = m \quad \frac{1}{M} \quad \frac{12d^2}{ML^2}$

or $m = \frac{ML^2}{L^2 \quad 12d^2}$

4.78. An arrow PQ of length $l = 75$ cm moving with a velocity of $v = 15$ m/sec strikes a smooth and a hard wall at an angle $\theta = 30°$ with the vertical as shown in Fig. 4.56. The end Q of the arrow slides downward along the wall without friction. Find the angular velocity of the arrow.

Solution: Angular momentum about $Q = mv \cos 60 \times \frac{l}{2} \cos 30$; m = mass of arrow.

Here the point O will move with new velocity v'.

The angular momentum of the arrow about Q after impact is given by

$$mv' \cdot \frac{l}{2} \cos 30 + \frac{ml^2}{12} \omega'$$

$$= \left[m\left(\frac{l}{2} \cdot \cos 30\right)^2 + \frac{ml^2}{12} \right] \omega'; \; v' = \frac{l}{2} \cos 30 \, \omega'$$

According to conservation of angular momentum we have

$$mv \cos 60 \times \frac{l}{2} \cos 30 = m \quad \frac{l^2}{4} \cos^2 30 \quad \frac{l^2}{12} \omega'$$

$$v \cos 60 \times \cos 30 = \left[\frac{l}{2} \cdot \cos^2 30 + \frac{l}{6} \right] \omega'$$

On substituting the values we have

$$15 \times \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 0.75 \left[\frac{3}{8} + \frac{1}{6} \right] \omega'$$

$$\frac{15 \quad \sqrt{3}}{4} = \frac{0.75 \quad 13}{24}$$

$$\therefore \omega' = \frac{15 \quad \sqrt{3} \quad 6}{0.75 \quad 13} = 15.98 = 16 \text{ rad/sec}$$

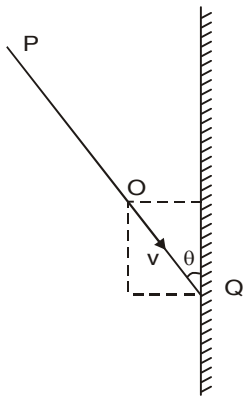


Fig. 4.56

4.79. A bar PQ of length $2l$ and mass $2m$ is bent at 90° at its middle point O as shown in the Fig. 4.57. The bar is hanged from a hinge at the point O and is capable of rotating in a vertical plane. Initially the bar is at rest and the arm PO is horizontal. If the bar is released find the maximum angular velocity acquired by it.

Solution: According to conservation of energy we have

Loss in potential energy of PO = gain in kinetic energy of PO + gain in kinetic energy of OQ + gain in potential energy of OQ.

$$mg \cdot \frac{l}{2} \sin\theta = \frac{1}{2} m \frac{l^2}{3} \frac{1}{2} m \frac{l^2}{3} \omega^2 + mg \frac{l}{2} (1 - \cos\theta)$$

or $$g \sin\theta = \frac{2l}{3} \omega^2 + g(1 - \cos\theta)$$

On differentiating it with respect to θ and equating it to zero we have

$$g \cos\theta = \frac{4l}{3} \omega \cdot \frac{d}{d} + g \sin\theta$$

or $$g(\cos\theta - \sin\theta) \cdot \frac{3}{4l} = \frac{d}{d} = 0$$

This is possible with the value of θ = 45°

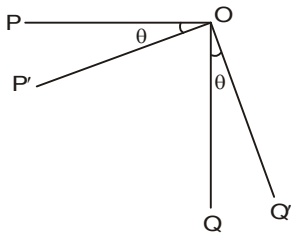


Fig. 4.57

On putting the value of θ as 45° we have

$$\frac{2l}{3} \cdot \omega^2 = g \frac{1}{\sqrt{2}} 1 \frac{1}{\sqrt{2}}$$

$$\omega = \sqrt{\frac{3g}{2l} \sqrt{2} 1}$$

On putting the value of $\sqrt{2}$ as 1.414 we have

$$\omega = \sqrt{\frac{0.414 \; 3g}{2l}} = \sqrt{0.621 g/l}$$

4.80. A uniform rod of length 2*a* stands vertically on a smooth horizontal plane as shown in Fig. 4.58. If it is just displaced from the vertical and released from rest it starts rotating. Find the angular velocity of the rod as a function of angle θ.

Solution: Rotational kinetic energy about the centre of the rod

$$= \frac{1}{2} m \cdot \frac{(2a)^2}{12} \cdot \omega^2$$

Kinetic energy corresponding to the vertical component of velocity *v* is given by

$$\frac{1}{2} mv^2 \sin^2\theta = \frac{1}{2} m\omega^2 a^2 \sin^2\theta$$

Total gain in kinetic energy

$$= \frac{ma^2 \; ^2}{6} + \frac{m \; ^2 a^2 \sin^2}{2}$$

Loss in potential energy = *mga* (1 – cos θ)

According to conservation of energy we have

$$mga(1 - \cos\theta) = \frac{m \; ^2 a^2}{6} (1 + 3\sin^2\theta)$$

or $$\omega = \sqrt{\frac{6g(1 \quad \cos \;)}{a(1 \quad 3\sin^2 \;)}}$$

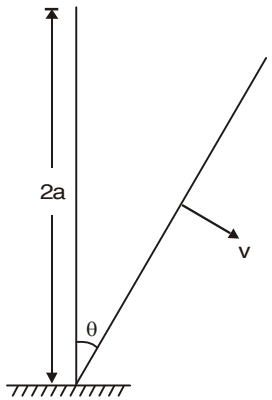


Fig. 4.58

4.81. A thin uniform bar PQ of mass m and length l is falling vertically at an angle θ with the horizontal without rotation. At the moment when the end P of the bar PQ strikes the surface as shown in the Fig. 4.59, the centre of gravity of the bar has velocity v_0. In the absence of rebound find the angular velocity of the bar immediately after the impact.

Solution: Before impact the angular momentum of the bar about P is equal to

$$m \cdot v_0 \cdot \frac{l}{2} \cos\theta$$

After impact the point O will move with a different velocity v'_0. The point P will move horizontally and as a result of that the bar will have an angular velocity ω. Angular momentum of the bar about P after impact is given by

$$mv'_0 \frac{l}{2} \cos\theta + \frac{ml^2}{12} \cdot \omega$$

According to conservation of angular momentum we have

$$mv_0 \frac{l}{2} \cos\theta = mv'_0 \frac{l}{2} \cos\theta + \frac{ml^2}{12} \cdot \omega$$

or $\omega = \frac{6}{l} v_0 v_0 \cos\theta$

Point C will be the centre of rotation after impact

$$v'_0 = \omega \frac{l}{2} \cos\theta$$

On putting the value of v'_0 we have

$$\omega = \frac{6}{l} v_0 \frac{l}{2} \cos \cos\theta$$

or $\omega (1 + 3\cos^2\theta) = \frac{6}{l} v_0 \cos\theta$

or $\omega = \frac{6v_0 \cos\theta}{l(1+3\cos^2\theta)}$

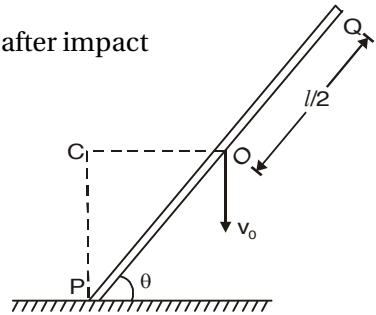


Fig. 4.59

4.82. The two ends of a string of length l is attached to a fixed point O and one end P of a horizontal bar of mass m and length $2l$ as shown in the Fig. 4.60. The bar is released from the rest. It starts rotating around a vertical axis through O. At what depth h below O a peg R be inserted in order that on striking it the bar comes exactly at rest.

Solution: Moment of inertia of the bar about $C = \frac{m(2l)^2}{12} = \frac{ml^2}{3}$

Moment of inertia of the bar about $O = \frac{ml^2}{3} + m \cdot (2l)^2$

$$= \frac{13ml^2}{3}$$

According to conservation of angular momentum we have

$$m.v.\,2l = \frac{13ml^2}{3} \times \omega$$

$$m.v.\,2l = \frac{13ml^2}{3} \cdot \frac{v}{h}$$

$$\therefore\ h = \frac{13ml^2}{6ml}$$

$$= \frac{13l}{6}$$

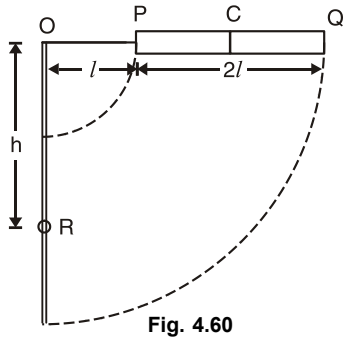


Fig. 4.60

4.83. The ends P and Q of the bar PQ of length $2l$ are constrained to move along y and x axis respectively. Find the path described by the two points A and B marked on the length of the bar.

Solution: The centre A of the bar is at a distance l from the origin O hence

$x = y = l \cos 45$ (see Fig. 4.61)

$\therefore\ x^2 + y^2 = l^2$ is the equation of path of the point A. This is a circle of radius l.

The coordinates of point B are

$$x = l \cos 45 + a \cos 45 = \frac{l\ \ a}{\sqrt{2}}$$

and $y = l \sin 45 - a \sin 45 = \dfrac{l\ \ a}{\sqrt{2}}$

$$\therefore\ \frac{x^2}{l\ \ a^{\ 2}} + \frac{y^2}{(l-a)^2} = 1$$

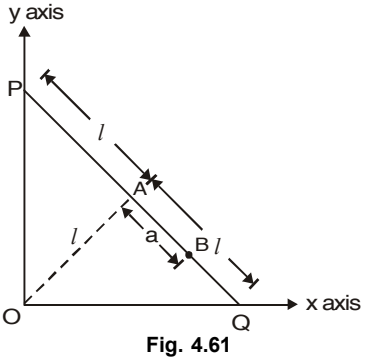


Fig. 4.61

The path of the point B is an ellipse with semi-major axis as $(l + a)$ and semi minor axis $(l - a)$.

4.84. A rod PQ of length l is fixed at the Crank pin P which moves in a vertical circle of radius l in xy plane as shown in Fig. 4.62. Find the path described by the point A marked on the length of the bar at a distance a from the point P.

Solution: In view of the figure $OP = PQ = l$

$\therefore\ \theta = 45°$

x coordinates of the point A is equal to

$x = l \cos 45 + a \cos 45 = (l + a) \cos 45$

y coordinates of the point A is equal to

$y = l \sin 45 - a \sin 45 = (l - a) \sin 45$

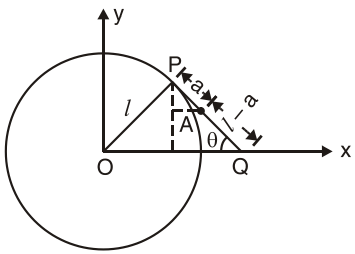


Fig. 4.62

The path of the point A is given by

$$\frac{x}{l\ \ a}^{2} + \frac{y}{l\ \ a}^{2} = 1$$

Which is an ellipse of the semi-major axis $(l + a)$ and semi-minor axis $(l - a)$.

4.85. Three slender bars PQ, QR and RS each of mass m and length l are freely joined at point Q and R. All the three bars are placed on a horizontal surface in the form of a straight line PS (see Fig. 4.63a). All the bars begin to move with a uniform velocity V on the horizontal surface by maintaining the line PS. During the course of motion the middle point of middle rod QR is suddenly fixed. Find (i) the angular velocity with which the two bars PQ and RS will move (ii) the time elapsed before the ends P and S of the two rods meets together.

Solution:

(i) Angular momentum of RS about R is equal to

$$mV \cdot \frac{l}{2} = \frac{ml^2}{3} \cdot \omega_1$$

$$\therefore\ \omega_1 = \frac{3V}{2l}$$ (see Fig. 4.63b)

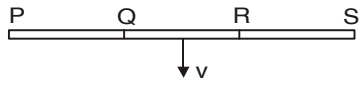


Fig. 4.63a

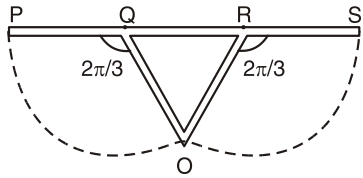


Fig. 4.63b

Similarly the angular momentum of PQ about Q is equal to $mV \cdot \frac{l}{2} = \frac{ml^2}{3} \cdot \omega_2$

$$\therefore\ \omega_2 = \frac{3V}{2l}$$

$$\therefore\ \omega_1 = \omega_2 = \omega = \frac{3V}{2l}$$

Thus the two bar PQ and RS will move with an angular velocity

$$\omega = \frac{3V}{2l}$$

(ii) The two bars will describe an angle $\frac{2}{3}$ in order to meet at O.

$$\therefore \text{ time taken } t = \frac{\frac{2}{3}}{\frac{3V}{2l}} = \frac{4\pi l}{9V}$$

4.86. Two bars PQ and QR each of mass m and length l are joined at Q with a hinge such that they can freely rotate with respect to the hinge. The two bars are placed on a horizontal frictionless table. An impulse F_0 is applied at a distance $\frac{l}{4}$ from centre of mass C_1 (see Fig. 4.64a). Find (i) the

velocity of centre of mass of the two bars (ii) the angular velocity of the two bars about the hinge (iii) the impulsive force F at the hinged end.

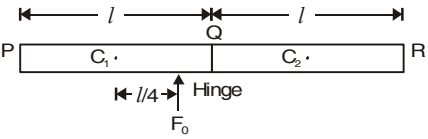


Fig. 4.64a

Solution: Change in momentum = Impulse

For the bar PQ, $mv_1 = F_0 - F$; v_1 = velocity of centre of mass C_1

For the bar QR; $mv_2 = F$; v_2 = velocity of centre of mass C_2

For bar PQ $\dfrac{ml^2}{12} \cdot \omega_1 = F_0 \cdot \dfrac{l}{4} - F \cdot \dfrac{l}{2}$ (see Fig. 4.64b & 4.64c)

For bar QR $\dfrac{ml^2}{12} \cdot \omega_2 = F \cdot \dfrac{l}{2}$

For the two bars velocity at the hinge will have the same value hence

$$v_1 + \omega_1 \frac{l}{2} = v_2 + \omega_2 \cdot \frac{l}{2}$$

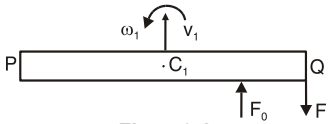


Fig. 4.64b

or $\dfrac{F_0 \quad F}{m} + \dfrac{3}{2} \quad \dfrac{F_0 \quad 2F}{m} = \dfrac{F}{m} + \dfrac{3F}{m}$

$$F = \frac{5F_0}{16}$$

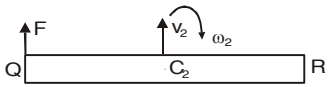


Fig. 4.64c

$$mv_1 = F_0 - \frac{5F_0}{16}$$

$$v_1 = \frac{11F_0}{16m}, \; v_2 = \frac{5F_0}{16m}$$

$$\omega_1 = \frac{9F_0}{8ml}, \; \omega_2 = \frac{15F_0}{8ml}$$

4.87. The rod OB of length l and mass m hinged at O to another small rod OA such that it can rotate freely about O (see Fig. 4.65). The rod OB is rotated at an angular velocity ω. Calculate the angle θ described by the rod.

Solution: Consider an element of length dx situated at a distance x from O.

Force acting on the element = $\dfrac{m}{l} \cdot dx\, \omega^2 \cdot x \sin\theta$

Taking the moment about O

$$mg \cdot \frac{l}{2} \sin\theta = \int_0^l dx \cdot \frac{m}{l} \omega^2 \cdot x \sin\theta \cdot x \cos\theta$$

$$\text{or } g\frac{l^2}{2} = \omega^2 \cos\theta \int_0^l x^2 dx$$

$$= \omega^2 \cos\theta \frac{l^3}{3}$$

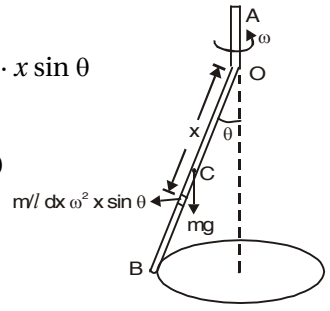


Fig. 4.65

$$\cos\theta = \frac{3g}{2\ ^2l}$$

or $$\theta = \cos^{-1}\ \frac{3g}{2\ ^2l}$$

4.88. A uniform bar PQ of length $l = 8$ m is held vertical on the ground by an auxiliary support. When the end P is released from rest the bar begin to rotate about Q. In the state of rotation a point C marked at a distance of $b = 0.8$ m above the centre of gravity G strikes the step as shown in the Fig. 4.66a. After the impact of bar rotates about C. Calculate the height of the step above the ground if the bar after the impact comes to rest for a moment in the horizontal position.

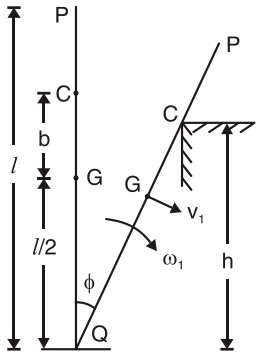


Fig. 4.66a

Solution: In view of conservation of energy we have

$$mg\frac{l}{2}(1-\cos\phi) = \frac{1}{2}\frac{ml^2}{3}\cdot\ ^2_1\ ;\ m \text{ mass of the bar}$$

$$\therefore\ \omega_1 = \sqrt{\frac{3g}{l}(1-\cos\phi)}$$

In moving to the horizontal position after impact we have gain in potential energy = loss in kinetic energy

or $$mgb\cos\phi = \frac{1}{2}I_C\ \ ^2_2\ ;\ I_C = \text{moment of inertia of the bar about } C.$$

$$= \frac{1}{2}\ \frac{ml^2}{12}\ \ mb^2\ \ ^2_2$$

$$\therefore\ \omega_2 = \sqrt{\frac{24\,g\,b\cos}{l^2\ \ 12b^2}}$$

According to conservation of angular momentum we have

$I\omega_1 - mv_1\,b = I\omega_2 + mv_2 b$; I = moment of inertia of the bar about $C.G.$

Further $v_1 = \omega_1 \cdot \frac{l}{2}$ and $v_2 = \omega_2 \cdot b$ (see Fig. 4.66b)

On substituting these values we have

$$\frac{ml^2}{12}\cdot\omega_1 - m\omega_1\frac{l}{2}\cdot b = \frac{ml^2}{12}\cdot\omega_2 + m\omega_2 b^2$$

or $$\frac{l^2}{12}\ \ \frac{lb}{2}\ \omega_1 = \frac{l^2}{12}\ \ b^2\ \ \omega_2$$

or $$\frac{l^2}{12}\ \ \frac{lb}{2}\ \ \sqrt{\frac{3g}{l}(1\ \ \cos\ )} = \frac{l^2}{12}\ \ b^2\ \ \sqrt{\frac{24\,gb\cos}{l^2\ \ 12b^2}}$$

On substituting the values we have

$$\frac{25.6}{12}\sqrt{3.675(1 \quad \cos\ )} = \frac{71.68}{12}\sqrt{\frac{188.16\cos}{71.68}}$$

$$25.6^2 \times 3.675\ (1 - \cos\phi) = 71.68 \times 188.16 \cos\phi$$

$$\text{or} \quad \frac{1 \quad \cos}{\cos} = \frac{1}{\cos} - 1$$

$$= \frac{71.68 \quad 188.16}{25.6^2 \quad 3.675} = 5.6$$

$$\frac{1}{\cos} = 6.6$$

$$\cos\phi = \frac{1}{6.6}$$

h = height of the step = $\frac{l}{2} \quad b \quad \cos\phi$

$$= \frac{4.8}{6.6}\text{m}$$

$$= 0.727\ \text{m}$$

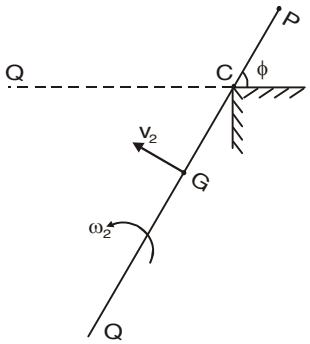


Fig. 4.66b

4.89. Two small spheres each of mass $m/2$ attached at the ends of a thin rod of length l forms a rigid dumbbell as shown in the Fig. 4.67. A ball of mass m moving with velocity v_0 suffers a head on elastic collision with one of the spheres of the stationary dumbbell. Find the proper angular momentum of the dumbbell after collision.

Solution: According to conservation of angular momentum we have

$$mv_0 = mv_1 + \frac{m}{2}v_2 \qquad (1);$$

v_1 = velocity of sphere of mass m after collision

v_2 = velocity of sphere of mass $m/2$ after collision

For perfectly elastic collision

$$\frac{v_2 \quad v_1}{v_0} = 1$$

$$\text{or} \quad v_2 - v_1 = v_0 \qquad (2)$$

$$mv_0 = m(v_2 - v_0) + \frac{m}{2}v_2$$

$$2v_0 = \frac{3}{2}v_2$$

$$\therefore\ v_2 = \frac{4v_0}{3}$$

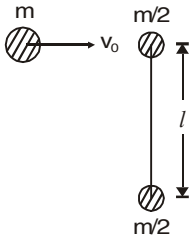


Fig. 4.67

Velocity of centre of mass of the dumbbell is equal to

$$v_{cm} = \frac{\frac{m}{2}\cdot\frac{4v_0}{3} + \frac{m}{2}\cdot\frac{4v_0}{3}}{\frac{m}{2} + \frac{m}{2}} = \frac{2v_0}{3}$$

L = proper angular momentum = angular momentum in the frame of centre of inertia $= r_1 \cdot p_1 + r_2 \cdot p_2$

$$= \frac{l}{2} \cdot \frac{m}{2} \cdot \frac{2v_0}{3} + \frac{l}{2} \cdot \frac{m}{2} \cdot \frac{2v_0}{3}$$

$$= \frac{mv_0 l}{3}$$

4.90. A small object of mass m attached to one end of a non-stretchable thread moves on a smooth horizontal plane. The other end of the thread passing through the hole O is being drawn at a constant velocity (see Fig. 4.68). Assuming the angular velocity of the thread equal to ω_0 when the distance r between the body and the hole is equal to r_0 find the tension in the thread as a function of r.

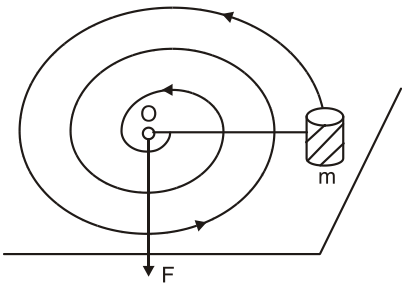


Fig. 4.68

Solution: $\frac{dr}{dt} = v$; v = velocity

or $r = vt$

Moment of inertia at a distance r_0 is given by

$I_0 = m r_0^2$

Initial angular momentum

$= m r_0^2 \omega_0$

Angular momentum after time $t = m (r_0 - vt)^2 \omega$

According to conservation law we have

$$m r_0^2 \omega_0 = m(r_0 - vt)^2 \omega$$

$$\therefore \omega = \frac{r_0^2 \omega_0}{r^2} ; r = r_0 - vt$$

$$T = m\omega^2 r = \frac{m \cdot r_0^4 \omega_0^2}{r^3}$$

4.91. A homogenous rod AB of length L = 1.8 m and mass M is pivoted of the centre O in such a way that it can rotate in the vertical plane. The rod is initially in the horizontal position. An insect S of the same mass falls vertically with speed V on the point C, midway between O and B. Immediately after falling, the insect moves towards the end B such that the rod rotates with constant angular velocity ω (a) Determine the angular velocity ω in terms of V and L (b) If the insect reaches the end B when the rod has turned through 90°, determine V. (IIT 1992)

Solution: (a) Applying the conservation of angular momentum of insect + rod about O we have

$$MV \cdot \frac{L}{4} = \left(\frac{1}{2} ML^2 + \frac{ML^2}{12} \right) \omega \text{ (see Fig. 4.69a)}$$

or $$\omega = \frac{12}{7} \cdot \frac{V}{L}$$

(b) When the insect moves the moment of inertia of the system changes (see Fig. 4.69b).

Instantaneous torque about $O = Mg\left(\frac{L}{4} - x\right)\cos\theta$; θ is the angle of rotation during time t.

Instantaneous moment of inertia about $O = M\left(\frac{L}{4} - x\right)^2 + \frac{ML^2}{12}$

Torque $T = \frac{d}{dt}(I\omega)$

or $Mg\left(\frac{L}{4} - x\right)\cos\theta = \frac{d}{dt}\left[\left\{M\left(\frac{L}{4} + x\right)^2 + \frac{ML^2}{12}\right\}\right]\omega$

or $Mg\left(\frac{L}{4} - x\right)\cos\theta = \omega \cdot 2M\left(\frac{L}{4} - x\right)\frac{dx}{dt}$

or $g\cos\theta = 2\omega\frac{dx}{dt} = 2\omega\frac{dx}{d}\cdot\frac{d}{dt} = 2\omega^2\frac{dx}{d}$

or $g\int_0^{\frac{\pi}{2}}\cos\theta\cdot d\theta = 2\omega^2\int_0^{\frac{L}{4}}dx$

or $g = 2\omega^2\frac{L}{4}$

or $g = \frac{^2L}{2}$

or $g = \frac{12V}{7L}^2 \times \frac{L}{2}$

or $V = \frac{7}{12}\sqrt{2gl}$

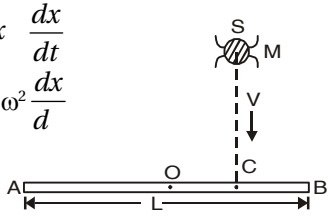


Fig. 4.69a

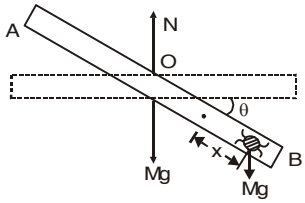


Fig. 4.69b

4.92. A ball, initially at rest is given an impulse by a cue which is held horizontally at a height h above the centre of the ball as shown in Fig. 4.70. The balls moves with a velocity v_0. Assuming the radius of the ball equal to r find the value of h if the ball acquires a final speed of $\frac{9v_0}{7}$.

Solution: According to conservation of angular momentum we have

$mv_0h = \frac{2}{5}mr^2\cdot\omega_0$; m = mass of the ball

ω_0 = angular velocity at the initial state.

Further force $= m\cdot a = -\mu mg$; a = acceleration

μ = Coefficient of friction

or $\frac{dv}{dt} = -\mu g$

or $dv = -\mu g\,dt$

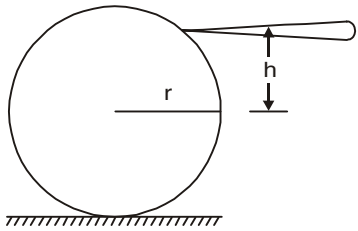


Fig. 4.70

or $v = -\mu g t + c$

at $t = 0$, $v = v_0$

$\therefore\ c = v_0$

or $v = v_0 - \mu g t$

We know that

$$I\alpha = \mu mgr$$

or $$\frac{d\omega}{dt} = \frac{mgr}{I}$$

or $$\int d\omega = \frac{mgr}{I} \cdot dt$$

or $$\omega = \frac{mgr}{I} t + c'$$

at $t = 0$, $\omega = \omega_0$ hence $c' = \omega_0$

or $$\omega = \omega_0 + \frac{mgrt}{I}$$

or $$\frac{v}{r} = \frac{I\omega_0 + \mu g t \cdot mr}{I}$$

or $$vI = I\omega_0 r + mr^2 (v_0 - v)$$

or $$v(I + mr^2) = \frac{2}{5} mr^2 \omega_0 \cdot r + mr^2 v_0$$

or $$\frac{7}{5} v = \frac{2\omega_0 r + 5v_0}{5}$$

or $$v = \frac{2 \quad _0 r \quad 5v_0}{7} = \frac{9v_0}{7}$$ (given)

$$\therefore\ 4v_0 = 2\omega_0 r$$

or $$\omega_0 = \frac{2v_0}{r}$$

or $$mv_0 h = \frac{2}{5} mr^2 \cdot \frac{2v_0}{r}$$

or $$h = \frac{4r}{5}$$

4.93. A solid sphere rolls without slipping on a table. What percentages of its total energy will in the form of translational kinetic energy and rotational kinetic energy?

Solution: The moment of inertia of a solid sphere of mass m and radius r about an axis passing through its diameter is given by

$$I = \frac{2}{5} mr^2$$

Total energy of the sphere = translational kinetic energy + rotational kinetic energy = $\frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$

$$= \frac{1}{2} mv^2 + \frac{1}{2} \times \frac{2}{5} mr^2\omega^2$$

$$= \frac{1}{2} mv^2 + \frac{1}{5} mv^2 = \frac{7}{10} mv^2$$

Now % translational energy = $\frac{100 \quad (1/2)\, mv^2}{(7/10)\, mv^2}$ 71.4%

and % rotational energy = $\frac{100 \quad (1/5)\, mv^2}{(7/10)\, mv^2}$ = 28.6%

4.94. A rolling solid sphere of mass 0.5 kg strikes a wall with a velocity of 10 m/s and rolls back with a velocity of 8 m/s. Calculate the amount of heat generated during the impact.

Solution: Total kinetic energy of the solid sphere of mass m moving with a velocity $v = \frac{1}{2} mv^2 + \frac{1}{5} mv^2 = \frac{7}{10} mv^2$ (see Q. 4.93)

Total kinetic energy of the solid sphere before impact = $\frac{7}{10} m v_1^2$

Total kinetic energy of the solid sphere after impact = $\frac{7}{10} m v_2^2$

Amount of heat generated during impact = $\frac{7}{10} \times m \quad v_1^2 \quad v_2^2$

On substituting the values of m, v_1 and v_2, we have

Amount of heat generated = $\frac{7}{10} \times 0.5 \times (10^2 - 8^2)$

$= 0.7 \times 0.5 \times 18 \times 2 = 12.6$ J

4.95. A mass of 4 kg is fixed at the midpoint of a 2.5 m long string whose ends are connected to a rod of length 2 m (see Fig. 4.71a). The system rotates about the length of the rod at a rate of 42 revolutions per minute. Find the tension in the two portions of the string.

Solution: Considering Fig. 4.71b, we have

$T_1 \cos\theta = mg + T_2 \cos\theta$; T_1 and T_2 are the tensions in the two portions of the string.

or $(T_1 - T_2) \cos\theta = mg$

or $(T_1 - T_2) \times \frac{1}{1.25} = 4g$

In view of Fig. 4.71a, $\cos\theta = 1/1.25$

or $T_1 - T_2 = 5g = 5 \times 9.8 = 49$ N

Further $(T_1 + T_2) \sin\theta = m\omega^2 r$; r is the radius of the circular path in which the mass is moving.

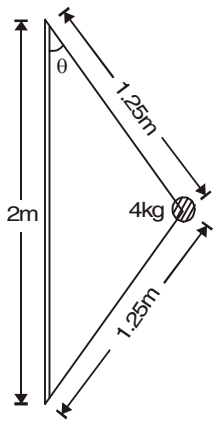


Fig. 4.71a

$$(T_1 + T_2) \times \frac{0.75}{1.25} = 4 \times \omega^2 \times 0.75$$

or $\frac{T_1 \quad T_2}{5} = \omega^2$

Now $\omega = 2\pi n$

$$\therefore n = \frac{\quad}{2}$$

or $\frac{42}{60} = \frac{\quad}{2}$

$$\therefore \omega = \frac{42 \quad 2}{60} \times \frac{22}{7} = 4.4 \text{ rad/s}$$

$$\omega^2 = 19.36 = \frac{T_1 \quad T_2}{5}$$

$$\therefore T_1 + T_2 = 96.80 \text{ N}$$

$$T_1 - T_2 = 49.0 \text{ N}$$

On solving the two we have $T_1 = 72.9$ N and $T_2 = 23.9$ N

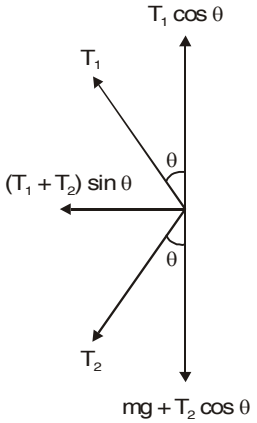


Fig. 4.71b

4.96. *ABC* is an inverted *L* shaped rod in which the length of the portion *BC* = 10.3 cm. A sphere of mass 500 g is suspended from the point *C* with the help of a string of length 100 cm (see Fig. 4.72a). At what frequency should the system rotate about *AB* for the string to make an angle of 60° with the vertical? What will be the tension in the string?

Solution: With reference to Fig. 4.72b, we have

$T \cos 60 = mg$

or $\frac{T}{2} = mg$

or $T = 2mg = 2 \times 0.5 \times 9.8 = 9.8$ N

Further $T \sin 60 = mr\omega^2$

or $2mg \times \frac{\sqrt{3}}{2} = mr\omega^2$

or $g\sqrt{3} = r\omega^2$; $r = BC + l \sin 60$;

l is the length of string

$$= 10.3 + 100 \cdot \frac{\sqrt{3}}{2} = 96.8 \text{ cm}$$

$$\therefore \omega = \sqrt{\frac{9.8 \quad 1.73}{96.8 \quad 10^{-2}}}$$

Now $n = \frac{\quad}{2} \times 60$

(here n in represented in rev/min)

$$= \frac{30}{} \sqrt{\frac{9.8 \quad 1.73 \quad 10^2}{96.8}} = 40 \text{ rev/min}$$

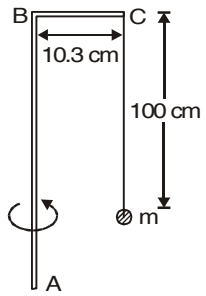


Fig. 4.72a

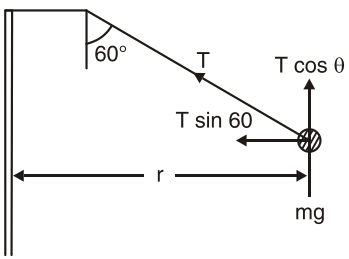


Fig. 4.72b

4.97. A solid sphere of radius $r = 5$ cm is rolling on a rough horizontal plane. Its centre O moves with constant acceleration $a = 5$ cm/s^2. After a lapse of time $t = 2s$, its position is represented by Fig. 4.73a. Find out the velocities and accelerations at point P, Q and R.

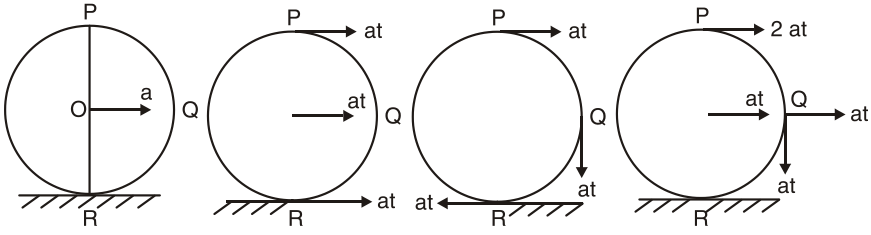


Fig. 4.73

Solution: From Fig. 4.73d the velocity of point

$P = 2at = 2 \times 5 \times 2 = 20$ cm/s

Velocity at point $Q = \sqrt{2}\ at = 1.41 \times 5 \times 2 = 14.1$ cm/s

Velocity at the point $R = 0$

Acceleration at point $P = \sqrt{(2a)^2 \quad \left(\frac{a^2t^2}{r}\right)^2}$

$= 2a\sqrt{1 \quad \left(\frac{at^2}{2r}\right)^2}$

$= 2 \times 5\sqrt{1 \quad \left(\frac{20}{10}\right)^2} = 10\sqrt{5} = 22.37$ cm/s^2

Acceleration at point $Q = \sqrt{a^2 \quad a \quad \left(\frac{a^2t^2}{r}\right)^2} = a\sqrt{1 \quad 1 \quad \left(\frac{at^2}{r}\right)^2}$

$= 5\sqrt{1 \quad 1 \quad \left(\frac{20}{5}\right)^2} = 5\sqrt{10} = 15.81$ cm/s^2

Acceleration at point $R = \frac{a^2t^2}{r} = \frac{25 \quad 4}{5} = 20$ cm/s^2

4.98. Two blocks of mass M_1 and M_2 are linked by a thread passing over a disc of mass M and radius R pivoted at O (see Fig. 4.74). The thread is slipping. Find the regular acceleration of the disc and the ratio of the two tensions.

Solution: Let us assume that the block of mass M_1 is going down. The equation of motion of the two blocks is given by

$M_1g - T_1 = M_1a$

and $T_2 - M_2 g = M_2 a$

where a is the common acceleration.

Due to the difference in tension, the disc will rotate. Now $(T_1 - T_2)\, R = I\alpha$ where I is the moment of inertia of the disc about an axis passing through O, $I = (1/2)\, MR^2$

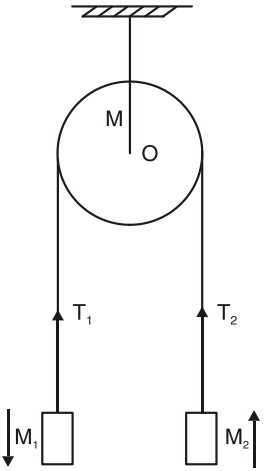


Fig. 4.74

α = angular acceleration = $\dfrac{a}{R}$

Now $(T_1 - T_2)R = [(M_1 - M_2)g - (M_1 + M_2)\, a]$

$R = \dfrac{1}{2}\, MR^2 \times \dfrac{a}{R}$

or $(M_1 - M_2)g = M_1 \quad M_2 \quad \dfrac{M}{2} \quad a$

or $a = (M_1 - M_2)g / \quad M_1 \quad M_2 \quad \dfrac{M}{2}$

or $\alpha = (M_1 - M_2)g / R \quad M_1 \quad M_2 \quad \dfrac{M}{2}$

Now $T_1 = M_1 g - \dfrac{M_1 (M_1 - M_2)g}{M_1 + M_2 + M/2}$

$$= \frac{M_1^2 g + M_1 M_2 g + M_1 (M/2)g - M_1^2 g + M_1 M_2 g}{M_1 + M_2 + M/2}$$

$$= [2M_1 M_2 + M_1 (M/2)]g / M_1 + M_2 + M/2$$

Similarly $T_2 = M_2 g + \dfrac{M_2 (M_1 - M_2)g}{M_1 + M_2 + M/2}$

$$= \frac{M_1 M_2 g + M_2^2 g + M_2 (M/2)g + M_1 M_2 g - M_2^2 g}{M_1 + M_2 + M/2}$$

$$= \frac{[2M_1 M_2 + M_2 (M/2)]g}{M_1 + M_2\ M/2}$$

$$\therefore \frac{T_1}{T_2} = \frac{M_1 \quad 4M_2 \quad M}{M_2 \quad 4M_1 \quad M}$$

4.99. A solid sphere and a solid cylinder having the same mass m and radius r roll down an incline plane. Initially they were at rest. (i) Which of the two bodies will reach the bottom first ? (ii) Which body will have the greater translational kinetic energy at the bottom.

Solution: (i) In view of Fig. 4.75 the equation of motion along the plane is given by: $mg \sin \theta - f_s = ma$; f_s = force of friction

Now $f_s\, r = I\alpha$ where I is the moment of inertia and α = angular acceleration = a/r

Further $a = r\alpha$

thus $f_s = \dfrac{Ia}{r^2}$

For sphere $I = \dfrac{2}{5} mr^2$

$\therefore f_s$ for sphere $= \dfrac{2}{5} ma$

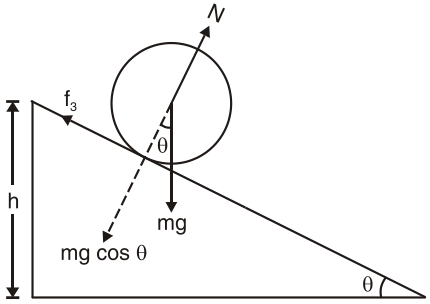


Fig. 4.75

Moment of inertia of a cylinder $= \dfrac{1}{2} mr^2$

$\therefore f_s$ for cylinder $= \dfrac{1}{2} ma$

The acceleration for the sphere can be obtained by the relation

$mg \sin\theta - \dfrac{2}{5}\ ma = ma$ or $a = \dfrac{5}{7}\ g \sin\theta$

Similarly we can determine the acceleration of cylinder as

$$a = \frac{2}{3} g \sin\theta$$

The acceleration of the sphere is more than that of the cylinder. Therefore, sphere will reach the bottom first.

(ii) In view of the law of conservation of energy

$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$$

For the sphere

$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} \times \frac{2}{5} mv^2 = \frac{7}{10} mv^2$$

For the cylinder

$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} \times \frac{1}{2} mv^2 = \frac{3}{4} mv^2$$

The rotational energy for the cylinder is more than that of the sphere. Thus the sphere will have more translational kinetic energy at the bottom.

4.100. A light string with a body of mass m tied to its end is wound on a pulley of radius $R = 10$ cm. Initially at rest, the body begins to descend with an acceleration $a = 5$ cm/s^2. (i) What will be the angular velocity of the pulley when the body has descended through a distance $s = 10$ cm? (ii) What will be the acceleration of a point O marked at the rim of the pulley at that moment (see Fig. 4.76).

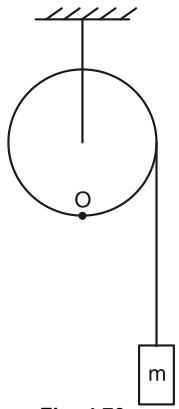


Fig. 4.76

Solution: The velocity v after a descent of 10 cm,

$$\sqrt{2as} = \sqrt{2 \quad 5 \quad 10} = 10 \text{ cm/s}$$

$$\text{Angular velocity } \omega = \frac{v}{R} = \frac{10}{10} = 1 \text{ rad/s}$$

$$\text{Acceleration of point } O = \sqrt{a^2 \quad \left(\frac{v^2}{R}\right)^2}$$

$$= \sqrt{25 \quad 100} = \sqrt{125} = 5\sqrt{5} \text{ cm/s}^2$$

4.101. A particle of mass $m = 500\ g$ resting on the top of a sphere of radius $R = 100$ cm begins to slide down along its surface. The sphere starts rotating about its vertical diameter with an angular velocity $\omega = 3$ rad/s. Calculate the centrifugal force at the time the particle breaks off from the surface of the sphere.

Solution: At the point of break-off there will be no reaction force and

$$mg\cos\theta = \frac{mv^2}{R}$$

$$\text{or} \quad g\cos\theta = \frac{v^2}{R};\ v = \sqrt{2gh}$$

$$\text{or} \quad \cos\theta = \frac{2h}{R}$$

$$\text{or} \quad \frac{R \quad h}{R} = \frac{2h}{R} \quad \text{(see Fig. 4.77)}$$

$$\text{or} \quad h = \frac{R}{3}$$

$$r = \sqrt{R^2 \quad \frac{4R^2}{9}} = R\sqrt{\frac{5}{9}}$$

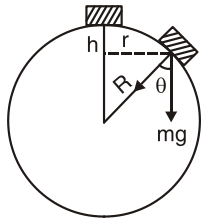


Fig. 4.77

Centrifugal force $mr\omega^2 = m\omega^2 R\sqrt{\frac{5}{9}}$

On substituting the values of m, ω and R, we have

$$\text{Centrifugal force} = 0.5 \times 9 \times 1 \times \frac{\sqrt{5}}{3} = 1.5 \times \sqrt{5} = 1.5 \times 2.237 = 3.35 \text{ N}$$

4.102. According to Bohr first postulate an electron of charge e and mass m moving around the nucleus experiences a force $F = ke^2/r^2$ where k is constant and r is electron nuclear distance. According to Bohr's second postulate, among infinite circular orbits only those are allowed in which the angular momentum L of the electron is equal to $\frac{nh}{2\pi}$. Calculate the possible radii of the circular orbits in which the electron can move.

Solution: The equation of motion of electron in the circular orbit is given by

$$F = \frac{ke^2}{r^2} = m\frac{v^2}{r}$$

or $$r = \frac{ke^2}{mv^2}$$

or $$1 = \frac{ke^2 mr}{(mvr)^2}$$

or $$(mvr)^2 = L^2 = \frac{n^2 h^2}{4\pi^2} = ke^2 mr$$

or $$r = \frac{n^2 h^2}{4\pi^2 kme^2}$$

4.103. A horizontal circular table of radius $r = 1$ m carrying a gun at its rim rotates about a vertical axis passing through O, the centre of the table (see Fig. 4.78). The combined moment of inertia about this axis is 19.9 kg m^2. During rotation of the table a bullet of mass $m = 100$ g is fired tangentially with a velocity $v = 200$ m/s. Calculate the increment in the angular velocity.

Solution: Let I be the combined moment of inertia of the circular table and gun about an axis passing through O. The initial angular momentum of the system is $(I + mr^2)\ \omega_1$

where ω_1 is the initial angular velocity.

After it is fired, the velocity of the bullet is $v - r\omega_2$

where ω_2 is the final angular velocity. Now final angular momentum of the system $= I\omega_2 - m\ (v - r\omega_2)r$

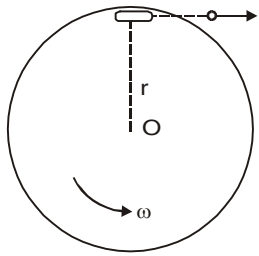


Fig. 4.78

In view of the law of conservation of angular momentum, we have

$$(I + mr^2)\ \omega_1 = I\omega_2 - m\ (v - r\omega_2)\ r$$

or $$(I + mr^2)\ \omega_1 = (I + mr^2)\ \omega_2 - mvr$$

or $$mvr = (I + mr^2)\ (\omega_2 - \omega_1)$$

$$\Delta\omega = \frac{mvr}{I \quad mr^2}$$

On substituting the values of m, v, r and I, we have

$$\Delta\omega = \frac{0.1 \quad 200 \quad 1}{19.9 \quad 0.1 \quad 1} = \frac{20}{20} = 1 \text{ rad/s}$$

4.104. A uniform wire is bent in the form of a circle of radius $r = 9.8$ cm. It is initially at rest and its diameter OA is horizontal. It is swinging about O in the vertical plane. Calculate its angular velocity ω when its diameter occupies the position OB (see Fig. 4.79).

Solution: On rotating the wire about O its centre of gravity is lowered by a distance r. The loss in potential energy = mgr; m being the mass of the wire.

This energy is balanced by the gain in kinetic energy $(1/2)\, I\omega^2$, where I is the moment of inertia about an axis passing through O.

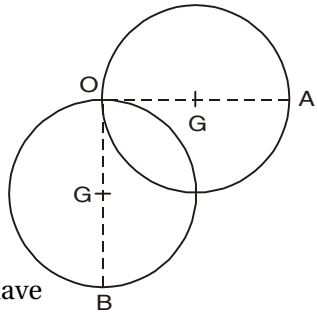


Fig. 4.79

$$I = mr^2 + mr^2 = 2mr^2$$

$$\therefore \quad \frac{1}{2} I\omega^2 = mr^2\omega^2 = mgr$$

or $$\omega = \sqrt{\frac{g}{r}}$$

On substituting the values of g and r, we have

$$\omega = \sqrt{\frac{9.8}{9.8 \quad 10^{\,2}}} = 10 \text{ rad/s}$$

4.105. Fig. 4.80 shows a rough track consisting of a straight portion and a curved portion of radius R. A solid sphere of mass m and radius r is placed on the straight portion of the track. What minimum speed should be imparted to the spheres at rest. So that it may go around the circle after rolling on the horizontal part of the track.

Solution:

$$\text{Total energy of the sphere at } A = \frac{1}{2} mv^2 + \frac{1}{2} \cdot \frac{2}{5} mr^2\omega^2$$

$$= \frac{1}{2} mv^2 + \frac{1}{5} mv^2$$

$$= \frac{7}{10} mv^2$$

Total energy of the sphere at B

= Translational kinetic energy + potential energy + rotational kinetic energy

$$= \frac{1}{2} mg(R-r) + mg\,2(R-r) + \frac{1}{2} \times \frac{2}{5} mg(R-r)$$

In view of conservation of energy we have

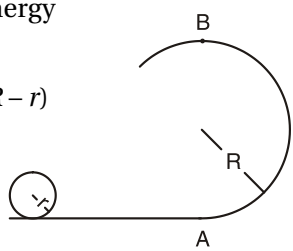


Fig. 4.80

$$\frac{7}{10} mv^2 = \frac{27g(R \quad r)}{10}$$

or $$v = \sqrt{\frac{27}{7} g(R \quad r)}$$

4.106. A ball of radius r initially at rest rolls without slipping down from the top of a sphere of radius R. Calculate the angular velocity of the ball at the instant it break off the sphere.

Solution: Total kinetic energy of ball $= \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2 = \frac{7}{10} mr^2\omega^2$

As the ball rolls down the gain in kinetic energy = loss in potential energy

or $\frac{7}{10} mr^2\omega^2 = mg(r + h)$ (see Fig. 4.81)

When the ball breaks off from the sphere, we have

$$\frac{mv^2}{R\ \ r} = mg\cos\theta$$

or $v^2 = (R + r)\, g \cdot \frac{x}{R\ \ r}$

$= gx$

$= g[R - h]$

Now $\frac{7}{10} mr^2\omega^2 = mgr + m[gR - v^2]$

$\frac{7}{10} mr^2\omega^2 + mr^2\omega^2 = mg(R + r);\ v = r\omega$

or $\omega = \sqrt{\frac{10g(R+r)}{17r^2}}$

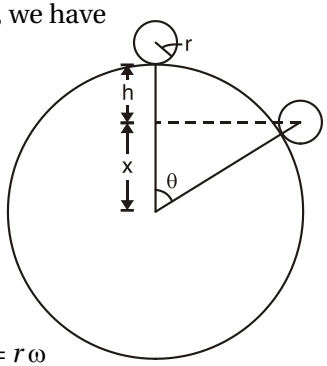


Fig. 4.81

4.107. A ball of mass m and radius $\frac{R}{4}$ is placed inside a spherical shell of mass m and radius R. The ball is released from the position a. Find the distance covered by the shell when the ball acquires the position b as shown in Fig. 4.82.

Solution: Here no external force is acting hence centre of mass will not change.

$$X_{cm} = \frac{mx \quad m\ \ x \quad R\ \ \frac{R}{4}}{2m}$$

$$= x - \frac{3R}{8}$$

Final centre of mass $X_{c_m} = \frac{mx \quad mx}{2m} = x'$

Now $x' = x - \frac{3R}{8}$

The displacement of the shell $= x - x' = \frac{3R}{8}$

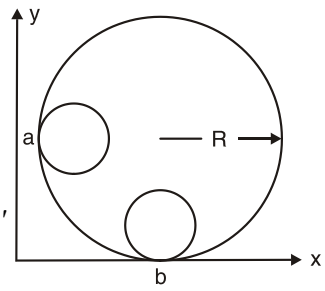


Fig. 4.82

4.108. A boy of mass m is at the edge of a horizontal disc of mass M and radius R. The disc is capable of rotating about a vertical axis passing through its centre. The boy starts moving along the edge of the disc with a velocity v which varies with time and stops after describing the angle θ relative to the disc. Find (i) the angle through which the disc has turned. (ii) the moment of the force with which the boy acted on the disc.

Solution: (i) Let ϕ_1 be the angle of rotation of the disc and ϕ_2 be the actual angle described by the boy, then $\theta = \phi_1 + \phi_2$

Further in the absence of the external force we have

$$\frac{\frac{1}{2}MR^2 \cdot \phi_1 - mR^2(\theta - \phi_1)}{\frac{1}{2}MR^2 + mR^2} = 0$$

$$\text{or} \quad \phi_1 = \frac{mR^2}{\frac{1}{2}MR^2 \quad mR^2}$$

$$= \frac{2m}{2m \quad M}$$

$$\text{(ii)} \quad \tau = \frac{d}{dt} \quad \frac{1}{2}MR^2 \quad = \frac{1}{2} MR^2 \cdot \frac{d^2}{dt^2}^{1}$$

$$= \frac{1}{2} MR^2 \cdot \frac{2m}{2m \quad M} \cdot \frac{d^2}{dt^2}$$

$$= \frac{mMR^2}{2m \quad M} \cdot \frac{d^2}{dt^2}$$

Now $\quad v = \frac{R \cdot d\theta}{dt}$

$$\therefore \frac{dv}{dt} = \frac{Rd^2}{dt^2}$$

$$\therefore \tau = \frac{mMR}{M + 2m} \cdot \frac{dv}{dt}$$

4.109. A man of mass 60 kg is standing at the edge of a disc of mass 100 kg and radius 1.5 m. The disc rotates at a rate of 10 rpm about a vertical axis passing through its centre. (i) With how many rotations per min the disc will rotate if the man moves from the edge of disc to its centre ? (ii) What work will be performed by the man in going from edge to the centre?

Solution:

(i) Here $I\omega$ should be constant.

$$\left(\frac{1}{2} \cdot 100r^2 + 60r^2\right) \times 2\pi \times 10 = \frac{1}{2} \times 100\, r^2 \times 2\pi n$$

$$n = \frac{110 \quad 10}{50} = 22 \text{ rpm}$$

$$\text{(ii)} \ \Delta E = \frac{1}{2} \times \frac{100\, r^2}{2} \times 4\pi^2 \times \quad \frac{22}{60}^{2} - \frac{1}{2} \quad \frac{1}{2} \quad 100r^2 \quad 60r^2 \quad \cdot 4\pi^2 \times \quad \frac{10}{60}^{2}$$

On putting $r = 1.5$ m

$\Delta E = 163.25$J

4.110. A boat is travelling with a speed of 27 km/hr due east. An observer is situated at 30 m south of the line of travel. Calculate the angular velocity of the boat relative to the observer in the position shown in Fig. 4.83.

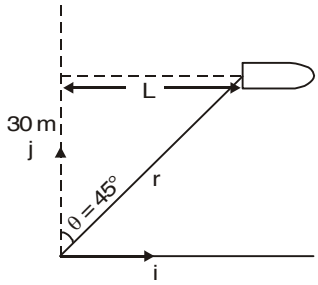


Fig. 4.83

Solution: $v = 27$ km/hr $= 7.5$ m/sec

In view of Fig. 4.83 we have

$r = Li + 30j$

$= 30 \tan\theta\, i + 30\, j$

Now $v = \dfrac{dr}{dt} = 30 \sec^2\theta \cdot \omega\, i$

$7.5 = 30 \times 2 \cdot \omega;\ \theta = 45°$

$\therefore\ \omega = \dfrac{7.5}{60} = 0.125$ rad/sec

4.111. A vehicle is moving towards south with an absolute velocity of 27 km/hr. An observer is positioned at O (to the east of line of travel) at a distance of 20 m. After moving 20 m it reaches point B see Fig. 4.84. What is the angular velocity relative to the observer.

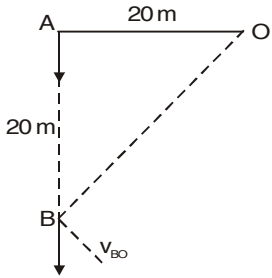


Fig. 4.84

Solution: 27 km/hr $= 7.5$ m/sec

$$v_{BO} = 27.5 \cos 45 = \frac{7.5}{\sqrt{2}}$$

$$v_{BO} = \frac{7.5}{\sqrt{2}} = OB \times \omega_{BO}$$

$$\therefore\ \omega_{BO} = \frac{7.5}{\sqrt{2}\ \ OB}$$

$$= \frac{7.5}{\sqrt{2}\ \ 20\sqrt{2}} = \frac{7.5}{40} = 0.188 \text{ rad/sec}$$

4.112. Two identical dumbells prepared by using four identical masses and two massless rods of length $l = 0.5$ m are arranged as shown in Fig. 4.85. They come towards each other with a linear velocity of $v = 2$ m/sec and collide inelastically. Find (i) angular speed of rotation (ii) the period during which the rotation occurs.

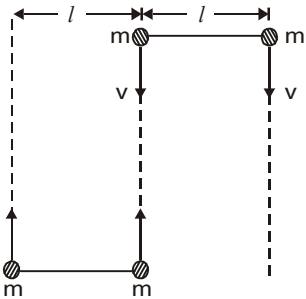


Fig. 4.85

Solution: (i) As a result of collision the two dumbells will rotate in the

clock-wise direction about their centres. The angular velocity of rotation is given by $\omega = \dfrac{v}{l/2} = \dfrac{2v}{l} = \dfrac{2\ \ 2}{0.5} = 8$ rad/sec

(ii) The time in which the rotation occurs $= \dfrac{\pi r}{v} = \dfrac{\pi r}{r\omega}$

$$= \frac{\pi}{\omega} = \frac{3.14}{8} = 0.4 \text{ sec}$$

4.113. Find the acceleration of an object *O* which moves without initial velocity along a helical grove of pitch *h* and radius *r* after completing *n* turns.

Solution: Velocity after *n* turns is given by relation

$$\frac{1}{2}\ mv^2 = mg \cdot nh$$

or $\qquad v = \sqrt{2gnh}$

Let θ be the inclination of the grove. The particle will execute circular motion with velocity $v \cos\theta$. The normal acceleration is given by

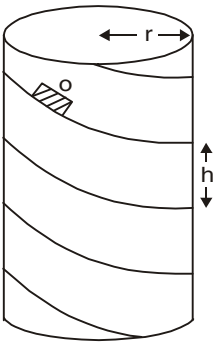


Fig. 4.86a

$$A_n = \frac{v^2 \cos^2\theta}{r}$$

In view of Fig. 4.86b; $\tan\theta = \dfrac{h}{2\pi r}$

$$A_n = \frac{v^2}{r} \cdot \frac{(2\pi r)^2}{h^2 + 4\pi^2 r^2}$$

$$A_t = g \sin\theta = \frac{gh}{\sqrt{h^2 + 4\pi^2 r^2}}$$

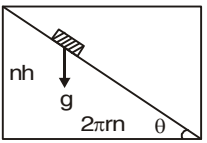


Fig. 4.86b

$$A = \sqrt{A_n^2 \quad A_t^2} = \sqrt{\frac{v^4 (2\pi r)^4}{r^2 \left(h^2 + 4\pi^2 r^2\right)^2} + \frac{g^2 h^2}{h^2 + 4\pi^2 r^2}}$$

$$= \frac{gh}{h^2 + 4\pi^2 r^2} \sqrt{64\,\pi^4 n^2 r^2 + h^2 + 4\pi^2 r^2}$$

4.114. A ball of radius $r = 0.25$ cm rolls without slipping on a horizontal table with a constant speed $v = \pi/10$ m/sec. The ball rolls off the edge of the table and falls vertically below a distance $h = 19.6$ cm. Calculate the number of revolution made by the ball while it is moving in air.

Solution: The time for free fall $= t = \sqrt{\dfrac{2h}{g}}$

The distance covered with velocity $v = v\sqrt{\dfrac{2h}{g}}$

In one revolution the distance covered by the ball = $2\pi r$

$\therefore$ no. of revolutions $n = \dfrac{v}{2\ r} \cdot \sqrt{\dfrac{2h}{g}}$

$$n = \frac{}{10} \times \frac{1}{2 \quad 0.25 \quad 10^{\,2}} \sqrt{\frac{2 \quad 19.6 \quad 10^{\,2}}{9.8}} = \frac{1}{0.25} = 4$$

4.115. A wheel of radius 2 m rolls to the right on a horizontal plateform with an angular velocity $\omega = 8$ rad/sec and an angular acceleration of $\alpha = 3$ rad/sec^2 in the anticlockwise direction. Calculate the linear velocity and acceleration of the point B marked on the rim.

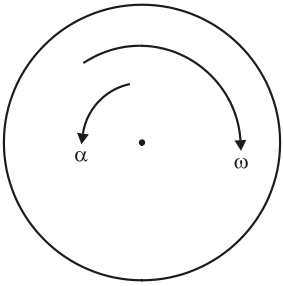


Fig. 4.87a

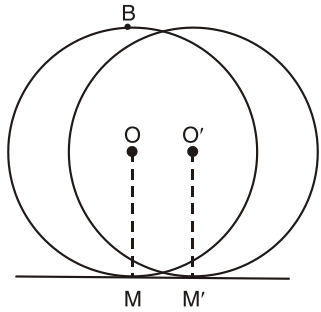


Fig. 4.87b

Solution: $OO' = r\theta$

Velocity of centre = $r\omega = 2 \times 8 = 16$ m/sec

$v_B = 2v_O = 32$ m/sec

$A_t = r\alpha = 2 \times 3 = 6$ m/sec^2

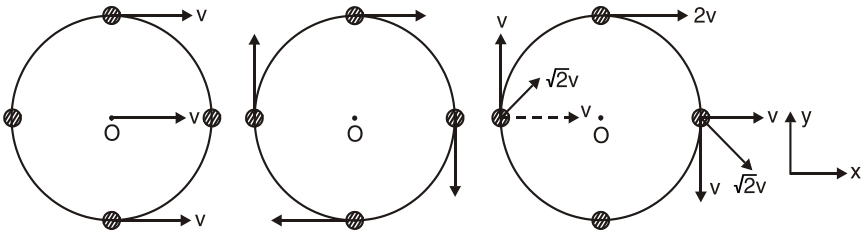
$A_n = r\omega^2 = 2 \times 64 = 128$ m/sec^2

Horizontal acceleration $A_h = A_o + A_t = 2 \times 3 + 6 = 12$ m/sec^2

Vertical acceleration $A_n = 128$ m/sec^2

$\therefore A = \sqrt{A_h^2 \quad A_n^2} = \sqrt{128^2 \quad 12^2} = 128.56$ m/sec^2

4.116. Four equal masses m are placed at quarter points of a thin massless rim of radius r. Find the angular momentum relative to the centre.



a b c

Fig. 4.88

Solution: $J = jr \times i\,2mv + ir \times -jmv + -ir \times jmv$

$= -2\,mvrk - mvrk - mvrk = -4\,mvrk$

4.117. An artificial satellite of mass m is moving in a circular orbit of radius r around earth. Express the angular momentum in terms of kinetic, potential and total energy of the satellite.

Solution: K.E. $= K = \frac{1}{2}\,mv^2$

$$\therefore\ v = \sqrt{2\frac{K}{m}}$$

$$J = \left(2mr^2K\right)^{\frac{1}{2}}$$

$$P.E. = U = -\frac{GM_e m}{r};\ \sqrt{\frac{U}{m}} = v$$

$$J = mr \cdot \sqrt{-\frac{U}{m}} = \left(-mr^2U\right)^{\frac{1}{2}}$$

Total energy $E = \frac{1}{2}\,mv^2 - mv^2 = -\frac{1}{2}\,mv^2$

$$J = mr \cdot \sqrt{\frac{-2E}{m}} = \left(-2mr^2E\right)^{\frac{1}{2}}$$

4.118. A solid sphere B of mass M and radius R is situated at the top of another sphere A of radius $2R$ as in Fig. 4.89. On displacing the sphere B it begins to roll down the surface of sphere A. Assuming the coefficient of sliding friction as μ find the angle θ at which the sphere B begins to slide on the surface of sphere A.

Solution: Let ϕ be the angle which describes the rotation of the sphere B about its own axis.

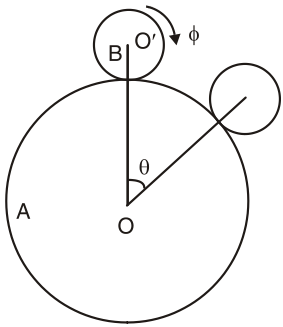


Fig. 4.89

The centre O' of sphere B moves in a circular path of radius $3R$.

The kinetic energy of sphere B is given by

$$E_K = \frac{1}{2}\,M(3R\dot{\ })^2 + \frac{1}{2}\cdot\frac{2}{5}\,MR^2\,\dot{\ }^2$$

$$= \frac{1}{2}\,MR^2\ \ 9\dot{\ }^2\ \ \frac{2}{5}\,\dot{\ }^2$$

Potential energy at an angle θ is given

$E_P = Mg \cdot 3R\cos\theta$

According to conservation of energy we have

$$3MgR = Mg \cdot 3R\cos\theta + \frac{1}{2}\,MR^2\ \ 9\dot{\ }^2\ \ \frac{2}{5}\,\dot{\ }^2$$

or $9\dot{\theta}^2 + \frac{2}{5}\dot{\theta}^2 = \frac{6g}{R}(1-\cos\theta)$

The centre of the sphere B moves with a velocity v_o given by

$v_o = 3R\dot{\theta} = R\dot{\theta}$

or $\dot{\theta} = 3\dot{\theta}$

$9\dot{\theta}^2 + \frac{2}{5}\cdot 9\dot{\theta}^2 = \frac{6g}{R}(1-\cos\theta)$

$\dot{\theta}^2 = \frac{10g}{21R}(1-\cos\theta)$ and $\ddot{\theta} = \frac{5g}{21R}\cdot\sin\theta$

Further $f_r \cdot R = \mu N \cdot R = \frac{2}{5}MR^2\ddot{\theta}$

The equation of motion of the sphere is

$Mg\cos\theta - N = M\cdot 3R\cdot\dot{\theta}^2$

$N = Mg\cos\theta - 3MR\cdot\frac{10g}{21R}(1-\cos\theta)$

$= \frac{Mg}{7}(17\cos\theta - 10)$

$\mu\cdot\frac{Mg}{7}(17\cos\theta-10) = \frac{6}{5}MR\ddot{\theta} = \frac{6}{5}\times MR\times\frac{5g}{21R}(\sin\theta)$

or $2\sin\theta = \mu(17\cos\theta - 10)$

$17\cos\theta - 10 = \frac{2\sin}{\quad} = 0$

$\theta = \frac{17}{10} = 54°$

4.119. A thin hoop of mass m and radius r starts from rest from A and rolls down along a cylindrical surface of radius R as in Fig. 4.90. Find the angle ϕ at which the hoop will begin to slip. Coefficient of friction $\mu = \frac{1}{3}$.

Solution: At the point B we will have

$mg\cos\phi - R_o = \frac{mv^2}{R\ \ r}$; R_o = reaction force

or $R_o = mg\cos\phi - \frac{mv^2}{R\ \ r}$

Force of friction = μR_o

$= \mu\ \ mg\cos\ \ \frac{mv^2}{R\ \ r}$

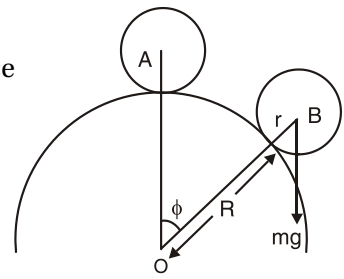


Fig. 4.90

Further $mg\sin\phi - \mu\ \ mg\cos\ \ \frac{mv^2}{R\ \ r} = mA$; A = acceleration

$\mu R_o \cdot r = I\alpha = mr^2\alpha$

$\mu R_o = mA$

In view of this

$$mg \sin \phi = 2 \quad mg \cos \quad \frac{mv^2}{R \quad r} \quad \mu$$

According to conservation of energy

$$\frac{1}{2} mv^2 + \frac{1}{2} \cdot I\omega^2 = mg\,(R + r)\,(1 - \cos \phi)$$

$$\frac{1}{2} mv^2 + \frac{1}{2} mv^2 = mv^2 = mg\,(R + r)\,(1 - \cos \phi)$$

$$3\, mg \sin \phi = 2\,[mg \cos \phi - mg\,(1 - \cos \phi)]$$

$$3 \sin \phi = 4 \cos \phi - 2$$

$$(3 \sin \phi + 2)^2 = 16\,(1 - \sin^2 \phi)$$

or $25 \sin^2 \phi + 12 \sin \phi - 12 = 0$

$$\sin \phi = -\frac{12 \pm \sqrt{144 + 25 \times 4 \times 12}}{50}$$

$$= 0.4932$$

$$\phi = 29° 32'$$

4.120. A cylinder of mass m and radius r rolls without slipping down an incline plane of mass M which lies on a frictionless horizontal table (see Fig. 4.91). The angle of inclination is θ. Find the acceleration of the plane while the rolling takes place.

Solution: Let the cylinder acquires a velocity v with respect to the wedge and V is the instantaneous horizontal velocity of the plane.

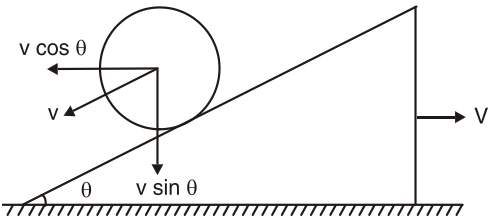


Fig. 4.91

$$MV = m\,(v \cos \theta - V)$$

$$\therefore v = \frac{(M \quad m)\,V}{m \cos}$$

Let the cylinder moves through x along the plane then

$$K = \frac{1}{2} MV^2 + \frac{1}{2} m \quad v \cos \quad V^{\,2} \quad v^2 \sin^2 \quad + \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} MV^2 + \frac{1}{2} mV^2 + \frac{3}{4} mv^2 - mvV \cos \theta$$

$$\frac{1}{2} (M + m)\, V^2 + \frac{3}{4} m \cdot \frac{(M \quad m)^2 V^2}{m^2 \cos^2} - \frac{mV \cos \theta\,(M + m)\,V}{m \cos \theta} = mgx \sin \theta$$

$$(M+m)V^2 \quad \frac{1}{2} \quad \frac{3(M \quad m)}{4m\cos^2} \quad 1 \quad = mgx\sin\theta$$

$$V^2 = \frac{4m^2 g\cdot\cos^2\theta\cdot\sin\theta\cdot x}{(M+m)\left[3(M+m)-2m\cos^2\theta)\right]}$$

$$= \frac{4m^2 g\cdot\cos^2\theta\cdot\sin\theta\cdot x}{(M+m)\left[3(M+m)-m(1+\cos 2\theta)\right]}$$

$$= \frac{4m^2 g\cdot\cos^2\theta\cdot\sin\theta\cdot x}{(M+m)(3M+2m-m\cos 2\theta)}$$

On differentiation we have

$$2V\cdot A = \frac{4m^2 g\cdot\cos^2\theta\cdot\sin\theta\cdot v}{(M+m)(3M+2m-m\cos 2\theta)}$$

$$= \frac{4m^2 g\cdot\cos^2\theta\cdot\sin\theta(M+m)V}{m\cos\theta\left[3M+2m-m\cos 2\theta\right](M+m)}$$

$$A = \frac{mg\sin 2\theta}{3M+2m-m\cos 2\theta}$$

4.121. A cylinder of mass m and radius r is rotated with an angular velocity ω_o about its own axis and is gently placed on the inclined plane. Find (i) the distance cylinder moves before sliding stops (ii) the maximum distance it moves on the slope. The coefficient of sliding friction between the cylinder and plane is μ ($\mu > \tan\theta$).

Solution: Along the plane we have

$\mu mg\cos\theta - mg\sin\theta = ma$

$\therefore a = (\mu\cos\theta - \sin\theta)\, g$

$V = at = g(\mu\cos\theta - \sin\theta)\, t$

$V > o$ or $\mu\cos\theta > \sin\theta$ or $\mu > \tan\theta$

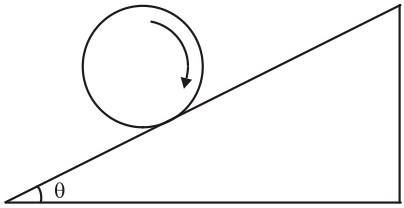


Fig. 4.92

Further $\mu\, mg\cos\theta\cdot r = \frac{1}{2} mr^2\cdot\alpha$

$$\text{or} \quad \alpha = \frac{2 \quad g\cos}{r}$$

now $\omega = \omega_o - a\, t_o$; t_o = the time at which the rolling begins.

$g(\mu\cos\theta - \sin\theta)\, t_o = \omega_0 r - 2\,\mu g\cos\theta\cdot t_o$

$$\therefore t_o = \frac{{}_o r}{g(3 \quad \cos \quad \sin \quad)}$$

The distance moved by the cylinder is given by

$$l_1 = \frac{1}{2}\, a \cdot t_o^2 = \frac{1}{2}\,(\mu \cos\theta - \sin\theta)\, g \cdot \frac{{}_o^2 r^2}{g^2\ 3\ \cos\ \ \sin^{\ 2}}$$

$$= \frac{{}_o^2 r^2\ \ \cos\ \ \sin}{2g\ 3\ \cos\ \ \sin^{\ 2}}$$

At the moment pure rolling set up the kinetic energy of the cylinder.

$$K.E. = \frac{1}{2}\, mv^2 + \frac{1}{4}\, mv^2 = \frac{3}{4}\, mv^2$$

$$v = at_o = (\mu \cos\theta - \sin\theta)\, g \cdot \frac{{}_o r}{g\ 3\ \cos\ \ \sin}$$

$$= \frac{\cos\ \ \sin\ \ {}_o r}{3\ \ \cos\ \ \sin}$$

$K.E. = mgl_2 \sin\theta$; l_2 = further moved distance

$$l_2 = \frac{3mv^2}{4mg\sin}$$

$$= \frac{3}{4} \cdot \frac{{}_o^2 r^2\ \ \cos\ \ \sin^{\ 2}}{3\ \ \cos\ \ \sin^{\ 2} g\sin}$$

$$l = l_1 + l_2 = \frac{{}_o^2 r^2\ \ \cos\ \ \sin}{2g\ 3\ \cos\ \ \sin^{\ 2}} + \frac{3}{4}\ \frac{{}_o^2 r^2\ \ \cos\ \ \sin^{\ 2}}{3\ \ \cos\ \ \sin^{\ 2} g\sin}$$

$$= \frac{{}_0^2 r^2\ \ \cos\ \ \sin}{4g\sin\ \ 3\ \ \cos\ \ \sin}$$

4.122. A hollow cylindrical disc of mass m and radius r moving with a constant speed v_o comes to an incline plane making an angle θ with the horizontal. If there is no slipping then to what height it will rise up the plane.

Solution: When it is going up the plane the equation of motion is given by

$$-(mg\sin\theta + f_s) = ma;\ a = \text{acceleration}$$

Further $mr^2 \cdot \alpha = f_s \cdot r$

$$\therefore f_s = ma$$

$$\therefore a = -\frac{g\sin}{2}$$

Now $$\frac{1}{2}\, mv_0^2 = m \cdot \frac{g\sin}{2} \cdot h$$

$$\therefore h = \frac{v_0^2}{g\sin}$$

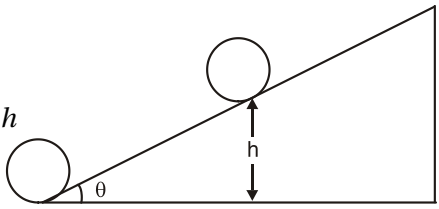


Fig. 4.93

4.123. In the above problem calculate the height h if the disc is replaced by solid sphere of mass m and radius r.

Solution: The equation of motion is given by

$$-(mg\sin\theta + f_s) = ma$$

Further $\frac{2}{5}\, mr^2 \cdot \alpha = f_s \cdot r$

$$\frac{2}{5}\, ma = f_s$$

$$a = \frac{5}{7}\, g\sin\theta$$

Now $\frac{1}{2}\, m v_0^2 = m \cdot \frac{5}{7}\, g\sin\theta \cdot h$

$$h = \frac{7v_0^2}{10g\sin\theta}$$

4.124. A solid sphere of radius r and mass m rolls down an incline without sliding. Find the amount of work done by the force of friction with the minimum value of μ.

Solution: Equation of motion of the sphere is given by

$$mg\sin\theta - \mu\, mg\cos\theta = ma$$

Further $\frac{2}{5}\, mr^2 \cdot \alpha = \mu\, mg\cos\theta \cdot r$

$$\frac{2}{5}\, ma = \mu\, mg\cos\theta$$

$$a = \frac{5}{7}\, g\sin\theta,\ \alpha = \frac{5}{7r}\, g\sin\theta$$

In view of conservation of energy

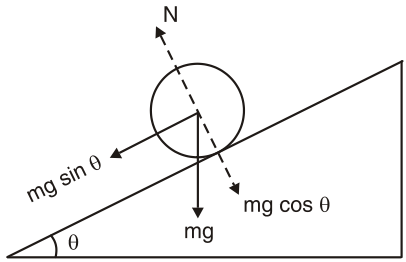


Fig. 4.94

$$W + mgh = \frac{1}{2}\, mv^2 + \frac{1}{2}\, I\omega^2 = \frac{7}{10}\, mv^2;\ W = \text{work due to force of friction}$$

$$W + mg\, l\sin\theta = \frac{7}{10}\, m \cdot 2\, al = \frac{7}{10}\, m \cdot 2\, l \times \frac{5}{7}\, g\sin\theta = mgl\sin\theta;$$

l = distance covered along the plane

$$\therefore\ W = 0$$

4.125. A particle of mass m is thrown at an angle α to the horizontal with an initial velocity v_o. The path of the particle lies in xy plane (see Fig. 4.95a). Find (i) the torque acting on the particle at an instant t (ii) the angular momentum of the particle. Evaluation should be made relative to the point of projection.

Solution:

(i) The displacement of the particle along x axis is $(v_o \cos\alpha \cdot t)\ i$

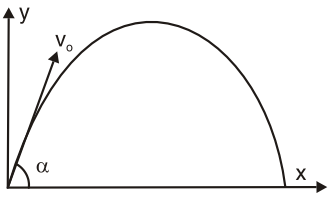


Fig. 4.95a

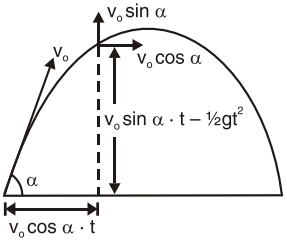


Fig. 4.95b

Force acting on the particle = $(mg) \cdot -j$

$\tau = r \times F = (v_o \cos \alpha \cdot t) \times mg\,(i \times -j)$

$= -k\, mgv_o \cos \alpha \cdot t$

(ii) In view of Fig. 4.95b

$L_1 = (v_o \cos \alpha \cdot t)\; i \times mv_o \sin \alpha \cdot j$

$= m v_0^2 \sin \alpha \cos \alpha \cdot t \cdot k$

$$L_2 = \left(v_o \sin \alpha \cdot t - \frac{1}{2} gt^2 \right) j \times mv_o \cos \alpha \cdot i$$

$$= -\left(mv_0^2 \sin \alpha \cdot \cos \alpha \cdot t - \frac{1}{2} mv_0 g \cos \alpha \cdot t^2 \right) k$$

Net angular momentum = $\frac{1}{2}\; mv_o\, g \cos \alpha \cdot t^2 \cdot k$

4.126. A body of mass m = 200 gm is thrown from a height h in a horizontal direction with a speed of v_o = 25 m/sec. Calculate the increase in the angular momentum of the body relative to the point of projection in the time t = 3 sec.

Solution: In the period of t sec it will fall through $\frac{1}{2}\; gt^2$.

$$\Delta L = \frac{1}{2}\; gt^2 \times -j \times mv_o i$$

$$= \frac{1}{2}\; mv_o\, gt^2 \cdot k$$

On substituting the values we have

$$\Delta L = \frac{1}{2} \times 0.2 \times 25 \times 9.8 \times 9$$

$= 220.5$ kg m^2/sec.

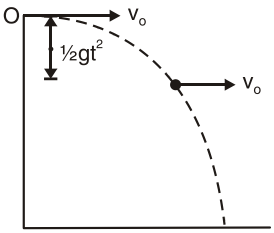


Fig. 4.96

4.127. A particle of mass 0.5 kg is thrown vertically upward along z axis with a speed of v = 20 m/sec from the point (0, 3, 0). Find the increase in angular momentum relative to the origin during the entire time of flight.

Solution:

Initial angular momentum = $3j \times (0.5 \times 20k) = 30i$

Final angular momentum = $3j \times (0.5 \times 20 \times -k) = -30i$

Increase $\Delta L = -30i - 30i = -60i$

4.128. An object of mass m slides down an incline without friction. Relative to the point O determine (i) the torque on the body (ii) the angular momentum after a lapse of time t.

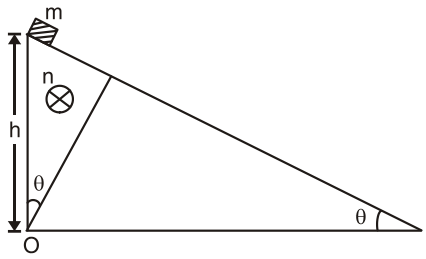


Fig. 4.97

Solution:

(i) $\tau = r \times F$

$= h \cos\theta \times mg \sin\theta \cdot n$;

n is the normal directed away from the paper.

$= \frac{1}{2} mgh \sin 2\theta \cdot n$

(ii) $J = h \cos\theta \times mg \sin\theta \cdot t \cdot n$

$= \frac{1}{2} mgh \sin 2\theta \cdot t \cdot n$

4.129. Two particles of mass m_1 and m_2 are moving rectilinearly in opposite direction with velocities v_1 and v_2 as shown in Fig. 4.98. The distance between the path followed by the two particles is L. Find the total angular momentum of the particle relative to the points 1 and 2.

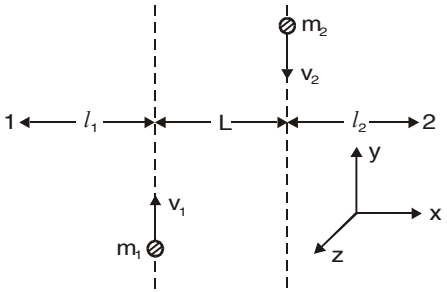


Fig. 4.98

Solution:

$L_1 = l_1\, i \times m_1 v_1 j + (l_1 + L)\, i \times m_2 v_2 \times -j$

$= (m_1 v_1 l_1 - m_2 v_2 l_1 - L\, m_2 v_2)\, k$

$[l_1 (m_1 v_1 - m_2 v_2) - L m_2 v_2]\, k$

$L_2 = -l_2\, i \times -m_2\, v_2\, j - (l_2 + L)\, i \times m_1 v_1 j$

$= [l_2 m_2 v_2 - l_2 m_1 v_1 - L\, m_1 v_1]\, k$

$= [l_2 (m_2 v_2 - m_1 v_1) - L m_1 v_1]\, k$

4.130. Due to cooling the linear dimensions of a solid homogenous sphere uniformly rotating about its diameter diminishes by 0.1%. What effect will this have on the angular velocity?

Solution: According to conservation of angular momentum

$$\frac{2}{5} mr^2\omega = \frac{2}{5} m (r - dr)^2 \cdot$$

$$\omega = 1 \quad \frac{2dr}{r}$$

$$= \frac{}{1 \quad \frac{2dr}{r}} = \omega\left(1+\frac{2dr}{r}\right)$$

Now $\frac{}{} \times 100 = 2 \times \frac{dr}{r} \times 100 = 0.2\%$

4.131. A cylinder of mass m and radius r is placed on a horizontal plateform which is moving with a constant acceleration A. In the absence of slip determine the acceleration of the centre of the cylinder. If $\mu = 0.3$ then what will be the maximum acceleration of the plateform without slip between them?

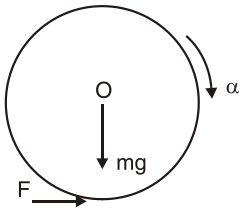


Fig. 4.99

Solution: $A_o = r\alpha + A$

Further $F = mA_o$

Taking the moment about O (see Fig. 4.99).

$$-mA_o r = \frac{1}{2} mr^2 \alpha$$

$$-2A_o = r\alpha = A_o - A$$

$$\therefore A = 3A_o$$

or $A_o = \frac{A}{3}$

$$F = 0.3\, mg = m \cdot \frac{A}{3}$$

$$\therefore A = 0.3 \times 3 \times 9.8$$

$$= 8.82 \text{ m/sec}^2$$

4.132. A solid sphere of mass m and radius r is released from rest from the top of a rough incline plane. The coefficient of friction is μ. What maximum angle the slope will have with the horizontal for the sphere to roll down the plane without sliding?

Solution: Let the required angle be θ. The equation of motion is given by

$$mg \sin\theta - \mu\, mg \cos\theta = ma$$

Further $\frac{2}{5} mr^2\alpha = \mu mg \cos\theta \cdot r$

or $ma = \frac{5}{2} \mu\, mg \cos\theta$

$$mg \sin\theta = \frac{7}{2} \mu\, mg \cos\theta$$

$$\tan\theta = \frac{7\mu}{2}$$

$$\theta = \tan^{-1}\left(\frac{7\mu}{2}\right)$$

4.133. A bar AB of length $2l$ and mass m is rotating with an angular velocity ω about the shaft OO' (see Fig. 4.100). The axis of the bar makes an angle α with the shaft. Find the torque acting on the shaft.

Solution: Consider elements of length dx at a distance x from the axis. Force acting on both the elements is equal to

$$\frac{m}{2l} \cdot dx \cdot x \cdot \sin\alpha \cdot \omega^2$$

Torque acting on the element

$$d\tau = \frac{m}{2l} \cdot dx \cdot x \cdot \sin\alpha \cdot \omega^2 \cdot 2x \cos\alpha$$

$$d\tau = \frac{m}{2l} \cdot \sin 2\alpha \cdot \omega^2 \cdot x^2 dx$$

Net torque $\tau = \int d\tau$

$$= \frac{m}{2l} \sin 2\alpha \cdot \omega^2 \int_0^l x^2 dx$$

$$= \frac{ml^2\omega^2}{6} \sin 2\alpha$$

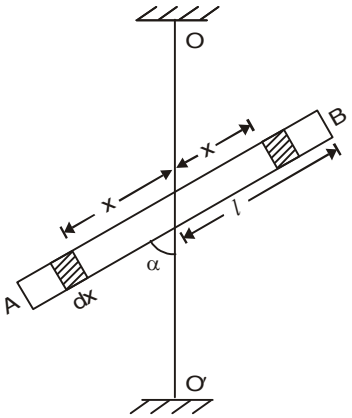


Fig. 4.100

4.134. A point mass m is attached at B at the rim of a disc of mass m and radius r. The disc rolls without sliding on a horizontal plane. During motion the angular velocity of the disc is 4ω when point B is directly below A. Find the angular velocity ω of the disc when the point B is directly above the centre A.

Solution: The centre of mass is at O and is at a distance of $0.5\,r$ from the centre A (see Fig. 4.101).

The total moment of inertia $= 2m \cdot (0.5\,r)^2 = 0.5\,mr^2$

According to conservation of energy

$$\frac{1}{2}[0.5\,mr^2]\,16\omega^2 + \frac{1}{2} \times 2m\,(0.5\,r \times 4\omega)^2$$

$$= \frac{1}{2}[0.5\,mr^2]\omega^2 + \frac{1}{2} \times 2m\,(0.5\,r\omega)^2 + 2mgr$$

$$4\,mr^2\omega^2 + 4\,mr^2\omega^2 - 0.25\,mr^2\omega^2$$

$$- 0.25\,mr^2\omega^2 = 2mgr$$

$$7.5\,mr^2\omega^2 = 2mgr$$

$$\omega = \sqrt{\frac{2g}{7.5r}}$$

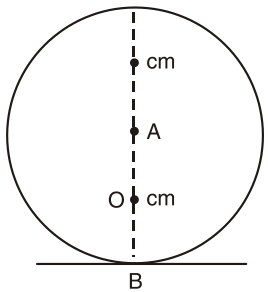


Fig. 4.101

4.135. A bar of mass m and L can freely rotate about an axis passing through the point O. The bar is deflected from the vertical by an angle ϕ

and released. Calculate the velocity of the lower end of the bar when it makes an angle θ from the vertical.

Solution: According to conservation of energy

$$mgh_1 = mgh_2 + \frac{1}{2} I\omega^2$$

$$mg \quad \frac{L}{2} \quad 1 \quad \cos \qquad \frac{L}{2} \quad 1 \quad \cos$$

$$= \frac{1}{2} \cdot \frac{mL^2}{3} \cdot \omega^2$$

$$mg\,[\cos\theta - \cos\phi] = \frac{mL}{3}\,\omega^2$$

$$\text{or} \quad \omega = \sqrt{\frac{3g}{L} \cdot (\cos\theta - \cos\phi)}$$

$$v = L\omega = \sqrt{3gL \quad \cos \qquad \cos}$$

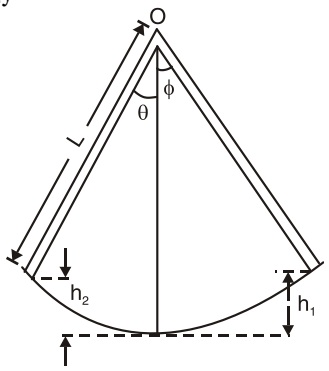


Fig. 4.102

4.136. A solid sphere of mass m and radius r rolls down an incline plane of length L and slope angle θ. Calculate the speed of the centre of mass of the sphere when it reaches at the bottom of the plane.

Solution: According to conservation of energy

$mg\,[L \sin\theta + r\cos\theta]$ – work against force of friction $= \frac{1}{2}\ mv^2 + \frac{1}{2} \times \frac{2}{5}\ mr^2\omega^2$

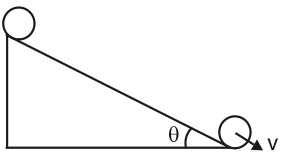


Fig. 4.103

$$mg\,[L\sin\theta + r\cos\theta] - \mu\, mg\cos\theta \cdot L = \frac{7}{10}\ mv^2$$

$$v = \sqrt{\frac{10g}{7}\ L\sin \qquad r\cos \qquad L\cos}$$

4.137. A fly wheel of mass m and radius r is rotating about its own axis with an angular velocity ω. What force should be applied to stop the wheel in t sec? Coefficient of friction is μ.

Solution: Loss in kinetic energy should be equal to work against friction.

$$\frac{1}{2} I\omega^2 = \frac{mr^2\omega^2}{4} = \mu \cdot F \cdot \frac{\omega r}{2} \cdot t;\ \frac{\omega r}{2} = \text{average velocity}$$

$$\therefore F = \frac{mr}{2\ t}$$

4.138. A linear velocity V is given to a hoop of mass m and radius r placed on a rough horizontal surface. Calculate the angular velocity of the hoop when it stops slipping.

Solution: $m\,(V - v) = at$; a = acceleration

$$\frac{mV^2}{2} - mv^2 = a\,\frac{V}{2}\cdot t$$

On dividing the two we have

$$\frac{V \quad v}{V^2 \quad 2v^2} = \frac{1}{V}$$

$$V^2 - Vv = V^2 - 2v^2$$

$$v = \frac{V}{2}$$

$$\omega = \frac{v}{r} = \frac{V}{2r}$$

4.139. A sphere of mass m and radius r is projected along a rough horizontal plane with an initial velocity v_o. Calculate its speed when it will roll without sliding.

Solution: $\frac{2}{5}\,mr^2\alpha = \mu mgr$

$$\alpha = \frac{5\ g}{2r}$$

acceleration $a = \dfrac{mg}{m} = \mu g$

now $v = v_o - \mu gt$ and $\omega = \omega_o + \alpha t$

$$v_o - \mu gt = r\,\omega_o + r\alpha t = \frac{5}{2}\ \mu gt\ ;\ \omega_o = 0$$

or $$t = \frac{2v_o}{7\ g}$$

$$v = v_o - \mu g\cdot\frac{2v_o}{7\ g} = \frac{5}{7}\ v_o$$

4.140. A solid sphere of mass m and radius r rotating with an angular velocity ω_o is placed gently on a rough incline plane of angle θ and coefficient of friction μ. The sphere moves up the plane. Determine (i) the time at which the sphere starts to roll without slipping up the plane (ii) the distance through which the ball subsequently rolls without slipping up the plane.

Solution: When the sphere makes contact with the plane then slipping occurs till $v = r\omega$

The equation of motion is given by

$\mu mg\cos\theta - mg\sin\theta = ma$

or $a = \mu\, g\cos\theta - g\sin\theta$

Further $\mu\ mg\cos\theta \times r = \frac{2}{5}\ mr^2\,\alpha$

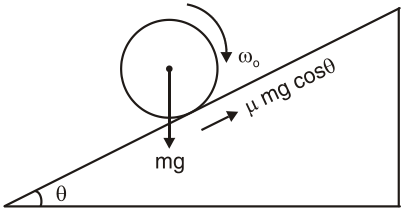


Fig. 4.104a

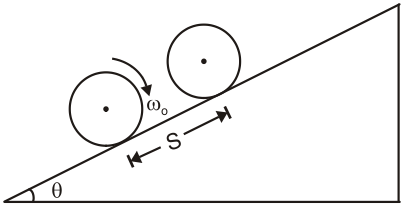


Fig. 4.104b

or $\mu g \cos\theta = \frac{2}{5} r\alpha$

$\omega = \omega_o - \alpha t$

or $\frac{o}{t} = \alpha$

$\mu \cos\theta - g \sin\theta = a = \frac{v}{t} = \frac{r}{t}$

$\mu g \cos\theta = \frac{2}{5} r \cdot \frac{o}{t}$

$= \frac{2}{5} \frac{_0 r}{t} \quad g\cos \quad g\sin$

or $\frac{7}{5} \mu g \cos\theta - \frac{2}{5} g \sin\theta = \frac{2}{5} \frac{_o r}{t}$

or $t = \frac{2 \; _o r}{g \; 7 \; \cos \quad 2\sin}$

$v = at = (\mu g \cos\theta - g \sin\theta) \times \frac{2 \; _o r}{g \; 7 \; \cos \quad 2\sin}$

$= \frac{2 \; _o r \quad \cos \quad \sin}{7 \quad \cos \quad 2\sin}$

The ball continues to move up till loss in K.E. = gain in P.E.

$mg S \sin\theta = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$

$2mg S \sin\theta = \frac{7}{5} mv^2$

or $S = \frac{7v^2}{10 g \sin\theta}$

$= \frac{7}{10 g \sin\theta} \left[\frac{2 \omega_o r (\mu \cos\theta - \sin\theta)}{7 \mu \cos\theta - 2 \sin\theta} \right]^2$

$= \frac{14}{5} \frac{^2_0 r^2}{g \sin} \cdot \frac{\cos \quad \sin}{7 \quad \cos \quad 2\sin}^2$

4.141. A solid disc of radius r has a circular hole of radius r_1 whose centre is at a distance l from the centre of the disc. Find its acceleration if it rolls down an incline plane of angle θ.

Solution: Let m be the mass of the disc

Mass of the hole created = $\frac{m}{\pi r^2} \pi r_1^2 = m \frac{r_1^2}{r^2}$

Mass of the remaining portion of the disc = $\frac{m}{r^2} r^2 r_1^2$

Moment of inertia of the portion taken out of the disc about its centre

$= \frac{1}{2} m \frac{r_1^4}{r^2}$

Moment of inertia of the portion taken out of the disc about the centre of the disc = $\frac{1}{2} m \frac{r_1^4}{r^2} + m \frac{r_1^2 l^2}{r^2}$

$= \frac{1}{2} m \frac{r_1^2}{r^2} r_1^2 l^2$

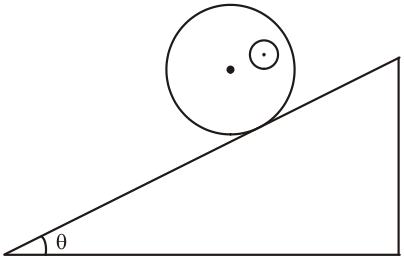


Fig. 4.105

Moment of inertia of the disc with hole = $\frac{1}{2} mr^2 - \frac{1}{2} m \frac{r_1^2}{r^2} r_1^2 l^2$

Equation of motion of the disc moving down the plane

$\frac{m}{r^2} r^2 r_1^2 g \sin\theta - \mu \frac{m}{r^2} r^2 r_1^2 g \cos\theta = \frac{m}{r^2} r^2 r_1^2 A;$

A = acceleration

Further $\mu \frac{m}{r^2} r^2 r_1^2 g \cos\theta \cdot r = \frac{m}{2} \frac{r^4 r_1^4 r_1^2 l^2}{r^3} \alpha \cdot r$

$\therefore \mu \frac{m}{r^2} r^2 r_1^2 g \cos\theta = \frac{m}{2r^4} r^4 r_1^4 r_1^2 l^2 A$

Hence $\frac{m}{r^2} r^2 r_1^2 g \sin\theta = \frac{m}{r^2} r^2 r_1^2 \frac{m}{2r^4} r^4 r_1^4 r_1^2 l^2 A$

$A = \frac{2 r^2 r_1^2 r^2 g \sin}{3r^4 2r^2 r_1^2 r_1^2 l^2 r_1^4}$

4.142. A hemisphere of mass m and radius r is released from the position as shown in Fig. 4.106a and acquires the position as shown in Fig. 4.106b. Assuming that the hemisphere rolls without sliding determine (i) the angular velocity of the sphere after it has rolled through 90° (ii) the normal reaction force at the surface at the same instant.

Solution: (i) Moment of inertia of the hemisphere about an axis passing through $O = \frac{2}{5} mr^2$

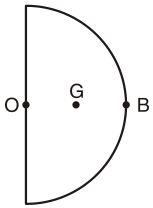


Fig. 4.106a

According to the theorem of parallel axis

$$I_G + m \cdot \left(\frac{3r}{8}\right)^2 = \frac{2}{5} mr^2; OG = \frac{3r}{8}$$

$$\therefore I_G = \frac{2}{5} - \frac{9}{64} mr^2 = \frac{83}{320} mr^2$$

Energy coservation leads to

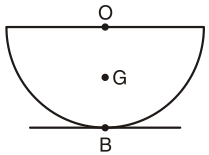


Fig. 4.106b

$$\frac{1}{2} \cdot \frac{83}{320} mr^2\omega^2 + \frac{1}{2} m \cdot \frac{25}{64} r^2\omega^2 = \frac{3}{8} r \cdot mg$$

$$\omega^2 = \frac{240}{208} \frac{g}{r}$$

or $\omega = 1.074 \sqrt{\frac{g}{r}}$

(ii) $N = mg + m \cdot \frac{3r}{8} \times \omega^2$

$$= mg + m \cdot \frac{3r}{8} \cdot \frac{240}{208} \cdot \frac{g}{r}$$

$$= 1.433 \text{ mg}$$

4.143. A half cylinder of mass m and radius r is released from rest from the position shown in Fig. 4.107. Assuming that half cylinder rolls without sliding determine (i) the angular velocity of half cylinder after it has rolled through 90° (ii) the normal reaction force at the surface at the same instant.

Solution:

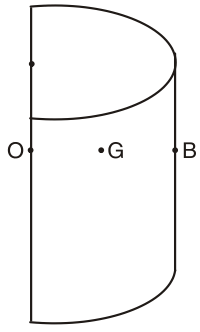


Fig. 4.107

(i) $OG = \frac{4r}{3\pi}$, $\therefore GB = \frac{(3\pi - 4)r}{3\pi}$

Now $I_G = \frac{1}{2} mr^2 - m \cdot \frac{16r^2}{9\pi^2} = 0.32\, mr^2$

According to conservation of energy

$$\frac{1}{2} \times 0.32\, mr^2\omega^2 + \frac{1}{2} m \left(\frac{3\pi - 4}{3\pi}\right)^2 r^2\omega^2 = \frac{4r}{3\pi} \cdot mg$$

$$\omega^2 = \frac{0.4246}{1.67} \frac{g}{r}$$

$$\omega = 0.5 \sqrt{\frac{g}{r}}$$

(ii) $N = mg + m \cdot \frac{4r}{3\pi} \cdot \omega^2$

$= mg + m \cdot \frac{4r}{3\pi} \cdot \frac{0.4246}{1.67} \cdot \frac{g}{r}$

$= 1.1\ mg$

4.144. A hemisphere of mass m and radius r is released from rest in the position shown in Fig. 4.108. Find (i) the minimum value of μ for which the hemisphere starts to roll without sliding, (ii) the corresponding acceleration of point B.

Solution:

Fig. 4.108

(i) $\mu mg \times \frac{3r}{8} = \frac{2}{5} mr^2\alpha$

$\frac{3\mu g}{8} = \frac{2}{5} a$; a = acceleration of the centre.

$I_G = \frac{2}{5} mr^2 - m \left(\frac{3r}{8}\right)^2 = \frac{83}{320} mr^2$

$I_A = \frac{83}{320} \cdot mr^2 + \frac{73}{64} mr^2 = \frac{448}{320} mr^2 = 1.4\ mr^2$

Now $1.4\ mr^2\alpha = \frac{3}{8} mgr$

$\therefore r\alpha = \frac{30g}{14 \quad 8} = \frac{15g}{56}$

With this value of $a = r\alpha = \frac{15g}{56}$ we have

$\mu = \frac{16}{15g} \times \frac{15g}{56} = \frac{2}{7} = 0.2857$

(ii) Acceleration of the point $B = 2r\alpha = \frac{15g}{28}$

4.145. Solve the above problem by using a half cylinder of length l, mass m and radius r instead of hemisphere.

Solution:

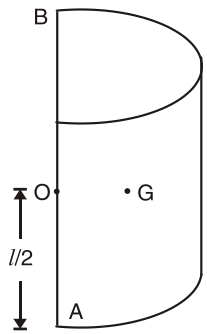


Fig. 4.109

(i) $OG = \frac{4r}{3}$

$\mu mg \cdot \frac{4r}{3} = \frac{1}{2} mr^2\alpha$

$\therefore \mu = \frac{3 \quad r}{8g}$

Now $I_G = \frac{1}{2} mr^2 - m \cdot \left(\frac{4r}{3}\right)^2$

$= mr^2 \left(0.5 - \frac{16}{9^2}\right) = 0.32\, mr^2$

$I_A = 0.32\, mr^2 + \frac{l^2}{4}\left(\frac{16r^2}{9^2}\right) m = 0.5\, mr^2 \quad \frac{ml^2}{4}$

Now $0.5\, mr^2 \quad \frac{ml^2}{4} \quad \alpha = mg \cdot \frac{4r}{3}$

$\therefore \alpha = \frac{16gr}{3 \quad 2mr^2 \quad ml^2}$

With this value of α we have

$\mu = \frac{3}{8g} \cdot \frac{r \quad 16gr}{3 \quad 2mr^2 \quad ml^2} = \frac{6r^2}{2mr^2 \quad ml^2}$

(ii) Acceleration of the point $B = l\alpha = \frac{16grl}{3 \quad 2mr^2 \quad ml^2}$

4.146. A rotor of weight W and radius r rests in a V shaped support as shown in the Fig. 4.110. The coefficient of friction at the contact is μ. Find the couple required to produce rotation (i) by way on maintaining the contact at the point O, (ii) in the absence of contact at point O.

Solution:

(i) Torque $T_1 = (\mu R_1 + \mu R_2)\, r$; R_1 and R_2 are reactions.

$W = (R_1 + R_2) \sin\theta + \mu\,(R_2 - R_1) \cos\theta$

In the horizontal direction $(R_2 - R_1) \cos\theta = \mu\,(R_1 + R_2) \sin\theta$

$\therefore W = (R_1 + R_2)\,(1 + \mu^2) \sin\theta$

$\therefore T_1 = \frac{\mu \cdot W \cdot r}{\left(1 + \mu^2\right) \sin\theta}$

(ii) $R_1 = 0 \;\therefore T = \mu R_2 r$

$W = R_2\,(\sin\theta + \mu \cos\theta)$

or $R_2 = \frac{W}{\sin \quad \cos}$

In the horizontal direction

$R_2 \cos\theta = \mu\, R_2 \sin\theta$

or $\mu = \cot\theta$

$\therefore T = \frac{\cot\theta \cdot W \cdot r}{\sin\theta + \cos^2\theta / \sin\theta} = Wr \cos\theta$

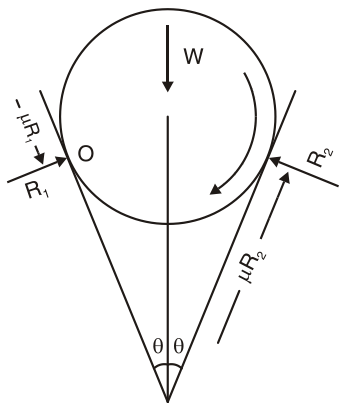


Fig. 4.110

4.147. A uniform disc of radius r_0 lies on a smooth horizontal plane. A similar disc spinning with the angular velocity ω_0 is gently lowered on the first one. How soon the two discs spins with the same velocity.

Solution: Consider a ring of radius r and $r + dr$ (Fig. 4.111)

Mass of the ring $= \dfrac{m}{r_0^2} \cdot 2\pi r dr$

Force of friction $df = \dfrac{m}{r_0^2} \cdot 2r dr \cdot \mu g$

Torque acting on it $= d\tau = r \cdot df = \dfrac{2m\ g}{r_0^2} \cdot r^2 dr$

Net torque $\tau = \quad d \quad = \dfrac{2m\ g}{r_0^2} \cdot \int_0^{r_0} r^2 dr = \dfrac{2m\ g r_0}{3}$

$$\tau = \frac{2m\ g r_0}{3} = \frac{1}{2}\ m r_0^2 \alpha$$

$$\therefore \qquad \alpha = \frac{4\ g}{3 r_0}$$

According to conservation of angular momentum

$\frac{1}{2} m r_0^2 \quad \frac{1}{2} m r_0^2 \quad \omega = \frac{1}{2}\ m r_0^2 \cdot \omega_0;$

ω = final and common angular velocity of the two discs.

$\omega = \frac{\omega_0}{2}$

Further $\omega = \omega_0 - \alpha \cdot t$

$$\frac{\omega_0}{2} = \omega_0 - \frac{4\mu g}{3 r_0} \cdot t$$

$$t = \frac{3 r_0\ \omega_0}{8\ g}$$

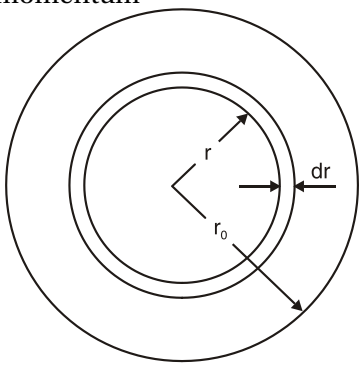


Fig. 4.111

4.148. A coin of radius r is spinning about a vertical axis as in Fig. 4.112a with an angular velocity ω. Find the condition in which its motion will be stable.

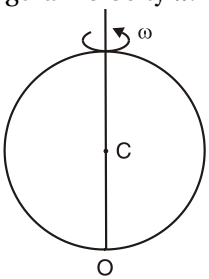


Fig. 4.112a

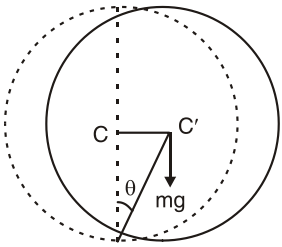


Fig. 4.112b

Solution: In Fig. 4.112b if we take the moment about C' then

$$\frac{mr^2}{4} \cdot \alpha = mgr \sin\theta = mgr\,\theta. \text{ For small angles.}$$

$$\omega^2 = \frac{4g}{r}$$

For equilibrium we must have

$$\omega^2 > \frac{4g}{r}$$

4.149. A semi-circular thick disc of radius r and mass m rests on a horizontal surface and is pulled by a horizontal force F applied at the end B. Find the angle α that the diameter AB makes with the horizontal just before sliding begins if the coefficient of friction is μ.

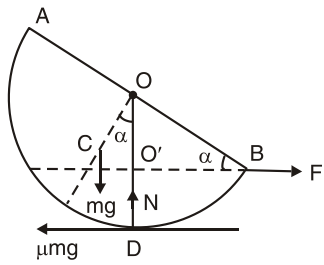


Fig. 4.113

Solution: Taking the moment about O'

$$\mu\, mg\, O'D = mg \cdot OC \sin\alpha$$

$$\mu\, mg\, r\,(1 - \sin\alpha) = mg \cdot \frac{4r}{3} \sin\alpha$$

$$3\pi\mu = (3\pi\mu + 4)\sin\alpha$$

$$\sin\alpha = \frac{3}{3 \quad 4}$$

4.150. A homogenous and uniform bar of length L is pivoted about a point at a distance l from one end as shown in Fig. 4.114. The bar is released from rest when it makes an angle θ with the vertical. Find the angular speed when the bar becomes vertical.

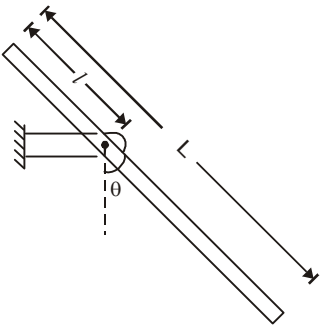


Fig. 4.114

Solution: In view of conservation of energy we have

$$\frac{1}{2}\left[\frac{m}{3L} \cdot l \cdot l^2 + \frac{m}{3L}(L-l)(L-l)^2\right]\omega^2$$

$$= \frac{m}{L} \quad \frac{L \quad l^{\ 2}}{2} \cdot g\,[1 - \cos\theta] - \frac{m}{L} \cdot \frac{l^2}{2} \cdot g\,[1 - \cos\theta]$$

$$\frac{1}{2} \cdot \frac{m}{3L} \quad L^3 \quad 3L^2 l \quad 3l^2 L \quad \omega^2 = \frac{m}{L} \cdot \frac{g}{2} \,(1 - \cos\theta) \quad L^2 \quad 2Ll$$

$$\omega = \sqrt{\frac{3g \quad L \quad 2l \quad 1 \quad \cos}{L^2 \quad 3\,Ll \quad 3l^2}}$$

4.151. Two portions of a rod of lengths l_1 and l_2 are rigidly fixed at O at an angle of 90°. The vertex O is connected to a vertical shaft which rotates

with an angular velocity ω. For what value of ω the rod of length l_1 will make an angle θ with the vertical?

Solution: Consider an element of length *dx* situated at a distance *x* from *O* on the bar of length l_1. Force acting on the element = $\frac{m}{l_1}\, dx \cdot x \sin\theta \cdot \omega^2$

Taking in moment about *O*

$$mg \cdot \frac{l_1}{2} \sin\theta = \int_0^{l_1} \frac{m}{l_1}\, dx \cdot x \sin\theta \cdot \omega^2 \cdot x \cos\theta$$

$$\frac{3g l_1^2}{\cos} = 2\ l_1^3 \cdot \omega^2$$

Similarly for the other rod of length l_2

will have $3g \cdot \frac{l_2^2}{\sin} = 2 l_2^3 \cdot \omega^2$

On substracting the two we have

$$\omega^2 = \frac{3g}{2\ l_2^3\ \ l_1^3}\ \ \frac{l_2^2}{\sin}\ \ \frac{l_1^2}{\cos}$$

or $\omega = \sqrt{\frac{3g}{2\ l_2^3\ \ l_1^3}\ \ \frac{l_2^2}{\sin}\ \ \frac{l_1^2}{\cos}}$

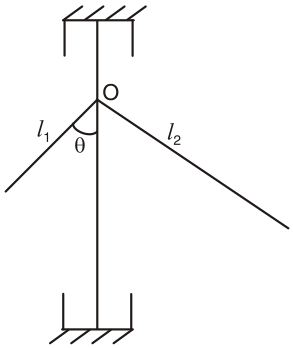


Fig. 4.115

4.152. A uniform bar of length *L* and mass *m* is rotating at a constant angular velocity ω about a vertical axis through a point at a distance *l* from one end. Determine the horizontal and vertical components of the reaction of the support on the bar.

Solution: Consider an element of length *dx* situated at a distance *x* from *O*. Force acting on the element is given by

$$dF = \frac{m}{L} \cdot dx \cdot x \sin\theta \cdot \omega^2$$

$$F = R_h = -\frac{m}{L} \sin\theta \cdot \omega^2 \int_l^{L-l} x\, dx$$

$$= -m\ \ \frac{L}{2}\ \ l\ \ \omega^2 \sin\theta$$

$$R_V = mg$$

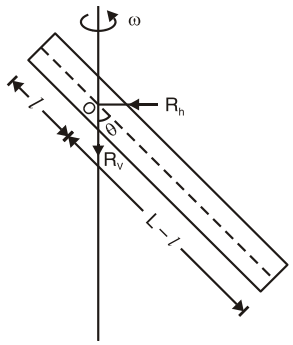


Fig. 4.116

4.153. A uniform bar of length *l* is pivoted about a horizontal axis through its lower end and is allowed to fall from its vertical position. Find (i) its angular velocity (ii) the normal and tangential acceleration.

Solution: Taking the moment about point *O*.

$$mg \cdot \frac{l}{2} \sin\theta = \frac{ml^2}{3}\alpha$$

or $$\alpha = \frac{3g}{2l} \cdot \sin\theta$$

$$\alpha = \frac{d}{dt} = \frac{d}{d} \cdot \omega = \frac{3g}{2l} \cdot \sin\theta$$

$$d = \frac{3g}{2l} \sin \cdot d$$

$$\frac{^2}{2} = -\frac{3g}{2l} \cos\theta + C$$

At $\theta = 0, \omega = 0$

$$\therefore \quad C = \frac{3g}{2l}$$

or $$\omega = \sqrt{\frac{3g}{l} \; 1 \quad \cos}$$

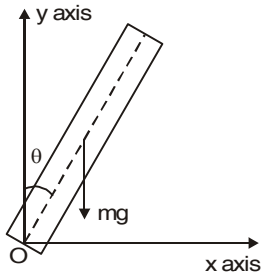


Fig. 4.117

a_t = tangential acceleration = $l\alpha = \frac{3g}{2} \sin\theta$,

a_n = normal acceleration = $l\omega^2 = 3g(1 - \cos\theta)$

4.154. A sphere of mass m and radius r rotating with an angular velocity ω_0 is placed on a horizontal plateform so that its rotational axis is at an angle θ with the vertical (see Fig. 4.118). Find the velocity v and the angular velocity ω of the sphere as soon as it stops slipping on the plateform.

Solution: Here $I\omega_0 \cos\theta$ = constt. since the moment of all external forces is zero.

$$I\omega_0 \sin\theta = I\omega + mvr$$

$$= (I + mr^2)\omega_{\text{horizontal}}$$

or $$\omega_{\text{horizontal}} = \frac{I_0 \sin}{I \quad mr^2} = \frac{2}{7}\omega_0 \sin\theta$$

$$\omega_{\text{vertical}} = \omega_0 \cos\theta$$

$$\text{velocity} = \frac{2}{7}\omega_0 r \sin\theta$$

and $$\omega = \left(\omega^2_{\text{horizontal}} + \omega^2_{\text{vertical}}\right)^{\frac{1}{2}}$$

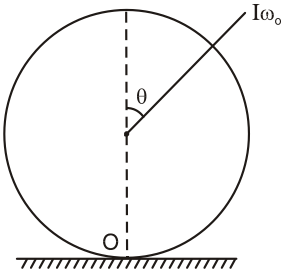


Fig. 4.118

4.155. A solid and hollow cylinder of mass m and radius r are placed on a rough inclined plane making an angle θ with the horizontal. Find the linear acceleration of the cylinder. The coefficient of friction between them is equal to μ.

Solution: For hollow and solid cylinders the equation of motion is

$$mg \sin\theta + f_1 - \mu_1 mg \cos\theta = ma$$

and $mg \sin\theta - f_2 - \mu_2 mg \cos\theta = ma$

f_1 and f_2 are the forces acting between the cylinders, μ_1 = coefficient of friction between the plane and hollow cylinder and μ_2 = coefficient of friction between the plane and solid cylinder.

On adding the two we have

$2mg \sin\theta + f_1 - f_2 = 2ma + (\mu_1 + \mu_2)\, mg \cos\theta$

Further $r \cdot f_1 = \mu \times \frac{1}{2} mr^2\alpha$

$\therefore f_1 = \frac{}{2} \cdot ma$

and $r \cdot f_2 = \mu \times mr^2\alpha$

$\therefore f_2 = \mu \cdot ma$

Now $\mu_1 mg \cos\theta \cdot r = mr^2\alpha$

or $\mu_1 mg \cos\theta = ma$

and $\mu_2 mg \cos\theta \cdot r = \frac{1}{2} mr^2\alpha$

or $\mu_2 mg \cos\theta = \frac{1}{2} ma$

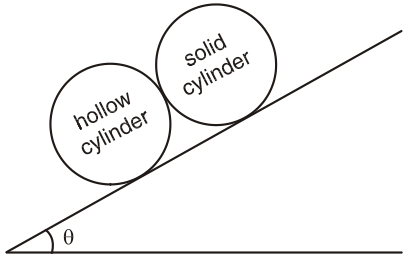


Fig. 4.119

With these values now

$2mg \sin\theta + \frac{}{2} mg - \mu ma = 2ma + ma + \frac{ma}{2}$

$2mg \sin\theta = \frac{}{2} \frac{7}{2} ma$

$\therefore a = \frac{4g \sin}{7}$

4.156. Two identical discs are placed as shown in the Fig. 4.120. Find the velocity of the centre of mass of the falling disc as a function of height h through which it has fallen.

Solution:

$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} I \left(\frac{}{2}\right)^2 + \frac{1}{2} I \left(\frac{}{2}\right)^2$$

$$mgh = \frac{1}{2} mv^2 + I \left(\frac{}{4}\right)^2$$

$$= \frac{1}{2} mv^2 + \frac{1}{2} \cdot \frac{mv^2}{4}$$

$$mgh = \frac{5mv^2}{8}$$

$$\therefore v = \sqrt{\frac{8gh}{5}}$$

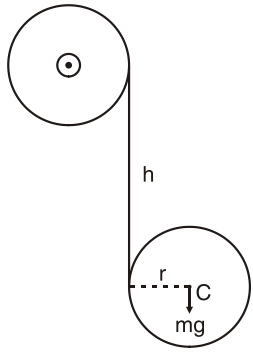


Fig. 4.120

4.157. A bar held at an angle ϕ with the horizontal is released. For what value of μ the lower end of the bar will not slip.

Solution: $\frac{1}{3} ml^2 \cdot \alpha = mg \cdot \frac{l}{2} \cos\phi$;

l = length of the bar

m = mass of the bar

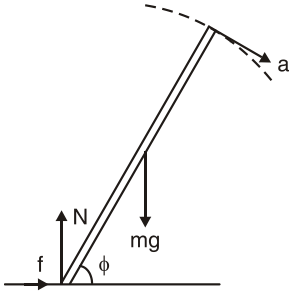


Fig. 4.121

$$\alpha = \frac{3g}{2l} \cos\phi$$

Tangential acceleration $a = \alpha \cdot \frac{l}{2}$

$$= \frac{3}{4} g \cos\phi$$

Now $mg\cos\phi - N\cos\phi + f\sin\phi = ma = \frac{3mg}{4}\cos\phi$

and along the length of the bar we have

$mg\sin\phi - N\sin\phi - f\cos\phi = 0$

$N = mg - f\cot\phi$

$= mg - \frac{3}{4} \cdot mg\sin\phi\cos\phi \cdot \cot\phi$; $f = a\sin\phi = \frac{3}{4} mg\cos\phi\sin\phi$

$= \frac{mg}{4} [4 - 3\cos^2\phi]$

$f = \mu N = \mu \cdot \frac{mg}{4} [4 - 3\cos^2\phi] = \frac{3}{4} \cdot mg \cdot \cos\phi \cdot \sin\phi$

$\mu \geq \frac{3 \sin \quad \cos}{1 \quad 3\sin^2}$

4.158. A block of mass m = 20 gm is attached by a weightless spring of force constant K = 100 N/m and of length 19.6 cm. At time t = 0 a constant angular acceleration α = 0.5 rad/sec^2 is imparted to the cone. Find (i) the position of the block from the top of the cone when the contact is lost with the surface of cone (ii) the time when the block will leave the contact with the surface.

Solution:

(i) Let l be the elongation of the spring. On equationg the forces we have

$mr\omega^2 \cos 30 + mg \cos 60 = Kl$

$mg \sin 60 - mr\omega^2 \cos 60 = N = 0$

therefore $\sqrt{3} \cdot g = r\omega^2$

With this identity we have

$m \cdot \sqrt{3} g \cdot \frac{\sqrt{3}}{2} + mg \cdot \frac{1}{2} = Kl$

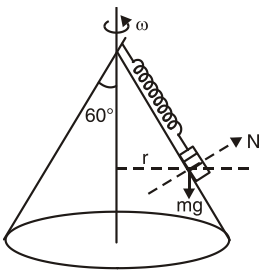


Fig. 4.122

$$\text{or} \quad l = \frac{2mg}{K} = \frac{2 \quad 20 \quad 10^{3} \quad 9.8}{100} = 0.392 \text{ cm}$$

$$\therefore \text{ total length} = 19.6 + 0.392 = 19.992 = 20 \text{ cm}$$

(ii) $\sin 60 = \dfrac{r}{0.2} \quad \therefore r = 0.1 \times \sqrt{3}$

$$\omega^2 = \frac{\sqrt{3} \cdot g}{0.1 \times \sqrt{3}} = 10g = 98$$

$$\therefore \omega = \sqrt{98}$$

$$\alpha \cdot t = 0.5\, t = \sqrt{98}$$

$$\therefore t = \frac{\sqrt{98}}{0.5} = 19.8 \text{ sec}$$

4.159. A thin rod PQ of length L and mass m is pivoted at P to an axle OO' which rotates with an angular velocity ω. The end Q of the rod is connected to the axle with the help of thin wire QR. Under these conditions the rod makes an angle ϕ with the horizontal (see Fig. 4.123). Find the tension in the wire QR.

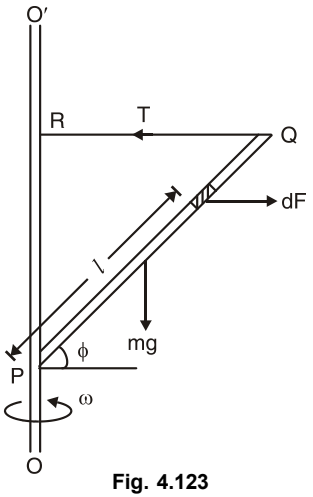


Fig. 4.123

Solution: Consider an element of length dl situated at a distance l from P.

dF = Force acting on the element $= \dfrac{m}{L} \cdot dl \cdot l \cos\phi \cdot \omega^2$

Torque acting on the element $= \dfrac{m}{L} \cdot dl \cdot l \cos\phi \cdot \omega^2 \cdot l \sin\phi$

Torque acting on the rod $= \dfrac{m}{L} \cdot \omega^2 \sin\phi \cos\phi \cdot \int_{O}^{L} l^2 dl$

$$= \frac{mL^2 \quad ^2}{3} \sin\phi \cos\phi$$

On taking the moment about P, we have

$$T \cdot L \sin\phi = mg \cdot \frac{L}{2} \cos\phi + \frac{m}{3} L^2 \omega^2 \sin\phi \cos\phi$$

$$\therefore T = \frac{1}{2} mg \cot\phi + \frac{m}{3} L\omega^2 \cdot \cos\phi$$

4.160. One end of a homogenous rod PQ of mass m and length l rests on a rough horizontal surface. The rod is maintained at an angle of 60° by means of a string attached to its other end Q. The string is suddenly cut. Assuming that the friction between end P of the rod and the surface is large enough to prevent sliding. Determine (i) the angular acceleration

of the rod immediately after the string is cut, (ii) the normal reaction and force of friction at P, (iii) the minimum value of coefficient of friction.

Solution:

(i) On taking the moment about P we have

$$\frac{ml^2}{3}\alpha = mg\cdot\frac{l}{2}\cos 60 = mg\,\frac{l}{4}$$

$$\therefore\ \alpha = \frac{3g}{4l}$$

(ii) Acceleration of the centre $a = \frac{l}{2}\alpha = \frac{3g}{8}$

Intertial force $= \frac{3mg}{8}$

$$N = mg - \frac{3mg}{8}\cos 60 = mg - \frac{3mg}{16} = \frac{13mg}{16}$$

$$\text{Force of friction} = ma\cos 30 = m\cdot\frac{3g}{8}\cdot\frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}\,mg}{16}$$

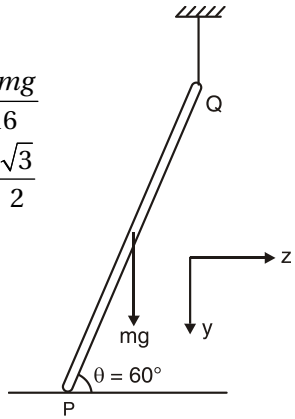


Fig. 4.124

(iii) μN = force of friction

$$\therefore\ \mu = \frac{\text{Force of friction}}{N} = \frac{3\sqrt{3}\,\frac{mg}{16}}{\frac{13mg}{16}}$$

$$= \frac{3\sqrt{3}}{13} = 0.4$$

4.161. A rectangular plate of dimensions $l_1 \times l_2$ and mass m is suspended from two pins A and B (see Fig. 4.125a). If the pin B is suddenly removed determine (i) the angular acceleration of the plate (ii) the components of the reaction force at A.

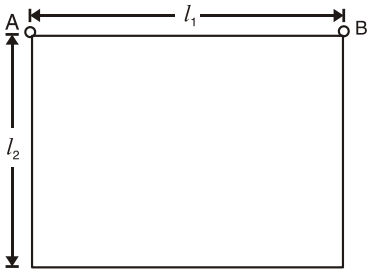


Fig. 4.125a

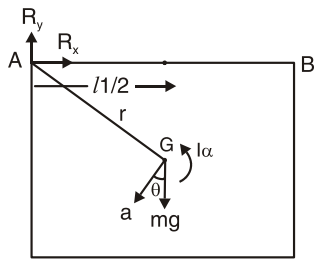


Fig. 4.125b

Solution: (i) In view of Fig. 4.125b, we have

$$mar + I\alpha - mg \cdot \frac{l_1}{2} = 0$$

$$(mr^2\alpha + I\alpha) = mg \cdot \frac{l_1}{2}$$

$$\therefore \alpha = \frac{mg\frac{l_1}{2}}{I \quad mr^2}$$

On putting $r = \sqrt{\frac{l_1^2 + l_2^2}{2}}$ and $I = \frac{m}{12} \quad l_1^2 \quad l_2^2$

$$\alpha = \frac{mg\frac{l_1}{2}}{\frac{m}{12} \cdot \left(l_1^2 + l_2^2\right) + m\left(\frac{l_1^2 + l_2^2}{4}\right)}$$

$$= \frac{3mgl_1}{2m \quad l_1^2 \quad l_2^2} = \frac{3g\,l_1}{2 \quad l_1^2 \quad l_2^2}$$

Inertial force of gravity $= ma = mr\alpha$

$$= \frac{m}{2} \quad \sqrt{l_1^2 \quad l_2^2} \cdot \frac{3g\,l_1}{2 \quad l_1^2 \quad l_2^2}$$

$$= \frac{3mg\,l_1}{4\sqrt{l_1^2 \quad l_2^2}}$$

(ii) $R_x + ma \sin\theta = 0$

$R_x = -ma \sin\theta$

$$= -\frac{3mg \cdot l_1}{4\sqrt{l_1^2 + l_2^2}} \cdot \frac{l_2}{\sqrt{l_1^2 \quad l_2^2}}$$

$$= -\frac{3mg \cdot l_1 l_2}{4\left(l_1^2 + l_2^2\right)}$$

$R_y + ma \cos\theta - mg = 0$

$\therefore R_y = mg - ma \cos\theta$

$$= mg - \frac{3mgl_1}{4\sqrt{l_1^2 \quad l_2^2}} \cdot \frac{l_1}{\sqrt{l_1^2 + l_2^2}}$$

$$= mg \frac{4l_1^2 \quad 4l_2^2 \quad 3l_1^2}{4 \quad l_1^2 \quad l_2^2}$$

$$= \frac{mg}{4} \frac{l_1^2 \quad 4l_2^2}{l_1^2 \quad l_2^2}$$

4.162. A solid cylinder 1 m in diameter and having mass of 100 kg is rolled up a 30° incline by a force of 1200 N as in Fig. 4.126. Assuming no slipping, determine the speed of the cylinder after 5 sec if the initial speed is zero.

Solution: Let F be the force of friction then

$$1200 - F - mg \sin 30 = ma$$

Further $1200 \times r + F \times r = I\alpha = \frac{1}{2} mr^2 \cdot \frac{a}{r}$

$$\therefore 1200 + F = \frac{1}{2} ma$$

$$2400 - \frac{1}{2} ma - \frac{mg}{2} = ma$$

$$2400 - \frac{mg}{2} = \frac{3ma}{2}$$

$$\frac{dv}{dt} = \frac{2}{3} \quad \frac{2400}{m} \quad \frac{g}{2}$$

$$\therefore v = \frac{2}{3} \quad \frac{2400}{m} \quad \frac{g}{2} \quad \times t$$

$$= \frac{2}{3} [24 - 4.9] \times 5$$

$$= \frac{19.1 \quad 10}{3} = 63.7 \text{ m/sec}$$

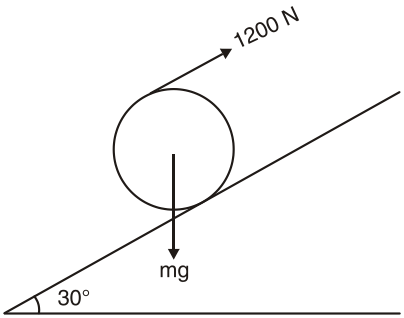


Fig. 4.126

4.163. A solid cylinder of mass 100 kg and radius 150 mm is attached to a spring of force constant K = 500 N/m and of length 500 mm. The cylinder can roll without sliding on the horizontal surface. Find the angular speed of the cylinder when its centre has moved by 500 mm toward right from the position shown in the Fig. 4.127.

Solution:

Initial *K.E.* = 0

Final *K.E.* $= \frac{3}{4} mv^2 = \frac{3}{4} mr^2\omega^2$

$= 0.75 \times (0.15\ \omega)^2 \times 100$

$= 1.6875\ \omega^2$

Initial length of the spring

$= \sqrt{0.5^2 \quad 1} = 1.1180$ m

Initial *P.E.* $= \frac{1}{2} \times 500\ [1.118 - 0.5]^2$

$= 250 \times 0.618^2 = 95.48$ J

Final length of the spring $= \sqrt{0.5^2 \quad 0.5^2} = 0.707$

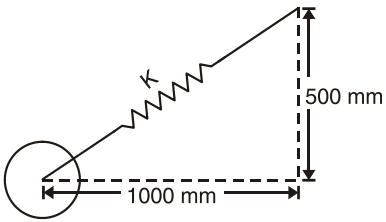


Fig. 4.127

Final *P.E.* of the spring $= \frac{1}{2} \times 500\ [0.707 - 0.5]^2$

$= 250\ [0.207]^2$

$= 10.71$ J

According to conservation of energy

$1.6875\ \omega^2 = 95.48 - 10.71$

$\therefore\ \omega^2 = \frac{84.77}{1.6875} = 50.23$

$\omega = 7.08$ rad/sec

4.164. A disc of radius r rolls without slipping and the inside of a fixed cylinder of radius R. The angular velocity ω of the disc is constant. Point P of the disc is in contact with the cylinder. Calculate the velocity and acceleration of point P at any time t.

Solution: Here the acceleration of the point P will be a coriolis acceleration given by $2\vec{\ } \times \vec{v}$

$\vec{\ } = \omega \vec{k}\ ;\ s = \vec{i}\ r \sin \omega t = \vec{k}\ r\ (1 - \cos \omega t)$

$v = r\omega \left[\vec{i} \cos \omega t + \vec{k} \sin \omega t\right]$

a_p = acceleration

$= 2\omega \cdot r\omega \cdot \vec{k} \times \left[\vec{i} \cos \omega t + \vec{k} \sin \omega t\right]$

or $a_p = 2r\omega^2 \cos \omega t\, \vec{j}$

$\frac{dv_p}{dt} = a_p$

$dv_p = 2r\omega^2 \cos \omega t \cdot dt$

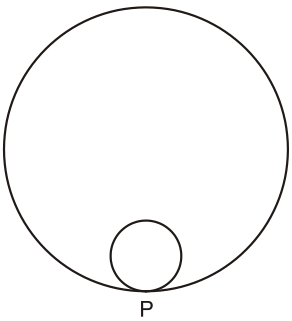


Fig. 4.128

On integration

$v_p = 2r\omega \sin \omega t + C$

At $t = 0$, $v_p = 0\ \therefore\ C = 0$

$\therefore\ v_p = 2r\omega \sin \omega t$

4.165. A bullet of mass 150 gm is fired horizontally in the north direction with a velocity of 400 m/sec at 30°N latitude. Calculate (i) the horizontal component of coriolis acceleration and the deflection of the bullet on hitting a target 300 m away (ii) the vertical displacement of the bullet due to gravity (iii) the coriolis force.

Solution: $v = 400\ \vec{k}$

$\vec{\ } = \omega\ \ \vec{k} \cos 30\ \ \vec{i} \sin 30$

$\omega = \frac{2\pi}{24 \times 60 \times 60} = 7.2 \times 10^{-5}$ rad/sec

Coriolis acceleration $= 2\omega \times v$

$= 2\omega \ \vec{k}\cos 30 \ \ \vec{i}\sin 30 \ \times 400 \ \vec{k}$

$= -2 \times 7.2 \times 10^{-5} \times \frac{400}{2} j$

$= 0.0288$ m/sec^2 towards west

Deflection $= \frac{1}{2} at^2 = \frac{1}{2} \times 0.0288 \times \left(\frac{300}{400}\right)^2$

$= 8.1 \times 10^{-3}$ m

(ii) Vertical displacement $= \frac{1}{2} gt^2 = 4.9 \times (0.75)^2$

$= 2.756$ m

(iii) Coriolis force $= -2\, m\omega \times v$

$= 2 \times 0.15 \times 7.2 \times 10^{-5} \times 400 \times \frac{1}{2} \ \vec{i}$

$= 4.32 \times 10^{-3} \ \vec{i}$ towards east

4.166. A stone is allowed to fall under gravity from the top of a h meter high tower at the equator. Find the horizontal displacement of the stone due to the rotation.

Solution: Let us choose z axis along the earth's axis, x axis vertical and y axis perpendicular to x and z.

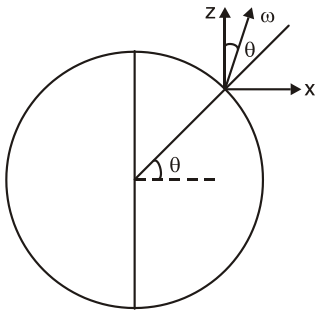


Fig. 4.129

$$-\vec{i}\cdot g = \vec{a_r} + 2\omega \times \vec{v_r}$$

$$\vec{a_r} = \vec{i}\ \frac{d^2x}{dt^2} + \vec{j}\ \frac{d^2y}{dt^2} + \vec{k}\ \frac{d^2z}{dt^2}$$

$$\vec{v_r} = \vec{i}\ \frac{dx}{dt} + \vec{j}\ \frac{dy}{dt} + \vec{k}\ \frac{dz}{dt}$$

$$-\vec{i}\cdot g = \vec{i}\ \frac{d^2x}{dt^2} + \vec{j}\ \frac{d^2y}{dt^2} + \vec{k}\ \frac{d^2z}{dt^2} + 2\omega\vec{k} \times \vec{i}\frac{dx}{dt}\ \ \vec{j}\frac{dy}{dt}\ \ \vec{k}\frac{dz}{dt}$$

$$-\vec{i}\cdot g = \vec{i}\ \frac{d^2x}{dt^2} + \vec{j}\ \frac{d^2y}{dt^2} + \vec{k}\ \frac{d^2z}{dt^2} + 2\omega\vec{j}\ \frac{dx}{dt} - 2\omega\vec{i}\ \frac{dy}{dt}$$

On neglecting $2\omega \frac{dy}{dt}$ we have

$$\frac{d^2x}{dt^2} = -g, \ \frac{d^2y}{dt^2} + 2\omega\frac{dx}{dt} = 0, \text{ and } \frac{d^2z}{dt^2} = 0$$

On integration $\frac{dx}{dt} = -gt + C$

At $t = 0$, $\frac{dx}{dt} = 0$, $\therefore C = 0$

$$\frac{dx}{dt} = -gt$$

On further integration $x = -\frac{gt^2}{2} + C$

At $t = 0$, $x = h \therefore C = h$

or $x = h - \frac{1}{2} gt^2$

At time $t = T$ the stone touches the ground i.e. $x = 0$

$$\therefore T = \sqrt{\frac{2h}{g}}$$

$$\frac{d^2y}{dt^2} = -2\omega \frac{dx}{dt} = 2\omega gt$$

On integration $\frac{dy}{dt} = \omega gt^2$

On further integration $y = \frac{gt^3}{3}$

displacement will attain a maximum value at $t = T$

$$\therefore y = \frac{g}{3} \cdot \left(\frac{2h}{g}\right)^{\frac{3}{2}}$$

4.167. A solid sphere of radius r and mass m is thrown on a plane horizontal surface with an initial velocity v_0 and angular velocity ω_0 (see Fig. 4.130). If $v_0 > r\omega_0$ then calculate the time t after which the sphere moves with a constant velocity.

Solution: The frictional force $F = \mu mg$

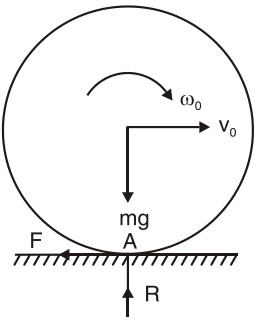


Fig. 4.130

The equation of motion is given by

$$m\ddot{x} = -\mu mg$$

or $\ddot{x} = -\mu g$

On integration we have

$$\dot{x} = -\mu gt + C$$

at $t = 0$

$$\dot{x} = v_0$$

$$\therefore \dot{x} = v_0 - \mu gt$$

For rotational motion

$$I\alpha = \mu mgr$$

or $\frac{2}{5} mr^2 \cdot \frac{d^2}{dt^2} = \mu mgr$

On integration we have $\frac{2}{5} r \cdot \frac{d}{dt} = \mu g t + C$

at $t = 0$, $\frac{d}{dt} = \omega_0 \therefore C = \frac{2}{5} r\omega_0$

or $\omega = \omega_0 + \frac{5}{2r} \mu g t$

These results are valid as long $\dot{x} = r \frac{d}{dt}$

or $\dot{x} = v_0 - \mu g t = r\omega_0 + \frac{5}{2} \mu g t$

$$\therefore t = \frac{v_0 \quad r_{\ 0}}{g \ \ 1 \ \ \frac{5}{2}}$$

$$= \frac{2}{7 \ g} (v_0 - r\omega_0)$$

4.168. A solid cylinder of radius r and mass m is placed on a horizontal surface. With an aim of pulling the cylinder on the surface a horizontal force F is applied to the end of the string wound over the surface of cylinder. Assuming no slip find (i) the acceleration of the centre (ii) the coefficient of friction between the cylinder and plane surface in order to prevent slipping.

Solution: (i) The applied force rotates the cylinder. It is the tendency of the point of contact to slip toward left therefore the static friction on the cylinder acts toward right (see Fig. 4.131).

Now $F + \mu mg = ma$

For rotation we have

$$(F - \mu mg)\, r = \frac{1}{2} mr^2\alpha$$

or $$F - \mu mg = \frac{1}{2} ma$$

$$F + \mu mg = ma$$

$$2F = \frac{3}{2} ma$$

$$\therefore a = \frac{4F}{3m}$$

(ii) $$\mu mg = ma - F$$

$$\mu mg = \frac{4F}{3} - F$$

or $$\mu = \frac{F}{3mg}$$

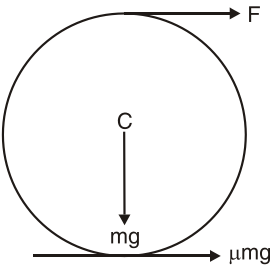


Fig. 4.131

4.169. A homogenous thin metallic hoop of radius r rolls without slip along a horizontal plane with a velocity v_c and strikes an incline plane $\theta = 45°$ (see Fig. 4.132). With what velocity V will the hoop starts moving up the incline plane?

Solution: Just before reaching the point B the hoop has a linear velocity v_c and angular velocity $\frac{v_c}{r}$. Point A is the instantaneous centre of rotation. In going up the plane the new centre of rotation is B.

In case of a body performing plane motion, the total angular momentum is made up of two parts (i) moment of momentum with respect to that axis of the body considered as a particle of mass m concentrated at centre of gravity C (ii) the angular momentum of the body with respect to a parallel axis through the moving centre of gravity of C.

Before moving up

$$J_A = mr^2 \frac{v_c}{r} + mv_c \cdot r \cos 45$$

After moving up $J_B = mr^2 \frac{V}{r} + mV \cdot r$

$$mrv_c + \frac{mv_c r}{\sqrt{2}} = 2mrV$$

$$\therefore V = 1 \quad \frac{1}{\sqrt{2}} \quad \frac{v_c}{2}$$

$$= 0.8536\, v_c$$

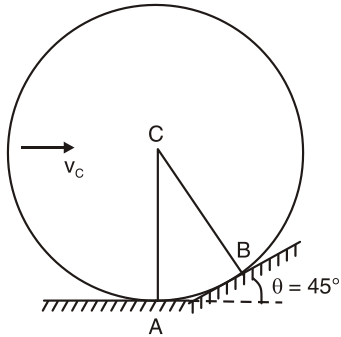


Fig. 4.132

4.170. A solid cylinder of mass m and radius r is placed on a $\theta = 30°$ incline plane. The cylinder is pulled up the incline by applying a constant force $F = mg/2$ at the end of the string wound over its circumference as shown in Fig. 4.133. There is no slip at the point of contact A. Find the acceleration of the centre of gravity C of the cylinder up the plane.

Solution: For motion, we have

$$F + \mu mg \cos\theta - mg \sin\theta = ma$$

and $$F - \mu mg \cos\theta = \frac{1}{2} ma$$

On adding the two we have

$$2F - mg \sin\theta = \frac{3}{2} ma$$

$$mg - \frac{mg}{2} = \frac{3}{2} ma$$

or $$a = \frac{g}{3}$$

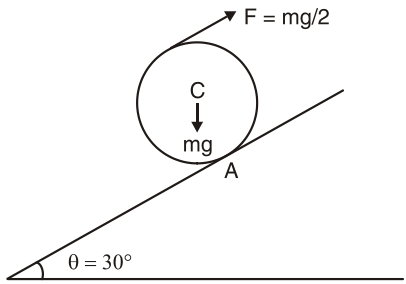


Fig. 4.133

4.171. A string is wound around the circumference of a solid cylinder of mass m and radius r. The cylinder is placed on a $\theta = 60°$ incline and is supported by fixing the end of the string at the point A of a rigid support (see Fig. 4.134). Assuming the coefficient of friction μ at the point of contact B equal to 1/3 find the acceleration of the cylinder down the plane.

Solution: The equation of motion is given by

$$mg \sin 60 - (T + \mu mg \cos 60) = ma$$

and $$T - \mu mg \cos 60 = \frac{1}{2} ma$$

On adding the two we have

$$mg \sin 60 - 2\mu mg \cos 60 = \frac{3}{2} ma$$

or $$a = \frac{2}{3} g \left[\frac{\sqrt{3}}{2} - 2 \times \frac{1}{3} \cdot \frac{1}{2} \right]$$

$$= \frac{2}{3} g \, [0.866 - 0.333]$$

$$= 0.355g$$

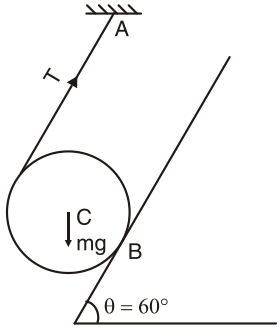


Fig. 4.134

4.172. A solid cylinder and a hoop of same mass m and radius r are placed on a $\theta = 30°$ incline by connecting them through a bar PQ whose mass is negligible (see Fig. 4.135). Assuming that there is no slip find (i) the acceleration of the system down the plane (ii) the tension in the bar.

Solution: The equation of motion of the cylinder and the hoop is given by

$$mg \sin \theta - T - \mu_1 mg \cos \theta = ma; \quad \mu_1 = \text{coefficient of friction between the incline plane and cylinder and}$$

μ_2 = coefficient of friction between the incline plane and the hoop

$$T + mg \sin \theta - \mu_2 mg \cos \theta = ma$$

Further we have

$$r \cdot \mu_1 mg \cos \theta = \frac{1}{2} mr^2\alpha$$

or $$\mu_1 mg \cos \theta = \frac{1}{2} ma; \; a = r\alpha$$

and $$r \cdot \mu_2 mg \cos \theta = mr^2\alpha$$

or $$\mu_2 mg \cos \theta = ma$$

On substituting these values we have

$$mg \sin \theta - T = \frac{3}{2} ma$$

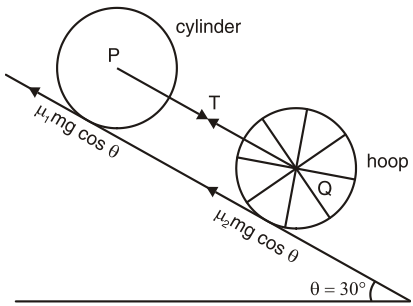


Fig. 4.135

$$T + mg \sin\theta = 2ma$$

On adding the two equations we have

$$2mg \sin\theta = \frac{7}{2} ma$$

or $$a = \frac{4}{7} g \sin\theta$$

On putting $\theta = 30°$ we have

$$a = \frac{2}{7} g$$

$$T = 2m \cdot \frac{2g}{7} - \frac{mg}{2}$$

$$= \frac{4}{7} \frac{1}{2} mg$$

$$= \frac{mg}{14}$$

4.173. A solid sphere of mass m and radius r is placed on a horizontal rough surface. A tangential force P as shown in the Fig. 4.136 is applied the sphere. Find the acceleration of the centre of the sphere.

Solution: The applied force P rotates the sphere. Here the point of contact slips towards the left therefore the frictional force acts toward the right.

For translational part of motion we have

$P + \mu mg = ma$; μ is the coefficient of friction beween the horizontal surface and the sphere.

For rotation

$$(P - \mu mg)\, r = I\alpha = \frac{2}{5} mr^2\alpha$$

or $$P - \mu mg = \frac{2}{5} ma$$

or $$2P = \frac{7ma}{5}$$

or $$a = \frac{10P}{7m}$$

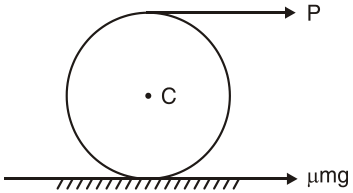


Fig. 4.136

4.174. A solid cylinder of radius r rotating about its own axis with an angular velocity ω_0 is placed into the corner of the two walls. Assuming the coefficient of friction between the two walls and the cylinder as μ find the number of rotations made by the cylinder before it stops.

Solution: Let R_1 and R_2 be the reactions at the point of contact then the required resisting torque is equal to $\mu(R_1 + R_2)r$ (see Fig. 4.137)

On equating the forces in the vertical direction we have

$(\mu R_2 + R_1) = mg$; m = mass of cylinder.

In the horizontal direction we have

$$\mu R_1 = R_2$$

$$\text{Torque} = \mu [R_1 + R_2] r = \mu [R_1 + \mu R_1] r$$

$$= \mu (\mu + 1) R_1 r$$

In the vertical direction we have

$$\mu^2 R_1 + R_1 = mg$$

$$\therefore R_1 = \frac{mg}{1+\mu^2}$$

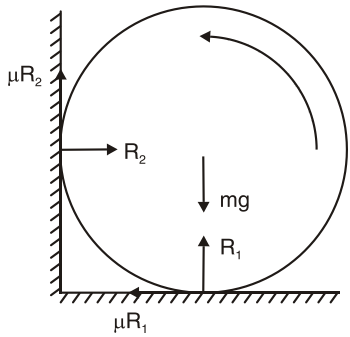


Fig. 4.137

$$\text{Torque} = \mu (\mu + 1) \cdot \frac{mg}{1+\mu^2} r$$

$$\text{Torque} = -\frac{\mu(\mu+1) mg \cdot r}{1+\mu^2} = I\alpha = \frac{1}{2} mr^2 \cdot \frac{d}{dt} = \frac{1}{2} mr^2 \cdot \frac{d}{d} \cdot \frac{d}{dt}$$

$$= \frac{1}{2} mr^2 \frac{d}{d} \cdot \omega$$

or $-\frac{2 (1)}{1 ^2} g \cdot \int_0^{\phi} d\theta = r \int_{\omega_0}^{0} \omega \cdot d\omega$; ϕ is the total angle described.

or $\frac{2 (1) g \cdot}{1 ^2} = \frac{r \cdot ^2_0}{2}$

$$\frac{2\mu(\mu+1) g \cdot 2\pi n}{1+\mu^2} = \frac{r\omega_0^2}{2}; \; n = \text{number of rotations made by the cylinder before it stops.}$$

$$\therefore n = \frac{r\omega_0^2 (1+\mu^2)}{8\pi\mu(\mu+1) g}$$

4.175. A solid spherical rotor of radius $r = 25$ cm and mass m is connected through a thin wire OA as shown in the Fig. 4.138. The rotor can freely rotate about an axis passing through its centre O with an angular velocity $\omega_0 = 60$ rad/sec. The rotor is sudden allowed to form a contact at C with a vertical wall AB. Assuming the angle $\theta = 20°$ and the coefficient of friction between the wall and the rotor as $\mu = 0.25$, find the time elapsed before the rotor comes to rest.

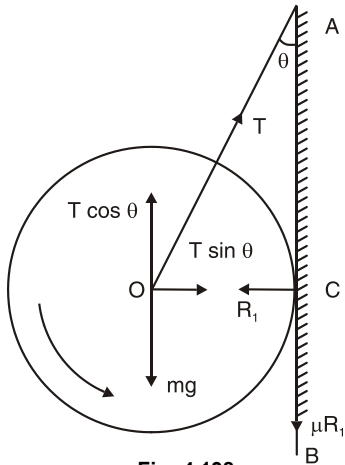


Fig. 4.138

Solution: On equating the forces in the horizontal and vertical directions we have

$T \sin\theta = R_1$; R_1 is the force of reaction.

T = tension in the wire

and $mg + \mu R_1 = T\cos\theta$

Now $mg + \mu T \sin\theta = T\cos\theta$

$$\therefore\ T = \frac{mg}{\cos \quad \sin}$$

and $$R_1 = \frac{mg \cdot \sin\theta}{\cos\theta - \mu \sin\theta}$$

$$\text{Torque} = -\mu R_1 \cdot r = -\frac{\mu \cdot mg \sin\theta}{\cos\theta - \mu \sin\theta} \cdot r = I\alpha = \frac{2}{5} mr^2 \cdot \frac{d}{dt}$$

$$\therefore\ d\omega = -\frac{5}{2} \frac{\mu \cdot g \sin\theta}{r(\cos\theta - \mu \sin\theta)}\, dt$$

or $$\int_0^{\omega_0} d\omega = -\frac{5}{2} \frac{\mu \cdot g \sin\theta}{r(\cos\theta - \mu \sin\theta)} \int_t^0 dt$$

or $$\omega_0 = \frac{5\mu g \sin\theta \cdot t}{2r(\cos\theta - \mu \sin\theta)}$$

or $$t = \frac{2r \cdot \omega_0 (\cos\theta - \mu \sin\theta)}{5\mu g \sin\theta}$$

On substituting the values we have

$$t = \frac{2 \quad 0.25 \quad 60\,(0.9397 \quad 0.3420 \quad 0.25)}{5 \quad 0.25 \quad 9.8 \quad 0.3420}$$

$= 6.116$ sec

4.176. A uniform disc of radius r and mass m rotating clockwise with an angular velocity ω_0 is carefully placed on a horizontal plane as shown in the Fig. 4.139. Coefficient of friction between the disc and plane is μ. Determine (i) the time t_1 at which the disc will start rolling without sliding (ii) the linear and angular velocities of the disc at time t_1.

Solution:

(i) $\mu mg = ma$; a = acceleration of the disc

$\therefore\ a = \mu g$

Further $I\alpha = -\mu mgr$

$$\frac{1}{2} mr^2 \cdot \alpha = -\mu mgr$$

$$\therefore\ \alpha = -\frac{2 \quad g}{r}$$

at $t = 0$, $v = 0 + \mu gt$ and

$$\omega = \omega_0 - \frac{2 \quad g}{r} \cdot t$$

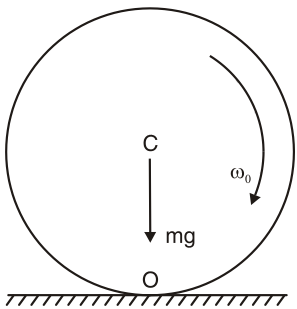


Fig. 4.139

The disc will start rolling without sliding when the velocity of the point of contact O is zero. At that time $t = t_1$ point O becomes the instantaneous centre of rotation and we have

$$\mu g \cdot t_1 = r\left(\omega_0 - \frac{2\mu g}{r} \cdot t_1\right)$$

or $3\mu g t_1 = r\omega_0$

or $t_1 = \frac{r_{\ 0}}{3\ g}$

(ii) velocity at that time

$$v_{t_1} = \mu g \cdot t_1 = \mu g \cdot \frac{r_{\ 0}}{3\ g} = \frac{r_{\ 0}}{3}$$

$$\frac{1}{3} r\omega_0 = r_{\ t_1}$$

or $_{t_1} = \frac{_0}{3}$

4.177. A spool with thread wound on it, of mass m rest on a rough horizontal plane. Its moment of inertia relative to its own axis is equal to $I = kmR^2$ where k is a positive constant and R is the outer radii of the spool. The radius of the wound thread layer is equal to r. The spool is pulled without sliding by a force F as shown in the Fig. 4.140. Find (i) the acceleration of the spool along x axis (ii) the work performed by the force F in time t_1.

Solution:

(i) For translational motion we have

$F\cos\alpha - \mu mg = ma$; a = acceleration and μ = coefficient of friction between spool and the surface

and $\mu mg \cdot R - F \cdot r = kmR^2 \cdot \beta$; β = angular acceleration

For no slip $\beta = \frac{a}{R}$

$$\therefore \mu mg = kma + F \cdot \frac{r}{R}$$

or $F\cos\alpha - kma - F \cdot \frac{r}{R} = ma$

or $a = \frac{F\cos\ \ F \cdot \frac{r}{R}}{m\ \ k\ \ 1}$

$$= \frac{F\ \cos\ \ \frac{r}{R}}{m\ \ k\ \ 1}$$

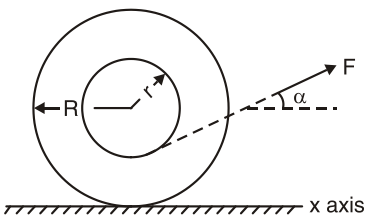


Fig. 4.140

(ii) $a = \dfrac{dv}{dt} = \dfrac{F \cos \quad \dfrac{r}{R}}{m(k \quad 1)}$

or $v = \dfrac{F\left(\cos\alpha - \dfrac{r}{R}\right)t_1}{m(k+1)}$

$$\text{Total work performed} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}k \cdot mR^2\,\omega^2$$

$$= \frac{1}{2}(1+k)\,mv^2$$

$$= \frac{\frac{1}{2}(1+k)\,m \cdot F^2\left(\cos\alpha - \frac{r}{R}\right)^2 t_1^2}{m^2(k+1)^2}$$

$$= \frac{\frac{1}{2}F^2t_1^2\left(\cos\alpha - \frac{r}{R}\right)^2}{m(k+1)}$$

4.178. A plank of mass m holding a solid sphere of mass M and radius r rests on a smooth horizontal surface. A constant horizontal force P is applied to the plank. Assuming no sliding between the plank and the sphere find (i) the acceleration of the plank (ii) the acceleration of the centre of the sphere.

Solution: For the plank (see Fig. 4.141) we have

$P - f = ma_1$; f = frictional force between the plank and the horizontal surface.

a_1 = acceleration of the plank.

For the sphere we have

$f = M \cdot a_2$; a_2 = acceleration of the sphere

and $f \cdot r = \dfrac{2}{5}Mr^2\alpha$

and $a_1 = a_2 + \alpha r$

On substituting the value of f we have

$$P - Ma_2 = ma_1$$

or $$P - Ma_2 = m\,(a_2 + \alpha \cdot r)$$

$$P = (M+m)a_2 + m \cdot r \times \frac{5}{2}\,\frac{f}{Mr}$$

$$= (M+m)a_2 + \frac{5m}{2M} \cdot Ma_2$$

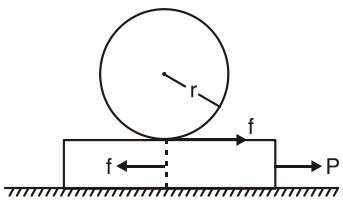


Fig. 4.141

or $2P = (2M + 7m)a_2$

or $a_2 = \dfrac{2P}{2M \quad 7m}$

$$a_1 = \frac{2P}{2M \quad 7m} + \frac{5}{2} \quad \frac{f}{M}$$

$$= \frac{2P}{2M \quad 7m} + \frac{5}{2} \cdot \frac{Ma_2}{M}$$

$$= \frac{2P}{2M \quad 7m} + \frac{5P}{2M \quad 7m}$$

$$= \frac{7P}{2M \quad 7m}$$

4.179. A small cylindrical tube of radius r and negligible mass is connected to two hoop of radius R by means of massless spokes. The mass of both the hoops is M. A string wound over the circumference of the tube passes over a weightless pulley. For the purpose of providing the tension in the string a weight W of mass m is attached to the end of the string as shown in the Fig. 4.142. Find (i) the acceleration of the weight (ii) the tension in the string (iii) the frictional force between the hoop and the surface by assuming that the hoop does not slip.

Solution:

(i) The velocity v of the weight and the velocity of the centre of gravity of the hoop v_0 are connected by the relation

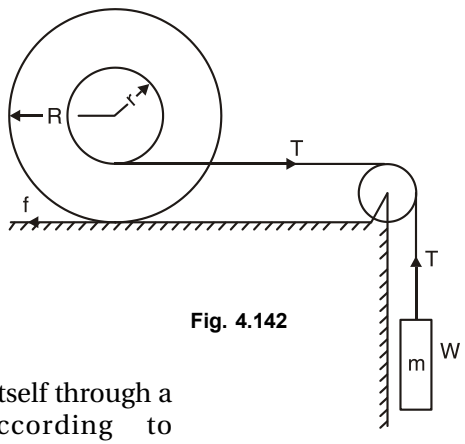


Fig. 4.142

$$\frac{v_0}{R} = \frac{v}{R \quad r}$$

or $v_0 = \dfrac{vR}{R \quad r}$

Let the weight W lowers itself through a distance h then according to conservation of energy we have

$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} Mv_0^2 + \frac{1}{2} MR^2\omega^2$$

$$= \frac{1}{2} mv^2 + Mv_0^2 ; v_0 = R\omega$$

or $mgh = \left[\dfrac{1}{2} m + \dfrac{M \cdot R^2}{(R-r)^2} \right] v^2$

or $$v = \sqrt{\frac{2mgh}{m\left(2M + \frac{R}{R-r}\right)^2}}$$

$$v^2 = 2ah = \frac{2mgh}{m\left(2M + \frac{R}{R-r}\right)^2}$$

$$\therefore a = \frac{mg}{m\left(2M + \frac{R}{R-r}\right)^2}$$

(ii) $mg - T = ma$; equation of motion of the weight W

or $T = m(g - a)$

$$= m\left(g - \frac{mg}{m\left(2M + \frac{R}{R-r}\right)^2}\right) = \frac{2m \cdot M \cdot g\left(\frac{R}{R-r}\right)^2}{m + 2M\left(\frac{R}{R-r}\right)^2}$$

(iii) $v_0 = v\left(\frac{R}{R-r}\right)$

Acceleration of the centre of gravity of the hoop $= \frac{dv_0}{dt} = \frac{R}{R-r} \cdot \frac{dv}{dt}$

$$= \frac{aR}{R-r}$$

$\therefore T - f = \frac{M \cdot aR}{R-r}$; f = force of friction between the hoop and the surface

$$\therefore f = T - \frac{MR}{R-r} \cdot a$$

$$= \frac{2mMg\left(\frac{R}{R-r}\right)^2}{m\left(2M + \frac{R}{R-r}\right)^2} - \frac{MR}{R-r} \cdot \frac{mg}{m\left(2M + \frac{R}{R-r}\right)^2}$$

$$= \frac{mMg\left(1 - \frac{r}{R}\right)}{m\left(1 - \frac{r}{R}\right)^2 2M}$$

4.180. A disc of mass m slides from rest down a plane AB inclined at an angle α with the horizontal and then moves through a distance l along the horizontal plane BC (see Fig. 4.143). Assuming the coefficient of friction between the disc and each of the two planes AB and BC as μ, find the work performed by the force of friction for the path AC.

Solution: Work done by the frictional force along $AB + BC$ is equal to $-\mu mg \cos\alpha \cdot x - \mu mgl$; x is the length of the plane AB

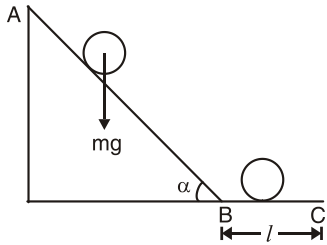


Fig. 4.143

Further

$mg\,[\sin\alpha - \mu\cos\alpha]\,x = \mu mgl$

or $x = \dfrac{l}{\sin \quad \cos}$

$$\therefore \text{Work done} = -\mu mgl - \mu mg\cos\alpha \cdot \frac{l}{\sin \quad \cos}$$

$$= -\frac{\mu mgl}{\sin\alpha - \mu\cos\alpha}\left[\sin\alpha - \mu\cos\alpha + \mu\cos\alpha\right]$$

$$= -\frac{\mu mgl\sin\alpha}{\sin\alpha - \mu\cot\alpha}$$

$$= -\frac{\mu mgl}{1 - \mu\cot\alpha}$$

4.181. Two thin walled cylinders of mass m and radius r are placed on a horizontal surface. A board of mass M is placed over these two cylinders see Fig. 4.144. Initially the system is at rest. A force P is applied to the board in the horizontal direction. Find (i) the accleration of the board (ii) the force of friction between the cylinders and the board (iii) the force of friction between the cylinders and the horizontal surface.

Solution:

(i) Let v be the velocity of the board. The velocity of centre of gravity of each of the cylinders is $\dfrac{v}{2}$

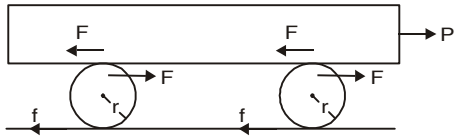


Fig. 4.144

Kinetic energy of the board + the two cylinders is

$$\frac{1}{2}Mv^2 + \frac{2mv^2}{4} = \frac{M \quad m \quad v^2}{2}$$

Let x be the distance covered then

$$\frac{M \quad m}{2} \cdot v^2 = 2\,Px$$

or $v^2 = \dfrac{2P}{M \quad m} \cdot x$

$\therefore$ acceleration of the board $a = \dfrac{P}{M \quad m}$

(ii) equation of motion of the board is given by

$M \cdot a = P - 2F$; F is force of friction between the board and the cylinders.

$$\therefore 2F = P - \frac{M \cdot P}{(M+m)}$$

$$F = \frac{mP}{2(M \quad m)}$$

(iii) $m \cdot \dfrac{a}{2} = F - f$ is the equation of motion of a cylinder

$$\therefore f = F - \frac{ma}{2}$$

$$= \frac{mP}{2 \ M \quad m} - \frac{m \cdot P}{2(M+m)} = 0$$

4.182. A uniform disc of radius R spinning about its own axis with an angular velocity ω is carefully placed on the horizontal rough surface. How long the disc will be rotating on the surface if the coefficient of friction is equal to μ.

Solution: Consider an element of radius r and $r + dr$. (See Fig. 4.145).

Area of the element $= 2\pi r dr$

Mass of the element $\dfrac{m}{R^2} \cdot 2\pi r dr$; m = mass of the disc.

Weight of the element $= \dfrac{m}{R^2} 2r dr \cdot g$

Frictional force on the element $= \dfrac{m}{R^2} \cdot 2r dr \cdot g$

Torque $d\tau$ on the element is given by

$$d\tau = \frac{m}{R^2} \cdot 2r dr \cdot g \cdot r$$

Net torque $\tau = \int d\tau = \dfrac{2 \ mg}{R^2} \int_0^R r^2 dr$

$$= \frac{2}{3} \frac{mgR^3}{R^2}$$

$$= \frac{2 \ mgR}{3}$$

$$\tau = I\alpha = \frac{1}{2} mR^2 \cdot \alpha = \frac{2 \ mgR}{3}$$

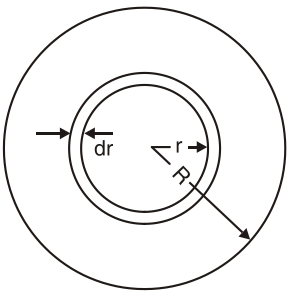


Fig. 4.145

$$\therefore \alpha = \frac{4}{3}\frac{g}{R}$$

$$\text{Time } t = \frac{}{} = \frac{3}{4}\frac{R}{g}$$

4.183. A spool with a thread wound on it, of mass m and moment of inertia relative to its own axis as I, is placed on a smooth plane inclined at an angle θ with the horizontal. The free end of the thread is attached to a rigid base as shown in the Fig. 4.146. Assuming the radius of the wound thread layer as r, find the accleration of the spool axis.

Solution: Equation of motion of the spool is given by

$$mg \sin\theta - T = ma$$

and $$T \cdot r = I\alpha = \frac{I \cdot a}{r}$$

$$\therefore T = \frac{I \cdot a}{r^2}$$

Now $$mg \sin\theta - \frac{Ia}{r^2} = ma$$

or $$a = \frac{mg \sin\theta}{m + \frac{I}{r^2}}$$

$$= \frac{g \sin\theta}{1 + \frac{I}{mr^2}}$$

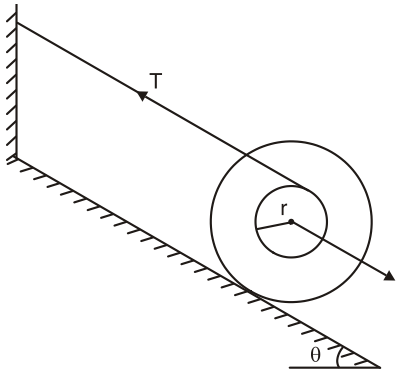


Fig. 4.146

4.184. Thin threads are tightly wound on the ends of a uniform solid cylinder of mass m. The free ends of the threads are connected to the ceiling of an elevator. The elevator is going up with an acceleration a_0. Find (i) the acceleration of the cylinder relative to the elevator (ii) the force exerted by the cylinder on the ceiling of the elevator.

Solution: The equation of motion of cylinder is given by

$mg + ma_0 = 2T + ma$; a is the acceleration of the cylinder relative to the elevator.

and $2T \times r = I\alpha$;

$r =$ radius of the cylinder

$I =$ moment of inertia of the cylinder about its own axis.

$$2T \times r = \frac{1}{2} mr^2 \cdot \frac{a}{r} \quad \text{(see Fig. 4.147)}$$

$$\therefore 2T = \frac{ma}{2}$$

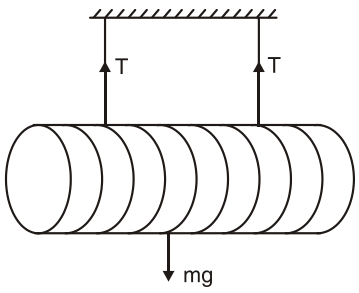


Fig. 4.147

On substituting the value of T we have

$$m(g + a_0) = \frac{ma}{2} + ma$$

$$\therefore a = \frac{2}{3}(g + a_0)$$

4.185. In the arrangement shown in the Fig. 4.148 two blocks of masses M_1 and M_2 are connected by a massless thread which passes over a frictionless pulley of mass m. The coefficient of friction between the block of mass M_1 and the horizontal plane is μ. Initially the system is at rest. At $t = 0$ the body of mass M_2 is decending. Find the work performed by the force of friction for a period of t sec from the beginning of the motion.

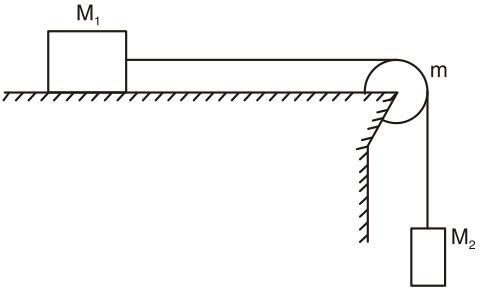


Fig. 4. 148

Solution: Let the block of mass M_2 descends to a distance x then according to conservation of energy we have

$$M_2 g \cdot x = \frac{1}{2} M_2 v^2 + \frac{1}{2} I\omega^2 + \frac{1}{2} M_1 v^2 + \mu M_1 g \cdot x$$

or $$(M_2 - \mu M_1) gx = \frac{1}{2} M_2 v^2 + \frac{1}{4} mv^2 + \frac{1}{2} M_1 v^2$$

On differentiating it with respect to t we have

$$(M_2 - \mu M_1)\, gv = M_2 v \cdot a + \frac{1}{2} mv \cdot a + M_1 va$$

or $$a = \frac{2\ M_2 \quad M_1\ g}{m \quad 2\ M_1 \quad M_2}$$

Now $$x = \frac{1}{2} at^2$$

$$= \frac{1}{2} \cdot \frac{2\ M_2 \quad M_1\ g}{m \quad 2\ M_1 \quad M_2} \cdot t^2$$

$$= \frac{(M_2 - \mu M_1) g \cdot t^2}{m + 2(M_1 + M_2)}$$

Work done by the force of friction $= -\mu M_1 gx$

$$= -\frac{\mu M_1 (M_2 - \mu M_1) g^2 t^2}{m + 2(M_1 + M_2)}$$

4.186. A man of mass m stands on the edge of a horizontal uniform disc of mass M and radius R which can rotate about an axis passing through its centre. At an instant the man moves along the edge through an angle relative to disc and then stops. Neglecting the dimensions of the man find (i) the angle through which the disc has turned by the time the man stopped (ii) the torque with which the man acted on the disc in the process of motion.

Solution:

(i) The distance travelled along the edge = R

Linear velocity = $R \cdot \dfrac{d}{dt}$

According to conservation of angular momentum we have

$$mR\frac{d}{dt} \cdot R + \frac{1}{2}MR^2 \quad mR^2 \quad \frac{d}{dt} = 0$$

or $$\frac{d}{dt} = -\frac{2m}{M \quad 2m} \cdot \frac{d}{dt}$$

or $$\phi = -\frac{2m}{M+2m} \cdot \phi'$$

(ii) $$R\frac{d}{dt} = -\frac{2m}{M+2m} \cdot R\frac{d}{dt}$$

$$v = \frac{2m}{M \quad 2m} \cdot v \; ; \; v = \text{velocity of the man}$$

$$\frac{dv}{dt} = -\frac{2m}{M+2m} \cdot \frac{dv}{dt}$$

or $$R\alpha = -\frac{2m}{M+2m} \cdot \frac{dv}{dt}$$

$$I\alpha = \text{Torque} = \frac{1}{2}\frac{M \cdot R^2}{R} \times -\frac{2m}{M+2m} \cdot \frac{dv}{dt}$$

$$= -\frac{mMR}{M+2m} \cdot \frac{dv}{dt}$$

4.187. A thin circular disc of mass m and radius r is rolling over a plane horizontal surface. Discuss the cases and determine the parameter involved.

Solution: Let v the linear velocity of the centre O of the disc and its plane AB be inclined at an angle θ with the vertical. The various torques acting on the disc are:

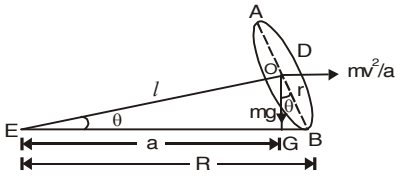


Fig. 4.149

(i) Gyrostatic torque T_1: This is due to its simultaneous rotation about its point of contact and about E (see Fig. 4.149). T_1 is given by

$T_1 = I\omega\phi$;

I = moment of inertia of the disc about OE

ω = angular velocity about OE

ϕ = rate of precession

$I = mk^2$; k = radius of gyration about OE

$v = r\omega$ and $\phi = \dfrac{v}{l}$

$$T_1 = -mk^2 \frac{v}{r} \cdot \frac{v}{l} = -mk^2 \frac{v^2}{r^2} \cdot \tan\theta$$

–ve sign shows that the torque acts in clockwise.

(ii) Gravitational torque T_2: This is due to the weight acting at O

$T_2 = -mgr \sin\theta$

It acts clockwise.

(iii) Centrifugal torque T_3: This is due to rotation about E.

$$T_3 = -\frac{mv^2}{a} \cdot OG = -\frac{mv^2}{l\cos\theta} \cdot r\cos\theta;\ EG = a.$$

In triangle OGE; $\cos\theta = \dfrac{a}{l} = \dfrac{OG}{r}$

$$T_3 = -mv^2 \cdot \frac{r}{l} = -mv^2 \tan\theta$$

It acts clockwise

For equilibrium we have

$$-mgr\sin\theta - (-mv^2\tan\theta) = -mk^2 \frac{v^2}{r^2} \cdot \tan\theta$$

or $$mv^2\tan\theta + mk^2 \frac{v^2}{r}\tan\theta = mgr\sin\theta$$

or $$v^2\tan\theta \left(1 - \frac{k^2}{r^2}\right) = gr\sin\theta$$

or $$v^2\left(1 - \frac{k^2}{r^2}\right) = gr\cos\theta$$

or $$\theta = \cos^{-1}\frac{v^2}{gr}\left(1 - \frac{k^2}{r^2}\right)$$

This angle θ is known as *angle of lean* of the disc for a given velocity v.

The minimum velocity known as *critical velocity* at which the disc can follow a straight path keeping its plane vertical i.e. $\theta = 0$ is given by

$$v_c = \sqrt{\frac{gr}{1 \quad \frac{k^2}{r^2}}}$$

For a disc $k^2 = \frac{r^2}{2}$

$$\therefore v_c = \sqrt{\frac{2gr}{3}}$$

When the velocity v is less than v_c the vertical position will be unstable and the disc will be tilted over by the force of gravity and attains an angle θ.

In the given figure $EB = R$ = radius of curvature of the path of the disc.

In right angled triangle EOB

$$\sin\theta = \frac{OB}{EB} = \frac{r}{R}$$

$$\therefore R = \frac{r}{\sin} = \frac{r}{\sqrt{1 \quad \cos^2}}$$

$$= \frac{r}{\sqrt{1 \quad \frac{v^4}{g^2r^2} \quad 1 \quad \frac{k^2}{r^2}^2}}$$

On putting $k^2 = \frac{r^2}{2}$, the radius of curvature R is given by

$$R = \frac{r}{\sqrt{1 - \frac{9v^4}{4g^2r^2}}}$$

4.188. A particle of mass m can slide without friction along the card AB of a circular disc of radius r rotating in a horizontal plane about an axis passing through its centre with an angular velocity ω (see Fig. 4.150). Find the period of oscillation of the particle if it is attached between the point A and B by two similar springs each of force constant $k/2$.

Solution: The base velocity and the base acceleration of the particle displaced through x from its original position O is given by

$$v_b = \omega\sqrt{h^2 \quad x^2}$$

$$a_b = \omega^2 \sqrt{h^2 \quad x^2}$$

The supplementary acceleration is equal to $2\omega\dot{x}$ and is perpendicular to AB. The equation of motion of the particle along x axis is given by

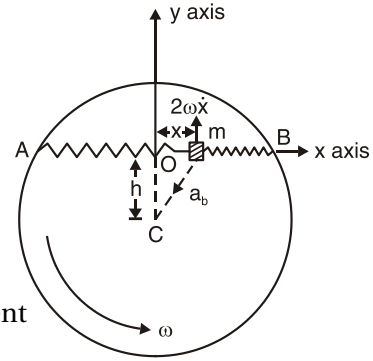


Fig. 4.150

$$m\ddot{x} = -kx + m\omega^2 x$$
$$= -(k - m\omega^2)\, x$$
$$\therefore \text{ accln.} = -\left(\frac{k - m\omega^2}{m}\right) \times \text{displacement}$$
$$\therefore \Omega = \frac{2}{T} = \sqrt{\frac{k \quad m^{\ 2}}{m}}$$

or T = period of oscillation $= 2\pi \sqrt{\dfrac{m}{k \quad m^{\ 2}}}$

4.189. A disc of radius $r = 1$ m has a cord AB at a distance $h = 0.5$ m from the centre. A particle moves with uniform relative velocity $v_r = 12$ m/sec along the cord of the disc when the disc rotates with uniform angular velocity $\omega = 20\pi$/sec. Assuming that the prticle is moving in the direction of rotation of the disc determine the absolute velocity and acceleration of the particle of the position shown in Fig. 4.151.

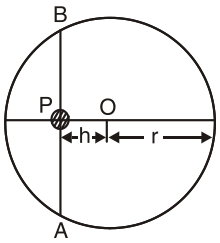


Fig. 4.151

Solution:

$$v = v_r + \omega h$$
$$= 12 + 20 \times 3.14 \times 0.5$$
$$= 12 + 31.4$$
$$= 43.4 \text{ m/sec}$$

Base acceleration $= \omega^2 h$

Supplementary acceleration $= 2\omega v_r$

$\therefore$ Net absolute acceleration $= \omega^2 h + 2\omega v_r$
$$= (20\pi)^2 \times 0.5 + 2 \times 20\pi \times 12$$
$$= 3479.1 \text{ m/sec}^2$$

4.190. A block X of mass 0.5 kg is held by a long massless string on a frictionless inclined plane of inclination 30° to the horizontal. The string is wound on a uniform solid cylindrical drum Y of mass 2 kg and of radius 0.2 m as shown in Fig. 4.152a. The drum is given an initial angular velocity, such that the block X starts moving up the plane: (i) Find the tension in the string during motion (ii) At a certain instant of time, the magnitude of the angular velocity Y is 10 rad/sec. Calculate the distance travelled by X from that instant of time until it comes to rest. (IIT 1994)

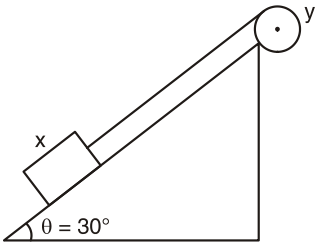


Fig. 4.152a

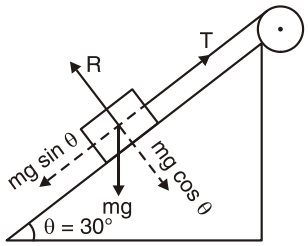


Fig. 4.152b

Solution: Equation of motion of block X along the plane is given by

$T - mg\sin\theta = ma$ (i) (see Fig 4.152b)

Torque $\tau = I\alpha = \frac{1}{2}MR^2\alpha$; M = mass of the drum and R is its radius

Further $T \cdot R = \frac{1}{2}MR^2 \cdot \frac{a}{R}$

$\therefore a = \frac{2T}{M}$ (ii)

On putting the value of a in the first equation, we have

$$T - mg\sin\theta = m \cdot \frac{2T}{M}$$

$$\therefore T = \frac{mg\sin}{1 \quad \frac{2m}{M}}$$

On substituting the values of various terms we have

$$T = \frac{0.5 \quad 9.8 \quad 0.5}{1 \quad \frac{2 \quad 0.5}{2}} = 4.9 \text{ N}$$

(ii) The block will stop when the kinetic energy of drum and mass is converted in gravitational potential energy of the block. Therefore

$\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = mg \cdot \frac{s}{2}$; s is the distance moved by the block.

$$\frac{1}{2} \quad \frac{1}{2}MR^2 \quad \omega^2 + \frac{1}{2}mR^2\omega^2 = \frac{mgs}{2}$$

or $\frac{1}{2}R^2\omega^2 \quad m \quad \frac{M}{2} = \frac{mgs}{2}$

or $s = \frac{M \quad 2m}{2mg}R^2\omega^2$

On putting the values we have

$$s = \frac{2 \quad 2 \quad 0.5 \quad 0.2^{\,2} \quad 10^2}{2 \quad 0.5 \quad 9.8} = 1.224 \text{ m}$$

4.191. A rectangular rigid fixed block has a long horizontal edge. A solid homogenous cylinder of radius r is placed horizontally at rest with its length parallel to the edge such that the axis of the cylinder and the edge of the block are in the same vertical plane. There is a sufficient friction present at the edge so that a very small displacement causes the cylinder to roll off the edge without slipping. Determine (a) the angle θ through which the cylinder rotates before it leave contact with the edge (b) the speed of the centre of mass of the cylinder before leaving contact with the edge and (c) the ratio of translational to rotational kinetic energies of the cylinder when its centre of mass is in horizontal line with the edge.

(IIT 1995)

Solution: When the cylinder rotates through an angle θ the loss in potential energy is equal to $mgr\,(1 - \cos\theta)$ and this will turn into kinetic energy, hence

$$mgr\,(1 - \cos\theta) = \frac{1}{2}\,mv^2 + \frac{1}{2}\,I\omega^2 \quad \text{(see Fig. 4.153)}$$

Further $I = \frac{1}{2}\,mr^2$ and $v = r\omega$

$$\therefore\ mgr\,(1 - \cos\theta) = \frac{3}{4}\,mv^2$$

or $\quad \frac{3}{4}\,v^2 = gr\,(1 - \cos\theta)$

For circular motion of cylinder about the edge we have

$$mg\cos\theta - R = \frac{mv^2}{r}\ ;\ R = \text{reaction force}$$

When the cylinder will leave the contact $R = 0$

$$\therefore\ g\cos\theta = \frac{v^2}{r}$$

or $\quad \frac{3}{4}\cdot gr\cos\theta = gr\,(1 - \cos\theta)$

or $\quad \cos\theta = \frac{4}{7}$

(a) or $\quad \theta = \cos^{-1}\ \frac{4}{7}$

(b) $v = \sqrt{\frac{4}{7}gr}$

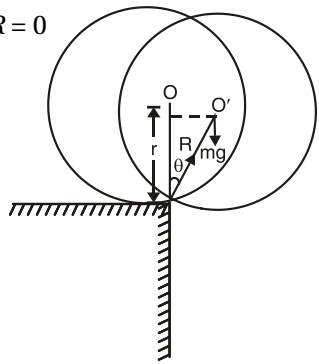


Fig. 4.153

(c) E_1 = rotational kinetic energy $= \frac{1}{2}\,I\omega^2 + \frac{1}{4}\,mv^2 = \frac{mgr}{7}$

Further $mgr = \frac{1}{7}\,mgr$ + Translational kinetic energy E_2

$$\therefore \quad E_2 = \frac{6}{7} mgr$$

$$\therefore \quad \frac{E_2}{E_1} = 6$$

4.192. A solid sphere of mass m and radius r, rotating with an angular velocity ω_0 is placed on a rough inclined plane as shown in Fig. 4.154. The ball moves up the plane. Assuming the coefficient of friction between the sphere and plane as μ discuss the motion of the ball and calculate the time at which the ball starts rolling up the plane without slip.

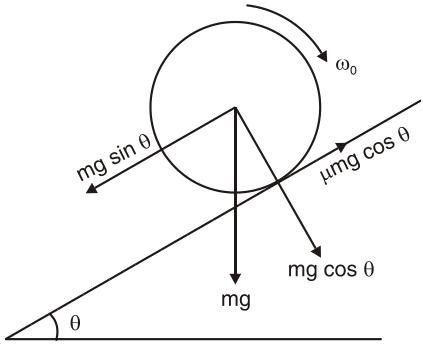


Fig. 4.154

Solution: The slipping will happen when $v = r\omega$

For linear motion we have

$$\mu mg \cos\theta - mg \sin\theta = ma; \; a = \text{acceleration}$$

or $\mu g \cos\theta - g \sin\theta = a$

For rotational motion we have

$$\mu mgr \cos\theta = \frac{2}{5} mr^2 \cdot \alpha$$

or $\frac{2}{5} r\alpha = \mu g \cos\theta$

now $v = at$ and $\omega = \omega_0 - \alpha t$

$$\mu g \cos\theta = \frac{2r}{5t} (\omega_0 - \omega) = \frac{2}{5} \quad \frac{r \quad _0}{t} \qquad g \cos \qquad g \sin$$

or $t = \dfrac{2 \quad _0 r}{g \quad 7 \quad \cos \qquad 2 \sin}$

$$v = at = (\mu g \cos\theta - g \sin\theta) \times \frac{2 \quad _0 r}{g \quad 7 \quad \cos \qquad 2 \sin}$$

$$= \frac{2 \quad _0 r \quad \cos \quad \sin}{7 \quad \cos \qquad 2 \sin}$$

5

FLUID PRESSURE AND BUOYANCY

5.1. A cubical block of wood 10 cm along each side floats at the interface between an oil and water with its lower surface 2 cm below the interface. The heights of the oil and water columns are 10 cm each. The density of oil is 0.8 gm/cm^3 and the density of water is 1 gm/cm^3. Find (i) the mass of the block, and (ii) the pressure at the lower surface of the block.

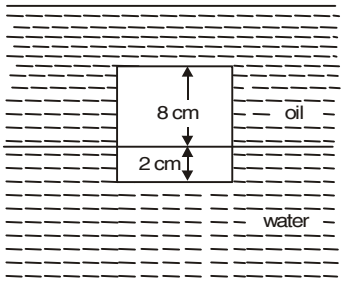


Fig. 5.1

Solution:

(i) In Fig. 5.1,

$$mg = 100 \times 8 \times 0.8 \times g + 100 \times 2 \times 1 \times g$$

or $\quad m = 640 + 200 = 840\ g$

(ii) Total pressure at the lower surface $= 10 \times 980 \times 0.8 + 2 \times 980 \times 1$

$= 980 \times 10 = 9800$ dyne/cm^2

5.2. A solid sphere of density ρ and volume V floats at the interface of two liquids (Fig. 5.2). The liquids are immiscible and have densities ρ_1 and ρ_2, $\rho_1 < \rho < \rho_2$.

What fraction of the volume of the sphere will lie in the two liquids ?

Solution: Let v_1 and v_2 be the volumes of the portions of sphere lying in the liquids of densities ρ_1 and ρ_2 respectively. Then $V = v_1 + v_2$.

In the condition of equilibrium, we have

$$(v_1 + v_2)\rho = v_1\rho_1 + v_2\rho_2$$

or $\quad v_1 (\rho - \rho_1) = v_2 (\rho_2 - \rho)$

Further $v_1 + v_2 = V$

or $\quad v_1 (\rho - \rho_1) = (V - v_1) (\rho_2 - \rho)$

or $\quad v_1 (\rho - \rho_1 + \rho_2 - \rho) = V (\rho_2 - \rho)$

or $\quad v_1 = \dfrac{V \quad 2}{2 \quad 1}$

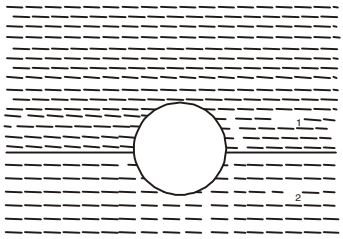


Fig. 5.2

Similarly $v_2 = \dfrac{V}{\quad}$ $\quad{}_2 \quad {}_1$

Fraction of volume lying in the liquid of density $\rho_1 = \dfrac{v_1}{V} = \dfrac{\quad}{\quad}$ $\quad{}_2 \quad {}_1$

Fraction of volume lying in the liquid of density $\rho_2 = \dfrac{v_2}{V} = \dfrac{\quad}{\quad}$ $\quad{}_2 \quad {}_1$

5.3. A tube floats vertically in water having density of 1 g/cm^3 such that 4 cm of the tube lies above the surface (see Fig. 5.3). What will be the length of the tube if it is filled completely with a liquid of density 0.9 g/cm^3.

Solution: Let l be the length of the tube and A its area of cross-section. On equalizing the pressure at the bottom of the tube, we have

$$l \times 0.9 \times A \times g = (l - 4) \times 1 \times A \times g$$

(All in the C.G.S. system.)

or $0.9l = l - 4$

or $4 = 0.1l$

or $l = 40$ cm

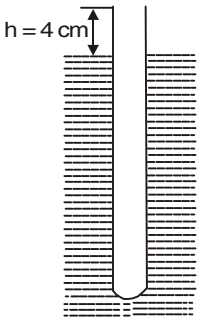


Fig. 5.3

5.4. Water is filled to a height H behind a width b of a dam. Determine the resultant force on it.

Solution: Pressure P at a depth y is given by

$P = \rho g\,[H - y]$; ρ = density of water

Consider a layer of thickness dy at a height y above the bottom.

$$dF = P \cdot b dy = \rho g b\,[H - y]\,dy \text{ (see Fig. 5.4)}$$

$$\int_0^F dF = \int_0^H \rho g b [H - y] dy$$

$$= \rho g b \quad H^2 \quad \frac{H^2}{2} = \frac{1}{2} \rho g b H^2$$

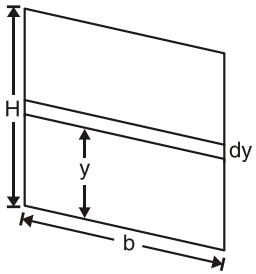


Fig. 5.4

5.5. A container full of water has a base area of $a_1 = 20$ cm^2 and top area as $a_2 = 10$ cm^2, height 25 cm contains 2 litre of water (see Fig. 5.5). Calculate the force at the base of the container. Atmospheric pressure = 10^5 N/m^2.

Solution: Net force at the base is given by

$F = Pa_2 + mg$

On putting the values we have

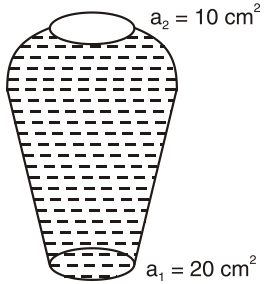


Fig. 5.5

$$F = 10^5 \times 10 \times 10^{-4} + 20 \times 10^{-4} \times 0.25 \times 1000 \times 9.8$$
$$= 100 + 4.9 = 104.9 \text{ N}$$

5.6. A tank is filled with a heterogeneous liquid whose density varies with depth y via relation $\rho = \rho_0 \frac{y}{k}$. Calcualte the total pressure at a distance y below the surface of the liquid. Atmospheric pressure is P_0.

Solution: Consider a thin layer of width dx at a depth x. Density at this depth $= \rho_0 \frac{x}{k}$

Pressure $dp = \frac{\rho_0 x dx g}{k}$

Net pressure at a depth y due to liquid $= P = \int dp$

$$= \int_0^y \frac{g\rho_0}{k} \cdot x dx = \frac{g \rho_0 y^2}{2k}$$

Total pressure $= P_0 + \frac{\rho_0 g y^2}{2k}$

5.7. A hollow metallic sphere of inner and outer radii as 3 cm and 5 cm respectively floats in water of density $\rho = 10^3$ kg/m^3 with 50% of its volume inside water. Calculate the density of the material of the sphere.

Solution: In equilibrium we have

$$\frac{4}{3}\pi [5^3 - 3^3] \times 10^{-6} \sigma \cdot g = 0.5 \times \frac{4}{3}\pi \cdot 5^3 \times 10^{-6} \times 10^3 \times g;$$

σ = density of the material of the sphere.

$$(125 - 27)\sigma = 62.5 \times 1000$$

$$\sigma = \frac{62.5 \quad 1000}{98} = 637.75 \text{ kg/m}^3$$

5.8. A jeweller prepares an ornament of weight 57.9 gm in air with a claim that he has used only 3 gm of copper. In view of the colour of the ornament his statement seems to be doubtful. In order to know the exact amount of copper the ornament was weighed in water and it was found to have a weight of 54.5 gm. Assuming the specific gravity of the gold and copper as 19.3 and 8.9 respectively. Calculate the amount of copper.

Solution: Let x be the mass of copper in the ornament then

$$\frac{57.9 \quad x}{19.3} + \frac{x}{8.9} = 57.9 - 54.5 = 3.4$$

or $\quad 3 + x \quad \frac{1}{8.9} \quad \frac{1}{19.3} = 3.4$

or $\quad x = \frac{0.4 \quad 8.9 \quad 19.3}{10.4} = 6.6 \text{ gm}$

5.9. A cubical block of ice having a piece of metal of mass 100 gm at its top floats in water. Calculate the minimum edge of the block so that it does not sink in water, specific gravity of ice = 0.9.

Solution: $100 \times 980 + L^3 \times 0.9 \times 980 = L^3 \times 1 \times 980$; $L =$ length of the block in cm.

$$0.1\,L^3 = 100$$
$$L^3 = 1000$$
$$L = 10 \text{ cm}$$

5.10. A piece of lead is fastened underneath of a wooden block of mass 184 gm. For what value of the mass of lead piece the wooden block just float in water? Specific gravity of wood = 0.8 and specific gravity of lead = 11.3.

Solution:

$$184 \times 980 + m \times 980 = \frac{184}{0.8} \times 1 \times 980 + \frac{m}{11.3} \times 1 \times 980;$$

m = mass of lead piece.

$$m\left(1 - \frac{1}{11.3}\right) = 184\left(\frac{1}{8} - 1\right) = 46$$
$$\frac{10.3m}{11.3} = 46$$
$$\therefore m = \frac{46 \times 11.3}{10.3} = 50.47 \text{ gm}$$

5.11. An iceberg floats in sea water of density $\rho = 1.03 \times 10^3$ kg/m^3. What percentage of its total volume is above the surface of water? Density of ice $\sigma = 0.92 \times 10^3$ kg/m^3.

Solution: In the conditions of equilibrium, we have

$V\sigma g = V'\rho g$; V = total volume of ice

V' = volume of ice inside water

$$1 - \frac{V'}{V} = 1 - \frac{V}{V}$$
$$\frac{V - V'}{V} \times 100 = \frac{V - V}{V} \times 100$$
$$= \frac{1.03 - 0.92}{1.03} \times 100$$
$$= 10.68\%$$

5.12. A piece of ice is floating in water. What will happen to the level of water when all the ice melts? What will happen if the beaker is filled not with water but with liquid (a) denser than water (b) lighter than water.

(IIT 1974, 1977; Roorkee 1976)

Solution: Let M mass of ice is floating in a liquid of density σ then in equilibrium

$Mg = V_L \sigma g$; σ is the density of liquid

V_L = volume of liquid displaced

$\therefore V_L = \frac{M}{\quad}$

If the ice melts to form water then

$V_W = \frac{M}{\quad}$; V_W = volume of water

ρ = density of water

Here $\sigma = \rho$ hence $V_L = V_W$ i.e. under this condition the level of water will be unchanged.

(a) If $\sigma > \rho$ then $V_W > V_L$

Due to this the level of liquid in the beaker will rise

(b) If $\sigma < \rho$ then $V_L > V_W$

Due to this the level of liquid in the beaker will fall.

5.13. A cubical block of iron 5 cm on each side is floating on mercury in a vessel. (a) What is the height of the block above mercury level? (b) Water is poured into the vessel so that it just covers the iron block. What is the height of water column? Relative density of mercury = 13.6 and relative density of iron = 7.2. (IIT 1973)

Solution: Let l be the length of the iron cube

(a) $l^3 \rho g = (l - h)\, l^2 \sigma g$ (see Fig. 5.6a & 5.6b)

$$\frac{\quad}{\quad} = \frac{l \quad h}{l} = 1 - \frac{h}{l}$$

$$\therefore h = l \quad 1 \quad \frac{\quad}{\quad}$$

On substituting the value we have

$$h = 5 \quad 1 \quad \frac{7.2}{13.6} \quad = \frac{5 \quad 6.4}{13.6} = 2.35$$

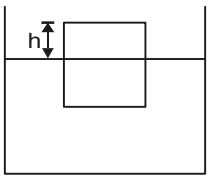


Fig. 5.6a

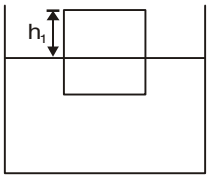


Fig. 5.6b

(b) $l^3 \rho g = l^2 h_1 \sigma_w g + l^2 (l - h_1) \sigma g$

$l\rho = h_1 \sigma_w + l\sigma - h_1 \sigma$

$h_1 (\sigma - \sigma_w) = l (\sigma - \rho)$

$$\therefore h_1 = \frac{l}{\quad}$$

w

On substituting the values we have

$$h_1 = \frac{5 \quad 13.6 \quad 7.2}{13.6 \quad 1} = 2.54 \text{ cm}$$

5.14. A piece of metal of mass 210 gm and density 7×10^3 kg/m^3 is placed in a glass filled with water. The glass is kept in a vacuum chamber (Fig. 5.7). Calculate the normal force exerted by the bottom of the glass on the metal. Take $g = 10$ m/sec^2.

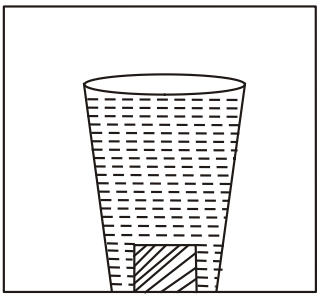


Fig. 5.7

Solution: Volume V of the metal

$$= \frac{210}{7} \frac{10^{-3}}{10^3} = 3 \times 10^{-5} \text{ m}^3$$

Upward thrust $= 3 \times 10^{-5} \times 10^3 \times 10 = 0.3$ N

Weight of metal $= 210 \times 10^{-3} \times 10 = 2.1$ N

Normal force $= 2.1 - 0.3 = 1.8$ N

5.15. To what height should a cylindrical vessel is filled with a homogeneous liquid to make the force, with which the liquid presses on the side of the vessel equal to the force exerted by the liquid on the bottom of the vessel. (IIT 1976)

Solution: Let r be the radius of the cylinder and h be the height of the liquid (See Fig. 5.8).

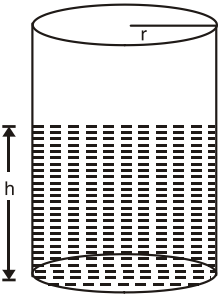


Fig. 5.8

Mass of the liquid in the cylinder $= \pi r^2 h\rho$;

ρ = density of the liquid.

Weight of the liquid $= \pi r^2 h\rho g$

Mean pressure on the wall $= \frac{1}{2} \rho g h$

Force on the wall $= \frac{1}{2} \rho g h \times 2\pi r h = \pi r \rho g h^2$

On equalising the two forces we have

$$\pi r^2 h\rho g = \pi r \rho g h$$

or $$r = h$$

i.e. the liquid should be filled upto a height equal to the radius of the cylinder.

5.16. A vertical spring of force constant $K = 60$ N/m is fixed between the bottom of a vessel and the underneath of a cubical block of wood of edge 4 cm (see Fig. 5.9). The vessel is filled with water and the wooden block floats in it. Calculate the maximum weight that can be put on the wooden block without wetting it. Density of wood = 800 kg/m^3 and $g = 10$ m/sec^2.

Solution: Let h be the height of block inside water then

$$4 \times 4 \times 4 \times 0.8 \times 980 = 4 \times 4 \times h \times 1 \times 980$$

$$\therefore \ h = 3.2 \text{ cm}$$

Height of the block outside water = 0.8 cm

Upward thrust = $64 \times 10^{-6} \times 1000 \times 10$

$= 0.64$ N

Force due to spring = $0.8 \times 10^{-2} \times 60$

$= 0.48$ N

Weight of the wooden block = $64 \times 10^{-6} \times 800 \times 10$

$= 0.512$ N

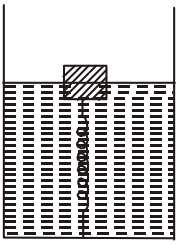


Fig. 5.9

Weight W placed = $0.64 + 0.48 - 0.512 = 0.608$ N

5.17. A piece of metal floats on mercury. The coefficient of volume expansion of the metal and mercury are γ_1 and γ_2 respectively. If the temperature of both mercury and metal are increased by an amount ΔT, by what factor the fraction of the volume of metal submerged in mercury changes. (IIT 1991)

Solution: In equilibrium we have

$V\rho g = V \sigma g$; V = volume of metal

ρ = density of metal

V = volume of metal inside mercury

σ = density of mercury

$\therefore f = \frac{V}{V} = —$

On increasing the temperature the fraction f changes due to change of the two densities.

$$f = —$$

$$\therefore \text{Change} = \frac{f}{f} = \frac{f \quad f}{f} = \frac{f}{f} - 1$$

$$= — \cdot — - 1$$

$$= \frac{1+\gamma_2 \Delta T}{1+\gamma_1 \Delta T} - 1$$

$$= \frac{(\gamma_2 - \gamma_1)\Delta T}{1+\gamma_1 \Delta T}$$

5.18. A body floating in water has volumes v_1, v_2, v_3 above the surface where the densities of the surrounding air are ρ_1, ρ_2 and ρ_3. Show that

$$\frac{2 \quad 3}{v_1} + \frac{3 \quad 1}{v_2} + \frac{1 \quad 2}{v_3} = 0$$

Solution: Let the body be allowed to float in water such that a volume v_1 is above the surface of water and let the density of the surrounding air be ρ_1. The weight of the body is given by

$$W = (v - v_1)\, m + v_1\, g\rho_1$$

Here m is the weight per unit volume of the water displaced and v is the volume of the body

$$\text{or} \quad \frac{W \quad vm}{v_1} + m - \rho_1 g = 0 \tag{i}$$

We can have similar equations when the volumes v_2 and v_3 of the body are above water and the densities of the surrounding air are ρ_2 and ρ_3 respectively.

$$\frac{W \quad vm}{v_2} + m - \rho_2 g = 0 \tag{ii}$$

$$\text{and} \quad \frac{W \quad vm}{v_3} + m - \rho_3 g = 0 \tag{iii}$$

On multiplying equation (i) by $(\rho_2 - \rho_3)$, equation (ii) by $(\rho_3 - \rho_1)$ and equation (iii) by $(\rho_1 - \rho_2)$ and then adding the three, we get

$$(W - vm) \quad \frac{2 \quad 3}{v_1} \quad \frac{3 \quad 1}{v_2} \quad \frac{1 \quad 2}{v_3} + m\,(\rho_2 - \rho_3 + \rho_3 - \rho_1 + \rho_1 - \rho_2)$$
$$- g\,[\rho_1\,(\rho_2 - \rho_3) + \rho_2\,(\rho_3 - \rho_1) + \rho_3\,(\rho_1 - \rho_2)] = 0$$

$$\text{or} \quad (W - vm) \quad \frac{2 \quad 3}{v_1} \quad \frac{3 \quad 1}{v_2} \quad \frac{1 \quad 2}{v_3} \quad 0$$

Since $W - vm \neq 0$,

$$\frac{2 \quad 3}{v_1} + \frac{3 \quad 1}{v_2} + \frac{1 \quad 2}{v_3} = 0$$

5.19. Three liquids whose densities are in *AP* series are filled in a semicircular tube whose bounding diameter is horizontal. Show that the depth *ON* of one of the common surfaces is double that of *OM* (see Fig. 5.10).

Solution: Let the densities of the liquid in the parts *AE*, *EF* and *FB* be ρ, $(\rho + \alpha)$ and $(\rho - \alpha)$ respectively. The depth of the common surfaces *F* and *E* are *OM* and *ON* respectively. On equating the pressure at *O* we have,

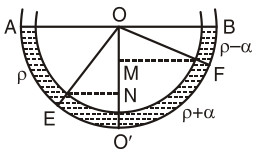


Fig. 5.10

$$OM \cdot g\,(\rho - \alpha) + O\,M \cdot g\,(\rho + \alpha) = O\,N \cdot g\,(\rho + \alpha) + ON \cdot g \cdot \rho$$

or $OM\,(\rho - \alpha) + (r - OM)\,(\rho + \alpha) = (r - ON)\,(\rho + \alpha) + ON\rho$

here r = radius of the tube

or $OM\,(\rho - \alpha - \rho - \alpha) = ON\,(\rho - \rho - \alpha)$

or $2\,OM = ON$

5.20. A tube in the form of a parabola held with its vertex downwards and axis vertical is filled with liquids of densities ρ_1 and ρ_2 as shown in Fig. 5.11. If the distances of the free surfaces of the liquids from the focus are l_1 and l_2 respectively, calculate the distance of their common surface from the focus of the parabola.

Solution: In Fig. 5.11, S and O represent focus and the vertex of the parabola.

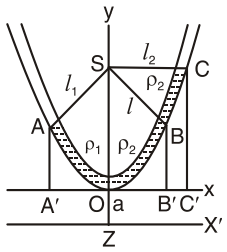


Fig. 5.11

Here, $l_2 = SC = CC + a$

$l = SB = BB + a$

and $l_1 = AS = AA + a$

On equating the pressure at O, we have

$$\rho_1 \cdot g \cdot AA = \rho_1 g BB = \rho_2 CC \quad BB$$

or $\rho_1 \quad AA \quad BB = \rho_2 \quad CC \quad BB$

or $\rho_1 (l_1 - l) = \rho_2 (l_2 - l)$

or $(\rho_1 l_1 - \rho_2 l_2) = l (\rho_1 - \rho_2)$

or $l = \dfrac{{}_1 l_1 \quad {}_2 l_2}{{}_1 \quad {}_2}$

5.21. A hollow cone whose axis is vertical and base downwards is filled with equal volumes of two liquids whose densities are in the ratio 3:1. Show that the thrust on the base is $\left(3 - 4^{\frac{1}{3}}\right)$ times as much as it is when the vessel is filled with the higher liquid.

Solution: In Fig. 5.12 let $AO = h_1$ and $OF = r_1$. The volume of the portion $AEF = (1/2) \times$ volume of the cone ABC, i.e.,

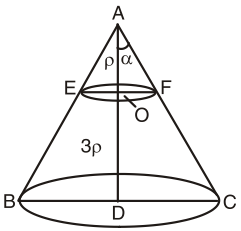


Fig. 5.12

$$\frac{1}{3}\pi h_1^2 \tan^2\alpha \cdot h_1 = \frac{1}{2} \cdot \frac{1}{3}\pi h^2 \tan^2\alpha \cdot h;$$

here $h = AD$ and $\angle DAC = \alpha$

or $h_1 = h\left(\dfrac{1}{2}\right)^{\frac{1}{3}}$

Total pressure of the liquid at the base is given by

$$P = (\rho g \cdot AO + 3\rho g \cdot OD)\, \pi h^2 \tan^2\alpha$$

$$= \pi \rho g h^3 \left[\frac{1}{2^{\frac{1}{3}}} + 3\left(1 - \frac{1}{2^{\frac{1}{3}}}\right)\right] \tan^2\alpha$$

or $P = \pi\rho g h^3 \left[3 - \frac{2}{2^{\frac{1}{3}}}\right] \tan^2 \alpha$

If the cone is filled by a liquid of density ρ alone then the pressure at the base is given by

$$P = \rho g h \cdot \pi h^2 \tan^2 \alpha$$

$$\therefore \frac{P}{P} = \left[3 - 2^{\frac{2}{3}}\right]$$

or $$\frac{P}{P} = \left[3 - 4^{\frac{1}{3}}\right]$$

5.22. Two identical cylindrical vessels with their base at the same level each contain a liquid of density ρ. The area of either base is A, but in one vessel the liquid height is h_1 and in the other h_2. Find the work done by gravity in equalizing the levels when the two vessels are connected.

Solution: Let us assume that $h_1 > h_2$. For the purpose of equalizing the levels let the liquid from the height h_1 come down through a distance x. In the other vessel the liquid column will go up through the same distance.

Thus we have $h_1 - x = h_2 + x$

or $\frac{h_1 \quad h_2}{2} = x$

Work done

$$= \text{ weight of the liquid column of height } \frac{h_1 \quad h_2}{2} \times \frac{h_1 \quad h_2}{2}$$

$$= \rho A \frac{h_1 \quad h_2}{2} \quad \frac{h_1 \quad h_2}{2} \quad g = \frac{1}{4} \rho A g (h_1 - h_2)^2$$

5.23. A body floats in a liquid of specific gravity ρ with as much of its volume out of the liquid as would be immersed in a second liquid of specific graity . If it floated in that liquid show that the specific gravity of the body is / .

Solution: Let V be the volume and σ be the specific gravity of the body when it floats in the first liquid of specific gravity ρ. Let v_1 be the volume of the body outside the liquid. In equilibrium

$V \sigma g = (V - v_1) \rho g$

When the body floats in the second liquid,

$V \sigma g = v_1 \quad g$

Now $V\sigma = V \quad 1 \quad — \quad \rho \quad \text{or} \quad \sigma\left(1 + \frac{\rho}{\rho'}\right) = \rho \quad \text{or} \quad \sigma = ——$

5.24. A uniform bore *U* tube *AOB* contains a column of mercury. In the arm *AO* water has been poured until it stands at a height h = 44.8 cm above the mercury level. The remaining space in the two arms of the tube is filled with kerosene. Calculate the difference in the level of the mercury column in the two arms, given the density of mercury = 13.6 g/cm^3 and density of kerosene = 0.8 g/cm^3.

Solution: Let h_1 and h_3 respectively be the lengths of mercury and kerosene columns in the arm containing water and h_2 be the length of the mercury column in the other arm. Now pressure at C = pressure at D (see Fig. 5.13).

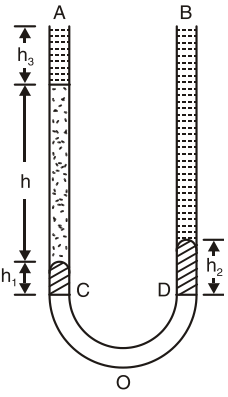


Fig. 5.13

Now $h_1 \times 13.6 \times g + h \times 1 \times g + h_3 \times 0.8 \times g$

$= h_2 \times 13.6 \times g + (h_1 + h + h_3 - h_2) \times 0.8 \times g$

or $h + 0.8h_3$

$= (h_2 - h_1) \times 13.6 + 0.8h_3 + 0.8h - (h_2 - h_1) \times 0.8$

or $0.2h = (h_2 - h_1)\ [13.6 - 0.8]$

or $h_2 - h_1 = \dfrac{0.2h}{12.8} = \dfrac{0.2 \quad 44.8}{12.8} = 0.7$ cm

5.25. A hollow cylinder of radius r and height h is half filled with liquid of density ρ_1 and the remaining half with a liquid of density ρ_2 ($\rho_1 > \rho_2$). Find the ratio of the pressure on the base and the total pressure on the curved surface.

Solution: Pressure on the base is given by

$$P = \pi r^2 \quad {}_1g\frac{h}{2} \quad {}_2g\frac{h}{2} \quad \text{(see Fig. 5.14)}$$

$$= \frac{r^2 gh}{2}\ (\rho_1 + \rho_2)$$

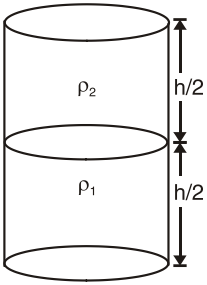


Fig. 5.14

Pressure on the curved surface of the lower half is given by

$$P_1 = \pi rh \quad {}_2g\frac{h}{2} \quad {}_1g\frac{h}{4}$$

Pressure on the curved surface of the upper half $P_2 = \pi rh\left[g\rho_2 \cdot \dfrac{h}{4}\right]$

Net pressure $P_3 = P_1 + P_2 = \pi rh\left[\rho_1 g\dfrac{h}{4} + \rho_2 g\dfrac{3h}{4}\right]$

$$= \frac{rh^2 g}{4}\ [\rho_1 + 3\rho_2]$$

$$\therefore \frac{P}{P_3} = \frac{\pi r^2 gh(\rho_1+\rho_2)\times 4}{2\pi r h^2 g\cdot(\rho_1+3\rho_2)}$$

$$= \frac{2r(\rho_1+\rho_2)}{h(\rho_1+3\rho_2)}$$

5.26. A solid sphere of radius r floats just immersed in three liquids whose densities are ρ, 2ρ and 3ρ. The thickness of each of the two upper liquids is equal to $r/2$. Calculate the density of the sphere.

Solution:

Volume of segment AEF of the sphere

$$= \frac{r^4}{4} \quad r \quad \frac{1}{3}\frac{r}{2}$$

$$= \frac{5r^3}{24}$$

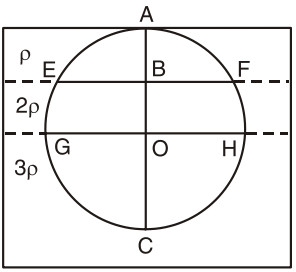


Fig. 5.15

Volume of segment $GCH = \pi r^2 \quad r \quad \frac{r}{3}$

$$= \frac{2}{3} r^3 \text{ (see Fig. 5.15)}$$

Volume of segment $EFHG = \frac{4}{3} r^3 - \frac{5}{24} r^3 - \frac{2}{3} r^3$

$$= \frac{11}{24} r^3$$

In equilibrium, we have

$$\frac{4}{3}\pi r^3\sigma g = \frac{5}{24} r^3 \rho g + \frac{11\pi r^3}{24}\cdot 2\rho\cdot g + \frac{2}{3} r^3 \cdot 3\rho g$$

$$\frac{4}{3}\sigma = \frac{8}{3}$$

$$\sigma = 2\rho$$

5.27. A cylinder of radius r, density 2σ and height h floats in two immersible liquids of densities σ and 3σ by keeping its axis vertical. The depth of the upper liquid is equal to h. Calculate the ratio of the pressure at the two ends of the cylinder.

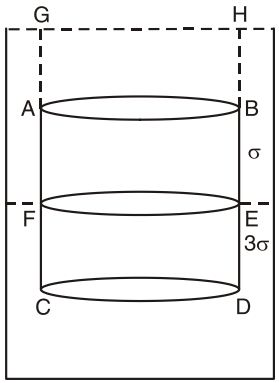


Fig. 5.16

Solution: $EH = BD = h$

Let $BH = x$, $BE = h - x$

In the conditions of equilibrium

$$\pi r^2 h\cdot 2\sigma\cdot g = \pi r^2 (h-x)\,\sigma\cdot g + \pi r^2\, 3\sigma\, xg$$

(see Fig. 5.16)

$$2h = (h - x) + 3x$$

or $$x = \frac{h}{2}$$

P_1 = pressure at the base $AB = \pi r^2 \cdot \frac{h}{2} \sigma g$

P_2 = pressure at the lower base CD

$$= \pi r^2 \quad h \quad g \quad \frac{h}{2} 3 \quad g \quad = \frac{5}{2} \pi r^2 h \sigma g$$

$$\therefore \frac{P_1}{P_2} = \frac{1}{5}$$

5.28. A cylinder of density ρ and height h floats with its axis vertical in a liquid of density σ_1; if now the density of the air increases from σ_2 to σ_3 find by much the cylinder will rise or sink.

Solution: Let x be the length of cylinder inside the liquid then we have

$\pi r^2 h \rho g = \pi r^2 x \sigma_1 g + \pi r^2 (h - x) \sigma_2 g$; r = radius of the cylinder

σ_2 = density of the surrounding air

or $h [\rho - \sigma_2] = x (\sigma_1 - \sigma_2)$

On changing the density of surrounding air from σ_2 to σ_3 we have a similar equation.

$\pi r^2 h \rho g = \pi r^2 x \ \sigma_1 g + \pi r^2 (h - x) \sigma_3 g$; x is length of cylinder inside the liquid

or $h [\rho - \sigma_3] = x \quad (\sigma_1 - \sigma_3)$

Now $x - x = h$ ——— ——— = ———

5.29. A, B & C are three balls of equal weight W. A balances B and C when all are suspended in a liquid of density σ_1; B balances C and A in a liquid of density σ_2; C balances A and B in a liquid of density σ_3. Find the specific gravity of A, B & C.

Solution:

$W \quad \frac{W}{d_1} \cdot \quad = \quad W \quad \frac{W}{d_2} \cdot \quad + \quad W \quad \frac{W}{d_3} \cdot \quad$; d_1, d_2 & d_3 are the specific gravity of the three balls.

or $$-\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} = \frac{1}{\quad} \qquad \text{(i)}$$

In liquids of densities σ_2 and σ_3 we have

$$W\left(\frac{W}{d_2}\right)_2 = W\left(\frac{W}{d_1}\right)_2 + W\left(\frac{W}{d_3}\right)_2$$

and $$W\left(\frac{W}{d_3}\right)_3 = W\left(\frac{W}{d_1}\right)_3 + W\left(\frac{W}{d_2}\right)_3$$

From these two relations we have

$$\frac{1}{d_1} - \frac{1}{d_2} + \frac{1}{d_3} = \frac{1}{\quad_2} \qquad \text{(ii)}$$

and $$\frac{1}{d_1} + \frac{1}{d_2} - \frac{1}{d_3} = \frac{1}{\quad_3} \qquad \text{(iii)}$$

On adding equations (ii) & (iii) we have

$$\frac{2}{d_1} = \frac{{}_2 \quad {}_3}{{}_2 \;\; {}_3}$$

or $$d_1 = \frac{2 \;\; {}_2 \;\; {}_3}{{}_2 \quad {}_3}$$

On similar lines $d_2 = \dfrac{2 \;\; {}_1 \;\; {}_3}{{}_1 \quad {}_3}$ and $d_3 = \dfrac{2 \;\; {}_1 \;\; {}_2}{{}_1 \quad {}_2}$

5.30. A hollow and weightless cone of height h and semi-vertical angle θ filled with water is suspended from a point B as shown in the Fig. 5.17. Calculate the pressure on the base.

Solution: A vertical through B will pass through centre of gravity of the cone G.

G divides OD such that $OG = \dfrac{h}{4}$

and $GD = \dfrac{3h}{4}$

Let BC makes an angle ϕ with the vertical then

$$\tan\phi = \frac{h}{4r} = \frac{h}{4h\tan} = \frac{1}{4\tan};$$

θ = semi-vertical angle of the cone.

$$\cos\phi = \frac{4\tan}{\sqrt{1 \quad 16\tan^2}} = \frac{4\sin}{\sqrt{1 \quad 15\sin^2}}$$

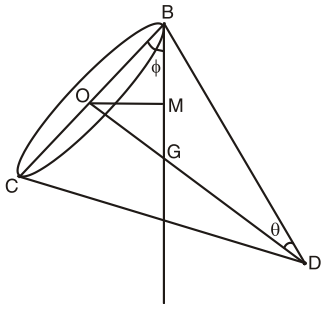


Fig. 5.17

Pressure on the base $= \rho g \pi r^2 \cdot BM$; ρ = density of water

$$= \rho g \pi r^2 r \cos\phi$$

$$= \pi \rho g r^3 \cdot \frac{4\sin}{\sqrt{1 \quad 15\sin^2}}$$

$$= \frac{4\pi\rho g h^3 \tan^3\theta \cdot \sin\theta}{\sqrt{1+15\sin^2\theta}}$$

5.31. A hollow weightless hemisphere with a plane base is filled with water and hung up by means of a string, one end of which is attached to a point of the rim of its base. Find the inclination to the horizontal of the resultant thrust on its curved surface.

Solution: Let r be the radius of the hemisphere. In view of Fig. 5.18

$OG = \frac{3r}{8}$

In ΔOAG

$\cot\theta = \frac{3}{8}$

$\therefore \sin\theta = \frac{8}{\sqrt{73}}$ and $\cos\theta = \frac{3}{\sqrt{73}}$

$OA = h = r\sin\theta = \frac{8r}{\sqrt{73}}$

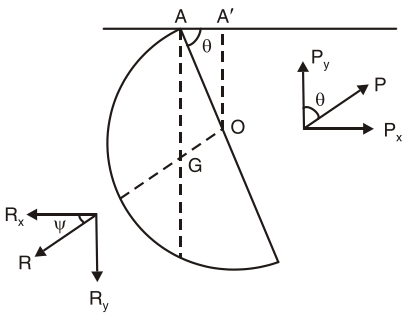


Fig. 5.18

Pressure on plane circular base is given by

$$P = \pi r^2 \cdot \frac{8r}{\sqrt{73}} \rho g$$

$$P_X = P\sin\theta = \pi r^2 \frac{64}{73} r \rho g$$

$$P_Y = P\cos\theta = \pi r^2 \cdot \frac{24}{73} r \rho g$$

Weight of the liquid in the hemisphere $\frac{2}{3}\pi r^3 \rho g$

Resultant thrust on the curved surfaces $= \frac{2}{3}\pi r^3 \rho g$

Further $P_X = R_X$ and $R_Y - P_Y = \frac{2}{3}\pi r^3 \rho g$

$$\therefore R_Y = \frac{24}{73}\pi r^3 \rho g + \frac{2}{3}\pi r^3 \rho g$$

$$= \frac{218}{219}\pi r^3 \rho g$$

$$\tan\psi = \frac{R_Y}{R_X} = \frac{218}{219}\frac{r^3 \ g\,73}{64 \ r^3 \ g} = \frac{109}{96}$$

$$\psi = \tan^{-1}\frac{109}{96}$$

5.32. W_1 and W_2 are the weight of a body in air and water respectively. If ρ_0 be the specific gravity of air, calculate the weight of the body in vacuum.

Solution: Let W be the required weight then the volume of the body

$$= \frac{W}{\cdot}$$

where ρ is the specific gravity of vacuum and ω is weight of unit volume.

$$\text{Apparent weight in air } W_1 = W - \frac{W}{\rho\omega} \cdot \rho_0 \quad \text{(i)}$$

$$\text{Apparent weight } W_2 \text{ in water} = W - \frac{W}{\rho\omega} \quad \text{(ii)}$$

$$\text{Now} \quad W_1 - W_2 = \frac{W}{} \quad \frac{1}{} \quad {}_0 \quad \text{(iii)}$$

$$\text{or} \quad \frac{W}{} = \frac{W_1 \quad W_2}{1 \quad {}_0} \quad \text{(iv)}$$

On substituting the value of $\frac{W}{}$ in equation (i) we have

$$W_1 = W - \frac{{}_0}{1 \quad {}_0} (W_1 - W_2)$$

$$\therefore W = W_1 + \frac{{}_0}{1 \quad {}_0} (W_1 - W_2)$$

5.33. A body of specific gravity σ when weighed against weight of specific gravity ρ in water, the whole balance being immersed appears to have a weight W, show that the true weight is

$$\frac{}{1} \cdot \frac{1}{} \cdot W$$

Solution: Let X be the actual weight of the body

Actual weight of the body – weight of water displaced = actual weight of the weights – weight of water displaced.

$$X - \frac{X}{} \cdot \omega = W - \frac{W}{} \cdot \omega \; ; \; \omega = \text{weight of unit volume}$$

$$X \quad 1 \quad \frac{1}{} \quad = W \quad 1 \quad \frac{1}{}$$

$$\therefore X = W \cdot \frac{}{1} \cdot \frac{1}{}$$

5.34. A rod of length 6 m has a mass of 12 kg. It is hinged at one end at a distance of 3 m below the water surface. (i) What weight must be attached to the other end of the rod so that 5 m of the rod is submerged? (ii) Find

the magnitude and the direction of the force exerted by the hinge on the rod. The specific gravity of the material of the rod = 0.5. (IIT 1976)

Solution: Let PQ be the rod of length 6 m. PC is the length of the rod submered under water ($PC = 5$ m). The buoyancy force F acts at G such that $PG = 2.5$ m. The weight W of the rod acts at G such that $PG = 3$ m. Let us suppose that a mass m is attached at the end Q. Then

$PG \cdot W + PQ \cdot mg = F \cdot PG$ (see Fig. 5.19).

Weight of the part PC = Force $F = \dfrac{\text{weight in air}}{\text{specific gravity}} = \dfrac{10g}{0.5} = 20g$

$3 \times 12g + 6 \times mg = 20g \times 2.5$

or $6m = 50 - 36 = 14$

or $m = \dfrac{7}{3}$ kg

Let R be the reaction at the hinge.

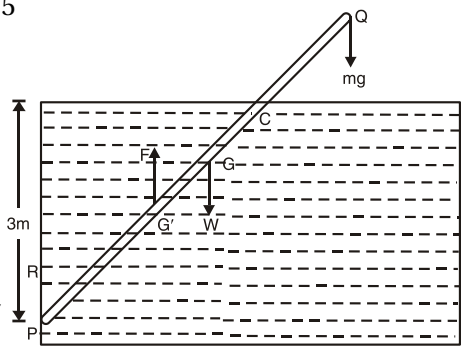


Fig. 5.19

Then $R + F = mg + W$

$\therefore R = mg + W - F$

$= \dfrac{7}{3}g + 12\,g - 20\,g$

$= \dfrac{43 \quad 60\ g}{3}$

$= -\dfrac{17g}{3} = -55.5$ N

Thus force exerted by the hinge = 55.5 N

5.35. A uniform rod of length $2a$ can turn freely about one end which is fixed at a height h above the surface of a liquid. In the conditions of equilibrium the rod makes an angle θ to the vertical. Show that

$$\cos\theta = \frac{h}{2a}\sqrt{}$$

where ρ and σ represent the densities of the rod and liquid respectively.

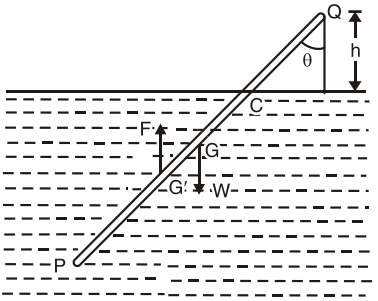


Fig. 5.20

Solution: Apart from the reaction force at Q there are two forces acting on the rod (see Fig. 5.20).

(i) Weight $W = 2a\,Ag\,\rho$ acting downward; A = area of cross-section of the rod.

(ii) Buoyancy force F on the portion PC lying in the liquid of density $\sigma = PC \cdot Ag\alpha = (2a - h \sec\theta)\,A \cdot g \cdot \sigma$

On taking moment about the point Q, we have

$$W \cdot QG \sin\theta = F \cdot QG \sin\theta$$

or $$W \cdot QG = F \cdot QG$$

$$2a\,A\,g\,\rho = (2a - h\sec\theta)\,A\,g\,\sigma \cdot \frac{1}{2}(2a + h\sec\theta)$$

or $$4a^2\rho = (4a^2 - h^2\sec^2\theta)\,\sigma$$

or $$\sigma h^2 \sec^2\theta = 4a^2(\sigma - \rho)$$

or $$\cos\theta = \frac{h}{2a}\sqrt{\qquad}$$

5.36. A uniform rod of length $2a$ and density ρ is moveable in a vertical plane about one end which is fixed in a liquid of density at a depth $2h$ below the free surface. A liquid of lower density σ is added on the top of the first liquid. If in the oblique position of equilibrium the rod is just covered by the liquid, show that its inclination to the vertical is

$$\cos^{-1}\ \frac{h}{a}\sqrt{\qquad /}$$

Solution: In view of Fig. 5.21 apart from the reaction force R at P there are three other forces acting on the rod.

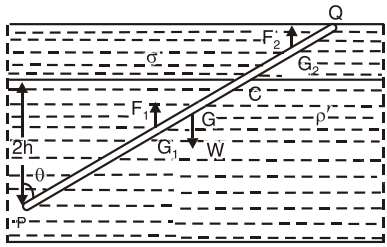


Fig. 5.21

(i) Weight $W = 2a\,A\,\rho\,g$ acting downward; A = area of cross-section of the rod.

(ii) Buoyancy force F_1 acts upward on the portion PC lying in the liquid of density $= PC \cdot Ag \quad = 2h\sec\theta\,g \quad A$

(iii) Buoyancy force F_2 acting upward on the portion CQ of the rod lying in the liquid of density $\sigma = CQ\,A\,g\,\sigma = (2a - 2\,h\sec\theta)\,A\,g\,\sigma$.

On taking the moment about P we have

$$W \cdot PG \sin\theta = F_1 \cdot PG_1 \sin\theta + F_2 \cdot PG_2 \sin\theta$$

or $$W \cdot PG = F_1 \cdot PG_1 + F_2 \cdot PG_2$$

$$2a^2\rho = 2\,h^2\sec^2\theta \cdot \quad + (2a^2 - 2h^2\sec^2\theta)\,\sigma$$

or $$a^2(\rho - \sigma) = h^2\sec^2\theta\,(\quad - \sigma)$$

or $$\cos^2\theta = \frac{h^2}{a^2} \quad \text{——}$$

$$\theta = \cos^{-1}\ \frac{h}{a}\sqrt{\qquad /}$$

5.37. A thin uniform rod of weight W has a particle of mass m attached to one end. It is floating in an inclined position in water with this end immersed. Show that the length of the rod above water is $\frac{mg}{mg \quad W}$ times its whole length and that the specific gravity of the rod is $\frac{W^2}{mg \quad W^{\,2}}$.

Solution: There are three forces acting on the rod :

(i) Weight W of the rod acting downward.

(ii) Buoyancy force F acting on the portion PC lying in the liquid.

(iii) Weight mg acting downward.

On taking the moment about P (see Fig. 5.22), we have

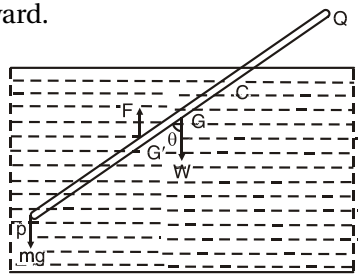


Fig. 5.22

$$W \cdot PG \sin\theta = F \cdot PG \sin\theta$$

or $$W \cdot PG = F \cdot PG$$

In equilibrium we have $F = W + mg$

$$\therefore\ W \cdot PG = (W + mg)\, PG$$

or $$\frac{W}{W \quad mg} = \frac{PC}{2a} = \frac{2a - l}{2a}$$; $2a$ = whole length of the rod and l is the part of the length of the rod outside the liquid.

$$\therefore\ 2a - 2a \quad \frac{W}{W \quad mg} \quad = l$$

or $$2a \quad \frac{W}{W \quad mg} \quad = l$$

Let us assume that ρ be the specific gravity of rod. The weight of the rod is given by $W = 2a\, A\, \rho\, g$

$$F = (2a - l)\, A \times 1 \times g$$

$$= \quad 2a \quad \frac{2a\, mg}{W \quad mg} \quad Ag$$

$$F = \frac{2a\, W}{W \quad mg}\ Ag = W + mg$$

or $$2a\, A\, Wg = (W + mg)^2$$

or $$\frac{W^2}{} = (W + mg)^2$$; on puting $2a \ \ Ag = \frac{W}{}$

or $$\rho = \frac{W^2}{W \quad mg^{\,2}}$$

5.38. A wooden plank of length 1 m and uniform cross-section is hinged at one end to the bottom of a tank. The tank is filled with water upto a height of 0.5 m. The specific gravity of the plank is 0.5. Find the angle θ that the plank makes with the vertical in the equilibrium position (exclude the case $\theta = 0$). (IIT 1984)

Solution: Let A be the area of cross-section of the plank. In view of Fig. 5.23, apart from the reaction force at P there are two forces acting on the rod.

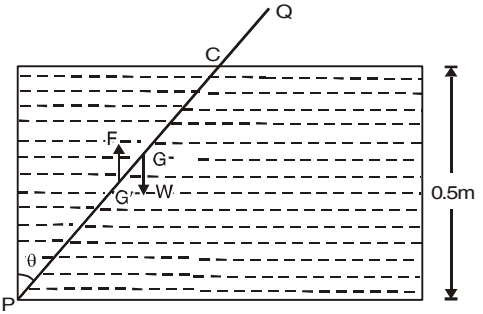


Fig. 5.23

(i) Weight W of the rod $= 1 \times A \times 0.5 \times g$ acting downward.

(ii) Buoyancy force F acting on the rod $PC = A \times 0.5 \sec\theta \times 1 \times g$ acting upward. On taking the moment about the point P we have

$$W \cdot PG \sin\theta = F \cdot PG \sin\theta$$

or $$W \cdot PG = F \cdot PG$$

or $$A \times 0.5 \times g \times 0.5 = 0.5\,Ag \sec\theta \times \frac{0.5 \sec}{2}$$

or $$2 = \sec^2\theta$$

or $$\cos^2\theta = \frac{1}{2}$$

or $$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = 45°$$

6

ELASTICITY

6.1. A uniform ring of radius $r_1 = 50$ cm of a wire of area of cross-section $a = 2$ mm^2 is tightly fitted on a circular steel rod of radius $r_2 = 50.15$ cm. Calculate the force which will cause the ring to expand. The Young's modulus of the material of the wire = 12×10^3 kgf/mm^2.

Solution: Stress = $\dfrac{\text{Force}}{\text{area}} = \dfrac{F}{a}$

Strain = $\dfrac{\text{change in length}}{\text{original length}} = \dfrac{2\pi(r_2 - r_1)}{2\pi r_1}$

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{F \cdot r_1}{a \cdot (r_2 - r_1)}$$

On putting the values of various terms, we have

$$F = Y \cdot a \cdot \frac{r_2 \quad r_1}{r_1} = \frac{12 \quad 10^3 \quad 2 \quad 0.15}{50} = 72 \text{ kgf}$$

6.2. A steel rod of length $l = 100$ m is suspended from the top of a tower through one of its ends. Calculate:

(i) the elongation of the rod by virtue of its own weight, and

(ii) the relative increment of its volume.

Young's modulus of steel = 20×10^{10} N/m^2

Density of steel = 7.8×10^3 kg/m^3 and Poisson's ratio = 0.29

Solution:

(i) Young's modulus $Y = \dfrac{\text{stress}}{\text{strain}} = \dfrac{F \cdot l}{a \cdot \ l}$; F = force = $a \cdot l \cdot \rho \cdot g$

a = area of cross-section of the rod

ρ = density of the material (steel)

Δl = elongation

$$\therefore \Delta l = \frac{al \ g}{aY} \cdot \frac{l}{2} = \frac{1}{2} \ \frac{g l^2}{Y}$$

The entire mass of the rod is concentrated at a length $\dfrac{l}{2}$.

On substituting the values of various terms, we have

$$\Delta l = \frac{1}{2} \times \frac{7.8 \quad 10^3 \quad 9.8 \quad 10^4}{20 \quad 10^{10}}$$
$$= 3.9 \times 4.9 \times 10^{-4} \text{ m}$$
$$= 0.39 \times 4.9 \text{ mm}$$
$$= 1.9 \text{ mm}$$

(ii) The initial volume of the rod, $V_1 = \pi r^2 l$; r is the radius of the rod. Let us assume that the radius of the wire is reduced by an amount Δr and increases the length by Δl. The new volume of the rod

$$V_2 = \pi (r - \Delta r)^2 (l + \Delta l)$$
$$= \pi(r^2 - 2r\Delta r + \Delta r^2) (l + \Delta l)$$
$$\cong \pi r^2 l - \pi 2r \Delta r l + \pi r^2 \Delta l$$

Change in volume $\Delta V = V_2 - V_1 = \pi r^2 \Delta l - 2\pi rl \Delta r$

$$\frac{V}{V_1} = \frac{l}{l} - 2\frac{r}{r}$$

Poisson's ratio is defined as $\sigma = \dfrac{\text{per unit decrease of diameter}}{\text{per unit increase of length}}$

$$= \frac{r/r}{l/l}$$

$$\therefore \frac{V}{V_1} = \frac{l}{l} - 2\sigma \frac{l}{l}$$
$$= (1 - 2\sigma) \frac{l}{l}$$
$$= (1 - 2 \times 0.29) \times \frac{3.9 \quad 4.9 \quad 10^{-4}}{10^2}$$
$$= 0.42 \times 3.9 \times 4.9 \times 10^{-6} = 8.026 \times 10^{-6}$$

6.3. A homogeneous plank of length L and surface area A moves over a horizontal frictionless surface under the action of a constant force F distributed uniformly on the end. Assuming the Young's modulus of the material of the plant as Y determine the strain in the direction of the force.

Solution: Consider an element of length dx at a distance x from O (see Fig. 6.1).

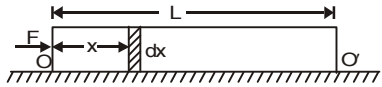


Fig. 6.1

Tension T at a distance x is given by

$\dfrac{F}{L} = \dfrac{T}{L \quad x}$; at the end O tension is F and at O tension is zero.

$$\frac{F}{L}(L - x) = T$$

$$\text{Strain on the element} = \frac{T}{AY} = \frac{F}{AY} \quad 1 \quad \frac{x}{L}$$

$$\text{Change in the length of the element } \Delta l = \frac{F}{AY} \quad 1 \quad \frac{x}{L} \quad dx$$

$$\text{Total change in length } l = \int_0^L \Delta l = \frac{F}{AY} \int_0^L \left(1 - \frac{x}{L}\right) dx$$

$$= \frac{FL}{2AY}$$

$$\therefore \text{Strain} = \frac{l}{L} = \frac{F}{2AY}$$

6.4. A uniform rod of length l and mass m is allowed to rotate in a horizontal plane about a vertical axis OO passing through one of its ends with an angular velocity ω. (i) Derive an expression for the tension in the rod in terms of distance x measured from the axis of rotation. (ii) If the rod is made of copper then find out the elongation in it. $l = 5$ m, density of copper 8.9×10^3 kg/m^3, $\omega = 30$ rad/s, and Young's modulus of copper $= 13 \times 10^{10}$ N/m^2.

Solution:

(i) Let A be the area of cross-section of the rod. Consider an element of length dr lying at a distance r from the axis of rotation OO (see Fig. 6.2). The tension is given by

$$dT = \frac{m}{l} dr \cdot r \cdot \omega^2$$

or $$T = dT = \frac{m}{l}^2 \int_x^l r dr$$

$$= \frac{1}{2} \frac{m}{l}^2 [l^2 - x^2]$$

$$= \frac{1}{2} m\omega^2 l \quad 1 \quad \frac{x^2}{l^2}$$

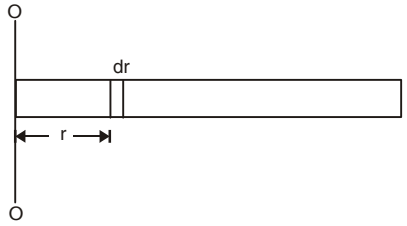


Fig. 6.2

(ii) Young's modulus $Y = \dfrac{\text{stress}}{\text{strain}} = \dfrac{dT}{A} \dfrac{r}{dl}$

On substituting the value of dT from the above, we have

$$Y = \frac{m}{l} \frac{rdr}{Adl}^2 r$$

or $$dl = \frac{m}{AlY}^2 r^2 dr$$

$$\text{Elongation of the rod} = l = dl = \frac{m}{AlY}^2 \int_0^l r^2 dr = \frac{1}{3} \frac{m}{AlY}^2 l^3$$

Now $l\,A\rho = m$ where ρ is the density of the rod.

$$\therefore\ l = \frac{1}{3}\,\frac{{}^{2}l^{3}}{Y}$$

On substituting the values of the various terms, we have

$$l = \frac{1}{3} \times \frac{8.9 \quad 10^{3} \quad 30^{2} \quad 5^{3}}{13 \quad 10^{10}}$$

$$= \frac{1}{3} \times \frac{8.9 \quad 9 \quad 125}{13} \times 10^{-5}$$

$$= 2.57 \text{ mm}$$

6.5. A uniform copper rod of length $l = 80$ cm is allowed to rotate in a horizontal plane about a vertical axis OO passing through its centre. Calculate the frequency of rotation per sec at which the rod ruptures. The ultimate strength of copper is 3000 kgf/cm^2 and its density is 8.9 g/cm^3.

Solution: Consider an element of length dx at a distance x from the axis of rotation (see Fig. 6.3).

Total mass of the rod = m

Mass per unit length of rod = $\frac{m}{l}$

Tension $dT = \frac{m}{l}\,dx \cdot x\omega^{2}$

$$T = \int dT = \frac{m}{l}{}^{2} \int_{0}^{l/2} x\,dx = \frac{ml}{8}{}^{2}$$

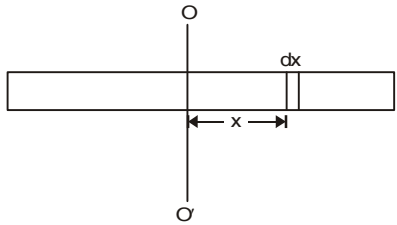


Fig. 6.3

Now $m = A\,l\,\rho$ where A is the area of cross-section of the rod and ρ its density.

$$\therefore\ T = \frac{Al^{2}}{8}{}^{2} = \frac{Al^{2} \quad 4 \quad {}^{2}n^{2}}{8}$$ where n is the frequency of rotation per second.

or $$n = \frac{8T}{Al^{2} \quad 4 \quad {}^{2}}^{1/2} = \sqrt{\frac{2T}{A}} \times \frac{1}{l};\ \frac{T}{A} = \sigma_{m} = \text{ultimate strength}$$

$$\therefore\ n = \sqrt{\frac{2 \quad {}_{m}}{}} \times \frac{1}{l}$$

$$= \sqrt{\frac{2 \quad 3000 \quad 980 \quad 10^{3}}{8.9}} \times \frac{1}{80\pi}$$

$$= 102.3 \text{ rps}$$

$$= 102 \text{ rps}$$

6.6. A lead ring of radius $r = 11.5$ cm rotates about a vertical axis OO passing through its centre and perpendicular to the plane of the

ring (Fig. 6.4). Calculate the number of rotations per second at which the ring breaks. The ultimate strength of lead is 150 kgf/cm^2 and its density is 11.3 g/cm^3.

Solution: Consider a part of the ring subtending an angle $\Delta\theta$ at the centre.

Mass of the element = $\frac{m}{l} \times r\,\Delta\theta$; m = mass of the ring.

Let T be the tension on the ring. Of the two components only T sin $\Delta\theta/2$ is operational. The net force is (Fig. 6.4b)

$$2T \sin \frac{\Delta\theta}{2} = T\Delta\theta = \frac{m}{l}\, r\,\Delta\,\theta\, r\,\omega^2$$

or $$T = \frac{m}{l} r^2\omega^2 = \frac{m}{2\ r} r^2\omega^2 = \frac{mr^{\ 2}}{2}$$

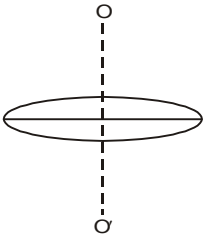


Fig. 6.4a

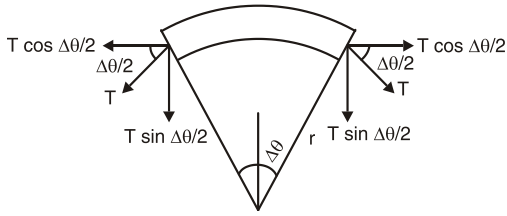


Fig. 6.4b

Mass of the ring $m = 2\,\pi\,r\,A\,\rho$ where A is the area of cross-section of the wire.

$$T = \frac{2\ r}{2} A\,\rho\, r\,\omega^2$$

$$\frac{T}{A} = r^2\rho\ 4\ \pi^2 n^2 = \sigma_m = \text{ultimate strength}$$

$$\therefore\ n = \sqrt{\frac{\ m}{}} \times \frac{1}{2\ r}$$

On substituting the values of various terms, we have

$$n = \sqrt{\frac{150\ \ 980\ \ 10^3}{11.3}} \times \frac{1}{2\ \ 11.5}$$

$$= \sqrt{\frac{1.5\ \ 98}{11.3}} \times 10^3 \times \frac{1}{23}$$

$$= 49.95$$

$$= 50 \text{ rps}$$

6.7. A stone of 0.5 kg mass is attached to one end of a 0.8 m long aluminium wire of 0.7 mm diameter and suspended vertically. The stone

is now rotated in a horizontal plane at a rate such that the wire makes an angle of 85° with the vertical. Find the increase in the length of the wire. Young's modulus of aluminium = 7×10^{10} N/m^2, sin 85° = 0.9962 and cos 85° = 0.0872. (Roorkee 1990)

Solution: On resolving the tension T in the two mutually perpendicular directions we have

$T \cos \theta = mg$ (see Fig. 6.5)

and $T \sin \theta = mr\omega^2$;

r = radius of the circular path.

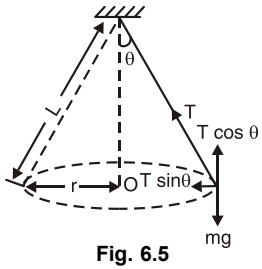


Fig. 6.5

Now $$Y = \frac{T}{A} \cdot \frac{L}{\Delta L}$$

$$\therefore \Delta L = \frac{TL}{AY}$$

$$= \frac{mg\,L}{r_0^2\, Y \cos}$$; r_0 = radius of the aluminium wire

On putting the values of various terms we have

$$\Delta L = \frac{0.5 \quad 9.8 \quad 0.8}{3.14 \quad 0.35 \quad 10^{3\ \ 2} \quad 7 \quad 10^{10} \quad 0.0872}$$

$$= 1.668 \times 10^{-3} \text{m}$$

6.8. What is the maximum load that a steel wire of diameter 2 mm can hold? The ultimate stress of steel 1.25×10^9 N/m^2.

Solution: The Young's modulus of the material of wire is given by

$$Y = \frac{T}{A} \frac{L}{l}$$

where T is the tension applied, A is the area of cross-section, l is the increase in length and L is the original length.

$Y = WL/l$ where W is the ultimate stress.

We can have another expression for Y when tension T has been applied on a wire of diameter d.

$$Y = \frac{4T}{d^2} \frac{L}{l}$$

On dividing the two equations for Y, we have

$$1 = \frac{W \quad d^2}{4T}$$

or $$T = \frac{W \quad d^2}{4}$$

On substituting the values of various terms, we have

$$T = \frac{1.25 \quad 10^9 \quad 3.14 \quad 2^2 \quad 10^{\ 6}}{4} = 1.25 \times 3.14 \times 10^3$$

$$= 3.924 \times 10^3 \text{ N}$$

6.9. A chandelier of mass 550 kg is attached to an aluminium rod with an ultimate stress of 1.1×10^8 N/m^2. What should be the cross-sectional area of the rod for the safety factor to be 3? What is the strain in the rod? Young's modulus of aluminium = 7×10^{10} N/m^2.

Solution: Stress $F_1 = \dfrac{mg}{A}$

Now safety factor $n = \dfrac{\text{ultimate stress}}{\text{admissible stress}} = \dfrac{F}{F_1}$

$$\therefore F_1 = \frac{F}{n} = \frac{mg}{A}$$

or $$A = \frac{mgn}{F} = \frac{550 \quad 9.8 \quad 3}{1.1 \quad 10^8}$$

$$= 14.7 \times 10^{-5} \text{ m}^2$$

Strain in the rod = $\dfrac{l}{L} = \dfrac{F}{nY} = \dfrac{1.1 \quad 10^8}{3 \quad 7 \quad 10^{10}} = 5.2 \times 10^{-4}$

6.10. The longitudinal waves starting from a ship returns from the buttom of the sea to the ship after 2.0 s. If the bulk modulus of water is 220 kg mm^{-2} and the density of sea water 1.1×10^3 kg/m^3, calculate the depth of the sea.

Solution: Velocity of sound = $v = \sqrt{\dfrac{B}{d}}$

where B is the bulk modulus of elasticity and d is density.

$$v = \sqrt{\frac{220 \quad 9.8}{10^{\ 6} \quad 1.1 \quad 10^3}}$$

$$= 1400 \text{ m/s}$$

Depth of the sea = $1400 \times 1 = 1400$ m

6.11. A wire of density 9 g/cm^3 is stretched between two clamps 100 cm apart while being subjected to an extension if 0.05 cm. What is the lowest frequency of transverse vibrations in the wire, assuming the Young's modulus of the material of wire to be 9×10^{11} dyne/cm^2.

Solution: Young's modulus = $\dfrac{\text{stress}}{\text{strain}} = \dfrac{T/A}{l/L}$

where T is the tension applied, A = area of cross-section of wire, l = change of length and L = original length.

$$Y = \frac{TL}{Al}$$

or $$\frac{Yl}{L} = \frac{T}{A}$$

or $\frac{Yl}{Ld} = \frac{T}{Ad}$ where d is the density of the material of the wire.

$$n = \frac{1}{2\ L\ \ l}\ \sqrt{\frac{T}{Ad}} = \frac{1}{2\ L\ \ l}\ \sqrt{\frac{Yl}{Ld}}$$

On substituting the values of the various terms, we have

$$n = \frac{1}{2\ \ 100.05}\ \sqrt{\frac{9\ \ 10^{11}\ \ 0.05}{100\ \ 9}}$$

$$= \frac{1}{200.1} \times 10^3\ \sqrt{50}$$

$$= 35.34 \text{ Hz}$$

6.12. A copper wire of negligible mass, 1 m length and cross-sectional area 10^{-6} m^2 is kept on a smooth horizontal table with one end fixed. A ball of mass 1 kg is attached to the other end. The wire and the ball are rotating with an angular velocity of 20 rad/sec. If the elongation in the wire is 10^{-3} m, obtain the Young's modulus. If on increasing the angular velocity to 100 rad/sec, the wire breaks down, obtain the breaking stress. (Roorkee 1992)

Solution: Force developed in the wire due to rotation.

$$F = mr\omega^2$$

On putting the values we have

$$F = 1 \times 1 \times 20^2 = 400 \text{ N}$$

$$\text{Stress} = \frac{400}{10^{\ 6}} = 4 \times 10^8 \text{ N/m}^2$$

$$\text{Strain} = 10^{-3} \text{ m}$$

$$\text{Young's modulus } Y = \frac{\text{stress}}{\text{strain}} = \frac{4\ \ 10^8}{10^{\ 3}} = 4 \times 10^{11} \text{ N/m}^2$$

$$\text{Breaking force } F = mr\ \ {}^{2}_{1}$$

On putting the value we have

$$F = 1 \times 1 \times 100^2 = 10^4 \text{ N}$$

$$\text{Breaking stress} = \frac{10^4}{10^{\ 6}} = 10^{10} \text{ N/m}^2$$

6.13. A load of 31.4 kg is suspended from a wire of radius 10^{-3} m and density 9×10^3 kg/m^3. Calculate the change in temperature of the wire if 75% of the work done is converted into heat. The Young's modulus and specific heat of the material of the wires are 9.8×10^{10} N/m^2 and 490 J/kg.K. respectively. (Roorkee 1993)

Solution: Work done $W = \frac{1}{2}\ Mg \cdot l$; l = increase in length.

$$\therefore Y = \frac{Mg \cdot L}{\pi r^2 \cdot l}; \quad L = \text{Length of the wire}$$
$$r = \text{radius of the wire}$$

$$W = \frac{1}{2} \cdot \frac{M^2 g^2 \cdot L}{\pi r^2 \cdot Y}$$

$$0.75\, W = m \cdot s \cdot \Delta\theta = (\pi r^2 L \rho)\, s\, \Delta\theta; \rho = \text{density of the material of the wire}$$

$$\Delta\theta = \frac{0.75\, W}{r^2\, L\, s} = \frac{0.75}{2} \frac{M^2 g^2 L}{\pi r^2 Y \cdot \pi r^2 L \rho s}$$
$$= \frac{0.75}{2} \frac{M^2 g^2}{{}^2 r^4 \; sY}$$

On substituting the values we have

$$\Delta\theta = \frac{0.75 \quad 31.4^2 \quad 9.8^2}{2 \quad 3.14^2 \quad 10^{3\ 4} \quad 9 \quad 10^3 \quad 490 \quad 9.8 \quad 10^{10}}$$
$$= \frac{1}{120}°\text{C}$$

6.14. A body of mass 3.14 kg is suspended from one end of a wire of length 10.0 cm. The radius of the wire is changing uniformly from 9.8×10^{-4} m at one end to 5×10^{-4} m at the other end. Find the change in the length of the wire. What will be the change in the length if the ends are interchanged? Young's modulus of the material of the wire is 2×10^{11} N/m^2. (Roorkee 1994)

Solution: $\tan\theta = \frac{r_2 \quad r_1}{L}$ (see Fig. 6.6)

$$r = r_1 + \frac{r_2 \quad r_1}{L} \cdot y$$

Stress on the wire at a depth y

$$= \frac{Mg}{r^2} = \frac{Mg}{\pi\left[r_1 + \frac{r_2 - r_1}{L} \cdot y\right]^2}$$

Consider an element of length dy.

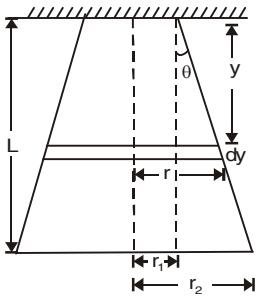


Fig. 6.6

Change in the length of the element $\Delta l = \frac{\text{Stress}}{Y}\, dy$

$$\Delta l = \frac{Mg \cdot dy}{Y \cdot \pi\left[r_1 + \frac{r_2 - r_1}{L} \cdot y\right]^2}$$

Net increase in length $= l = \int_0^L \Delta l = \frac{Mg}{\pi Y} \int_0^L \frac{dy}{\left[r_1 + \frac{r_2 - r_1}{L} \cdot y\right]^2}$

$$= \frac{MgL}{Y r_1 r_2}$$

On putting the values we have

$$l = \frac{3.14 \quad 9.8 \quad 0.1}{3.14 \quad 2 \quad 10^{11} \quad 5 \quad 10^{4} \quad 9.8 \quad 10^{4}}$$
$$= 10^{-5} \text{ m}$$

On changing the two ends change in length will have the same value.

6.15. A solid steel cylinder of length $l = 1$ m is placed on a horizontal surface. A vertically downward compression force of $F = 1500$ N distributed uniformly over the end face is applied on it. Calculate the change in volume of the cylinder. Young's modulus of steel $= 2 \times 10^{11}$ N/m^2. Poisson's ratio = 0.29.

Solution: Poisson's ratio $\sigma = \dfrac{\text{per unit increase in diameter}}{\text{per unit increase in length}} = \dfrac{r/r}{l/l}$

Initial volume of the cylinder $V_1 = \pi r^2 l$

Final volume of the cylinder $V_2 = \pi (r + \Delta r)^2 (l - \Delta l)$

$$= \pi (r^2 l + 2r \Delta r \cdot l - r^2 \cdot \Delta l)$$

$$\Delta V = V_1 - V_2 = -2\pi r \cdot \Delta r \cdot l + \pi r^2 \cdot \Delta l$$

$$= \pi r^2 l \quad \frac{2 \quad r}{r} \quad \frac{l}{l}$$

$$= \pi r^2 l \, [1 - 2\sigma] \quad \frac{l}{l}$$

$$= \pi r^2 \, l \, [1 - 2\sigma] \quad \frac{F}{r^2 Y}$$

$$= l \, [1 - 2\sigma] \quad \frac{F}{Y}$$

On substituting the values we have

$$\Delta V = 1 \, [1 - 2 \times 0.29] \times \frac{1500}{2 \quad 10^{11}}$$
$$= 3.15 \text{ mm}^3$$

6.16. Find the value of Poisson's ratio at which the volume of a wire does not change when the wire is subjected to a tension.

Solution: Consider a wire of length l and radius r. Initial volume

$$V_1 = \pi r^2 l$$

Let the wire be subjected to a tension. Its volume is given by

$$V_2 = \pi (r - \Delta r)^2 (l + \Delta l) = \pi (r^2 - 2r\Delta r + \Delta r^2) (l + \Delta l)$$

Now $V_1 = V_2$

$$\therefore \pi r^2 l = \pi r^2 l - 2\pi r \cdot \Delta r \cdot l + \pi r^2 \cdot \Delta l$$

or $$2\pi r \cdot \Delta r \cdot l = \pi r^2 \cdot \Delta l$$

$$\frac{2\ r}{r} = \frac{l}{l}$$

$$2\sigma\ \frac{l}{l} = \frac{l}{l}$$

or $$\sigma = \frac{1}{2} = 0.5$$

6.17. A steel wire of diameter d = 1.0 mm is stretched horizontally between two clamps located at a distance l = 2.0 m from each other. A load of m = 0.25 kg is suspended from the midpoint of the wire. What is the vertical distance through which the midpoint of the wire will move down? Young's modulus of steel = 20×10^{10} N/m^2 and g = 9.8 m/s^2.

Solution: In Fig. 6.7 the resolve part of T perpendicular to the length of the wire is $2T \sin\theta$ and is balanced by the weight, mg, i.e.,

$$T = \frac{mg}{2\sin}$$

Young's modulus $Y = \dfrac{Tl}{A\ \ l}$ where A is the area of cross-section of the wire and Δl is the change in length.

From the geometry of the figure, $\Delta l = \dfrac{2\ \ l}{2\cos} - l = \dfrac{l\ 1\ \ \cos}{\cos}$

$$\therefore\ Y = \frac{Tl\cos}{r^2 l\ \ 1\ \ \cos} = \frac{T\cos}{r^2\ \ 1\ \ \cos} = \frac{mg\cos}{2\ \ r^2\sin\ \ \ 1\ \ \cos};$$

r = radius of the wire

$$\therefore\ Y = \frac{2mg\cos}{d^2\sin\ \ \ 1\ \ \cos}$$

When θ is small,

$\sin\theta = \theta$, $\cos\theta = 1 - 2\sin^2\theta/2 = 1 - \theta^2/2$

$$\frac{\sin\ \ \ 1\ \ \cos}{\cos} = \frac{^3}{2} = \frac{2mg}{d^2Y}$$

$$\therefore\ \theta = \left(\frac{4mg}{d^2Y}\right)^{1/3} = \frac{2x}{l}$$

$$\therefore\ x = l\ \left(\frac{4mg}{8\ \ d^2Y}\right)^{1/3} = l\ \left(\frac{mg}{2\ \ d^2Y}\right)^{1/3}$$

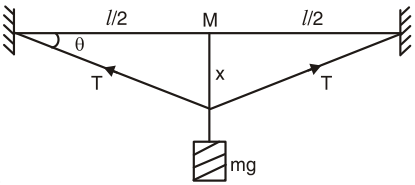


Fig. 6.7

On substituting the values of the various terms, we have

$$x = 200\ \left(\frac{250\ \ 980}{2\ \ 3.14\ \ 0.01\ \ 20\ \ 10^{11}}\right)^{1/3} \cong 2.5\ \text{cm}$$

6.18. A light rod of length 200 cm is suspended from the ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross-section 0.1 cm^2 and the other is of brass of cross-section 0.2 cm^2. Find out the position along the rod at which a weight may be hung to produce:

(i) equal stress in both wires,

(ii) equal strain in both wires,

Young's modulus of elasticity of brass and steel are 10×10^{11} dyne/cm^2 and 20×10^{11} dyne/cm^2 respectively. (IIT 1974)

Solution: (i) Let the weight W be hung at O (Fig. 6.8) such that $PO = x$.

On taking moment about O, we have

$$T_1 \cdot x = T_2 (200 - x);\ T_1 \text{ and } T_2 \text{ are the tensions in the two wires.}$$

or $$\frac{T_1}{T_2} = \frac{200 \quad x}{x}$$

For producing equal stress in both the wires we have $\frac{T_1}{A_1} = \frac{T_2}{A_2}$; A_1 and A_2 are the areas of cross-section of the steel and the brass wire respectively.

$$\therefore \frac{T_1}{T_2} = \frac{A_1}{A_2} = \frac{200 \quad x}{x}$$

or $$\frac{1}{2} = \frac{200 \quad x}{x}$$

or $$x = 400 - 2x$$

or $$x = 400/3 = 133.3 \text{ cm}$$

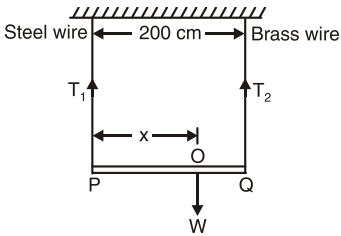


Fig. 6.8

(ii) The Young's moduli of the two materials can be represented by the following equations.

$$Y_1 = \frac{T_1 L_1}{A_1 l_1} \text{ and } Y_2 = \frac{T_2 L_2}{A_2 l_2}$$

where l_1 and l_2 are the increase in length of the two wires respectively.

Now $$\frac{l_1}{L_1} = \frac{T_1}{Y_1 A_1} \text{ and } \frac{l_2}{L_2} = \frac{T_2}{Y_2 A_2}$$

For there to be equal strains,

$$\frac{T_1}{Y_1 A_1} = \frac{T_2}{Y_2 A_2}$$

$$\therefore \frac{T_1}{T_2} = \frac{Y_1 A_1}{Y_2 A_2} = \frac{200 \quad x}{x}$$

$$\text{or} \quad \frac{20 \quad 10^{11} \quad 0.1}{10 \quad 10^{11} \quad 0.2} = \frac{200 \quad x}{x}$$

$$\text{or} \quad x = 200 - x$$

$$\text{or} \quad x = 100 \text{ cm}$$

6.19. A sphere of radius 10 cm and mass 25 kg is attached to the lower end of a steel wire which is suspended from the ceiling of a room, the point of support being 521 cm above the floor. When the sphere is set swinging as simple pendulum its lowest point just grazes the floor. Calculate the velocity of the ball at its lowest position. $Y = 20 \times 10^{11}$ dyne/cm^2, $L = 500$ cm, radius of the steel wire = 0.05 cm.

Solution: Length of the wire, $L = 500$ cm

$(L + l) = 501$ cm (see Fig. 6.9)

$\therefore\ l = 1$ cm (increase in length)

$$\text{Strain } \frac{l}{L} = \frac{1}{500}$$

$$Y = \frac{F}{Al} L$$

$$\text{or} \quad F = \frac{YAl}{L} = \frac{20 \quad 10^{11} \quad 3.14 \quad (0.05)^2 \quad 1}{500}$$

$$= 31.4 \times 10^6 \text{ dynes}$$

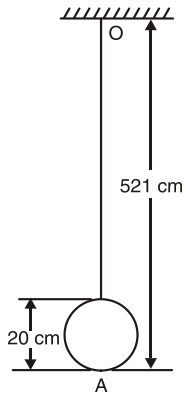


Fig. 6.9

At the lowest point A, the equation of motion is

$$F - mg = m \frac{v^2}{r}$$

$$\text{or} \quad F = mg + m \frac{v^2}{r}$$

On substituting the values of the various terms, we have

$$31.4 \times 10^6 = 25{,}000 \times 980 + \frac{25{,}000}{511} \times v^2;$$

$m = 25000\ g$, $r = 521 - 10 = 511$ cm

$v = 375$ cm/s

6.20. A copper cylinder is heated up to 100°C. What is the pressure required to keep its length constant? Young's modulus of copper $= 13 \times 10^{10}$ N/m^2, coefficient of linear expansion of copper $= 17 \times 10^{-6}$/°C.

Solution: $Y = \dfrac{Fl}{A \quad L}$

where A is the area of cross-section, Δl is the change in length, l = original length.

Increase in length, $\Delta l = l \alpha \Delta t$; α = coefficient of linear expansion

Δt = change in temperature

$$\therefore \frac{l}{l} = \alpha \Delta t$$

$$\text{Pressure } P = \frac{\text{force}}{\text{area}} = \frac{F}{A} = Y \frac{l}{l} = Y \alpha \Delta t$$

On substituting the values of various terms, we have

$$P = 13 \times 10^{10} \times 17 \times 10^{-6} \times 10^{2}$$
$$= 13 \times 17 \times 10^{6}$$
$$= 2.21 \times 10^{3} \times 10^{5}$$
$$= 2.21 \times 10^{3} \text{ atmospheric pressure (1 atmos} = 10^{5} \text{ N/m}^2\text{)}$$

6.21. A copper wire of diameter 0.4 mm heated up to 220°C is clamped at its ends in the same condition. On reducing its temperature a tension develops in it. Calculate the force exerted on the clamps when its temperature falls to 20°C. Young's modulus of copper = 13×10^{11} dyne/cm^2, coefficient of linear expansion of copper = 17×10^{-6}/°C.

Solution: Change in length = $l \alpha \cdot \Delta t = 200 \alpha l$

$$Y = \frac{F}{\text{area} \cdot \text{strain}}; F \text{ is force exerted on the clamps}$$
$$\therefore F = Y \times \text{area} \times \text{strain}$$
$$F = 13 \times 10^{11} \times \pi \times (0.02)^2 \times 17 \times 10^{-6} \times 200$$
$$= 13 \times 3.14 \times 17 \times 8 \times 10^{3} \text{ dynes}$$
$$= 5.55 \times 10^{6} \text{ dynes}$$

6.22. A uniform rod of area of cross-section 2 mm^2 is heated from 0°C to 40°C. Calculate the energy stored per unit volume. Coefficient of linear expansion of the rod $\alpha = 12 \times 10^{-6}$/°C and Young's modulus = 10^{11} N/m^2.

Solution: Strain produced in the rod = $\alpha t = 12 \times 10^{-6} \times 40 = 48 \times 10^{-5}$

Force required to prevent the rod from expanding = $Y \cdot A \cdot$ strain

$$= 10^{11} \times 2 \times 10^{-6} \times 48 \times 10^{-5} = 96 \text{ N}$$

$$\text{Energy stored per unit volume} = \frac{1}{2} \text{ stress} \times \text{strain}$$
$$= \frac{1}{2} \times \frac{96}{2 \quad 10^{\,6}} \times 48 \times 10^{-5}$$
$$= 11{,}520 \text{ J/m}^3$$

6.23. A heated steel tyre is fitted on a wheel of radius 0.4 m. If the tension produced in the tyre on cooling to 17°C is within elastic limits, then to what temperature should the tyre be heated? Calculate the tension developed in it if the tyre is 3 cm wide and 3 mm thick. Young's modulus

of steel = 2×10^4 kg/mm^2, coefficient of linear expansion = 1.1×10^{-5}/°C and the limit of elastic tensile stress = 28 kg/mm^2.

Solution: Let the tyre be heated up to t°C.

Strain produced = $\alpha\,(t - 17)$

Young's modulus $Y = \dfrac{\text{tensile stress}}{\text{strain}}$

$\therefore$ Strain = $\alpha\,(t-17) = \dfrac{\text{tensile stress}}{Y}$

$$\therefore\ t - 17 = \frac{\text{tensile stress}}{Y} = \frac{28 \quad 9.8 \quad 10^6}{1.1 \quad 10^{5} \quad 2 \quad 10^4 \quad 9.8 \quad 10^6}$$

$$= \frac{1400}{11} \cong 127.3°\text{C}$$

$$\therefore\ t = 144.3°\text{C}$$

Let F be the tension developed in the tyre. Then,

$$\frac{F}{3 \quad 3 \quad 10^{5}} = 28 \times 9.8 \times 10^6$$

or $$F = 9 \times 280 \times 9.8$$

$$= 2520 \times 9.8$$

$$= 2520 \text{ kg wt}$$

6.24. A copper wire is clamped at its two ends between two rigid supports. When the temperature is 30°C then there is no tension in the wire. What will be the speed of the transverse wave in the wire at 10°C. Young's modulus of copper = 1.3×10^{11} N/m^2, coefficient of linear expansion of copper = 1.7×10^{-5}/°C and density of copper = 9×10^3 kg/m^3.

(IIT 1979)

Solution: Let l be the length of the wire at 30°C.

Reduction in length = $l\,\alpha\,(30 - 10) = 20\,\alpha\,l$

Strain produced = $20\,\alpha = 34 \times 10^{-5}$

$$Y = \frac{T}{A \quad \text{strain}}$$

$$\therefore\ \frac{T}{A} = Y \times \text{strain} = Y \times 20\,\alpha = 1.3 \times 10^{11} \times 34 \times 10^{-5}$$

Speed of the transverse wave $\sqrt{\dfrac{T}{m}} = \sqrt{\dfrac{T}{Ad}}$

$$= \sqrt{\frac{1.3 \quad 10^{11} \quad 34 \quad 10^{5}}{9 \quad 10^3}}$$

$$= \sqrt{13 \quad 34} \times \frac{10}{3} = 70.08 \text{ m/s}$$

6.25. A composite rod is made by joining a copper rod to another rod of different material but of same area of cross-section. At 25°C the length of the composite rod is 1 m while that of copper rod 30 cm. At 125°C the length of the composite rod increases by 1.91 mm. Find the coefficient of linear expansion of the second rod (for copper, $\alpha = 1.7 \times 10^{-5}$/°C).

When the composite rod is not allowed to expand by holding it between two rigid walls, it is found that the length of the two constituents do not change with rise in temperature. Find the Young's modulus of the second rod (for copper $Y = 1.3 \times 10^{11}$ N/m^2). (IIT 1979)

Solution:

(i) Total length of the composite rod at 125°C is given by

$L = l_1 [1 + \alpha_1 (t_2 - t_1)] + l_2 [1 + \alpha_2 (t_2 - t_1)]$; l_1 is the length of other rod and l_2 is the length of copper rod.

$$\text{Increase in length} = L - (l_1 + l_2) = (l_1\alpha_1 + l_2\alpha_2)\,(t_2 - t_1)$$

$$\text{or} \quad \frac{L \quad l_1 \quad l_2}{t_2 \quad t_1} = \frac{0.191}{125 \quad 25} = (70\,\alpha_1 + 30 \times 1.7 \times 10^{-5})$$

$$\text{or} \quad 191 \times 10^{-5} = 70\,\alpha_1 + 51 \times 10^{-5}$$

$$\text{or} \quad 140 \times 10^{-5} = 70\,\alpha_1$$

$$\text{or} \quad \alpha_1 = 2 \times 10^{-5}/°\text{C}$$

(ii) In the second case, since there is no increase in length, i.e., no net force is acting; in other words, the same amount of force is exerted by the two rods.

Therefore,

$$F = \frac{Y_1 A \quad l_1}{l_1} = \frac{Y_2 A \quad l_2}{l_2}$$

$$\text{or} \quad Y_1 = Y_2 \frac{l_2}{l_1} \frac{l_1}{l_2}$$

$$\Delta l_1 = 70 \times 2 \times 10^{-5} \times 100 = 14 \times 10^{-2} \text{ cm}$$

$$\Delta l_2 = 30 \times 1.7 \times 10^{-5} \times 100 = 5.1 \times 10^{-2} \text{ cm}$$

$$Y_1 = \frac{1.3 \quad 10^{11} \quad 5.1 \quad 10^{4} \quad 0.7}{14 \quad 10^{4} \quad 0.3} = 1.1 \times 10^{11} \text{ N/m}^2$$

6.26. A steel wire of length 1 m, mass 0.1 kg and uniform cross-sectional area 10^{-6} m^2 is rigidly fixed at both ends. The temperature of the wire is lowered by 20°C. If transverse waves are set up by plucking the wire in the middle, calculate the frequency of the fundamental mode of vibration (Young's modulus of steel = 2×10^{11} N/m^2, coefficient of linear expansion of steel = 1.21×10^{-5}/°C). (IIT 1984)

Solution: The tension produced in the wire by lowering the temperature is given by

$$T = Y \cdot A \cdot \alpha \, \Delta T$$
$$= 2 \times 10^{11} \times 10^{-6} \times 1.21 \times 10^{-5} \times 20$$
$$= 48.40 \text{ N}$$

Fundamental frequency $n = \frac{1}{2l}\sqrt{\frac{T}{m}} = \frac{1}{2 \quad 1}\sqrt{\frac{48.4}{0.1}} = \frac{22}{2} = 11$ Hz

6.27. A rod AD consisting of three segments AB, BC and CD joined together, is hanging vertically from a fixed support A. The lengths of the segments are respectively 0.1 m, 0.2 m and 0.15 m. The cross-section of the rod is uniformly 10^{-4} m^2. A weight of 10 kg is hung from D. Calculate the displacements of points B, C and D. Neglect the weight of the rod. Given $Y_{AB} = 2.5 \times 10^{10}$ N/m^2, $Y_{BC} = 4 \times 10^{10}$ N/m^2 and $Y_{CD} = 1.0 \times 10^{10}$ N/m^2. (Roorkee, 1986)

Solution: The four points A, B, C and D are shown in Fig. 6.10.

$$Y_{AB} = \frac{10 \quad 9.8 \quad AB}{10^{-4} \quad l_1}$$

$$\therefore \Delta l_1 = \frac{10 \quad 9.8 \quad 0.1}{10^{-4} \quad 2.5 \quad 10^{10}} = 3.92 \times 10^{-6} \text{ m}$$

Thus, B will shifted downward by 3.92×10^{-6} m

$$Y_{BC} = \frac{10 \quad 9.8 \quad BC}{10^{-4} \quad l_2}$$

$$\therefore \Delta l_2 = \frac{10 \quad 9.8 \quad 0.2}{10^{-4} \quad 4 \quad 10^{10}} = 4.9 \times 10^{-6} \text{ m}$$

Therefore, C will be shifted downward by

$$\Delta l_1 + \Delta l_2 = 8.82 + 10^{-6} \text{ m}$$

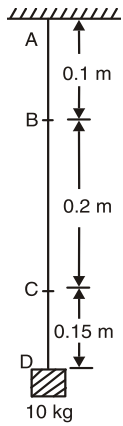


Fig. 6.10

$$Y_{CD} = \frac{10 \quad 9.8 \quad CD}{10^{-4} \quad l_3}$$

or $$\Delta l_3 = \frac{10 \quad 9.8 \quad 0.15}{10^{-4} \quad 1.0 \quad 10^{10}} = 14.7 \times 10^{-6} \text{ m}$$

Thus D will be shifted downward by $\Delta l_1 + \Delta l_2 + \Delta l_3 = 23.52 \times 10^{-6}$ m

6.28. Calculate the twisting couple per unit angle of twist of a hollow cylindrical tube of length l and having inner and outer diameters as d_1 and d_2 respectively.

Solution: Let the upper end of the tube be fixed and a couple be applied in a plane perpendicular to its length so as to twist it through an angle θ radian. In this case of twisting there is neither a change in the length nor the radius of the tube.

In view of Fig. 6.11. $BB' = l\,\phi = x \cdot \theta$

$$\therefore \phi = \frac{x}{l}$$

$$\eta = \frac{F}{\phi}; \eta = \text{modulus of rigidity}$$

or $$F = \eta\phi = \eta\frac{x}{l}$$

Consider a shell of radius x and $x + dx$

Face area $= 2\pi x dx$

Total shearing force $= 2\pi x dx\, \eta \cdot \frac{x}{l}$

$$= \frac{2}{l} \cdot x^2 dx$$

Moment of this force about $OO' = \frac{2}{l} \cdot x^3 dx$

Net twisting couple on the cylinder

$$N = \int_{\frac{d_1}{2}}^{\frac{d_2}{2}} \frac{2\pi\eta\theta}{l} \cdot x^3 dx$$

$$= \frac{}{32l} \quad d_2^4 \quad d_1^4$$

$$C = \frac{N}{} = \frac{}{32l} \quad d_2^4 \quad d_1^4$$

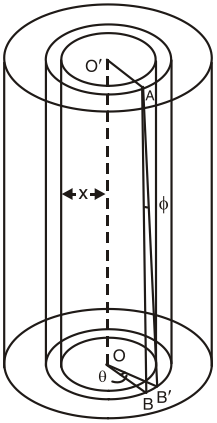


Fig. 6.11

6.29. Calculate the power that can be transmitted through a steel shaft of length $l = 1$ m, radius $r = 1.5$ cm, rotating about its own axis with an angular velocity of 90 rad/sec. The permissible torsional angle $\phi = 2°$. Modulus of rigidity of steel $\eta = 8.1 \times 10^{10}$ N/m^2.

Solution: A shaft is basically a cylinder. Torque acting on a cylinder of radius r and length l is given by

$C = \frac{r^4}{2l}$; ϕ = torsional angle.

Work done $dw = c \cdot d\theta$

Power $\frac{dw}{dt} = C \cdot \frac{d}{dt} = C \cdot \omega$

$\therefore P = \frac{r^4 \cdot}{2l}$

On substituting the values of various terms we have

$$P = \frac{3.14}{2} \times 8.1 \times 10^{10} \times 1.5^4 \times 10^{-8} \times \frac{2}{180} \times \frac{3.14}{} \times 90$$

$= 2.02$ watt

6.30. Calculate the internal pressure sustained by a glass tube of wall thickness $\Delta r = 1$ mm and radius $r = 2$ cm. Breaking stress of glass $\sigma = 0.05 \times 10^9$ N/m.

Solution: Let us consider a half of the cylindrical tube of length l

The force keeping the two half parts together = $2\sigma\ (\Delta r \cdot l)$

In case of equilibrium $p \cdot 2rl = 2\sigma \cdot \Delta r \cdot l$

$$\therefore\ p = \sigma \cdot \frac{\Delta r}{r}$$

On substituting the values we have

$$p = \frac{0.05 \quad 10^{9} \quad 10^{\ 3}}{2 \quad 10^{\ 2}}$$

$$= 2.5 \times 10^{6}\ \text{N/m}^2$$

6.31. Calculate the amount of work to be performed to construct a hoop out of a steel sheet of length l = 3 m, width h = 5 cm and thickness δ = 1.5 mm. Young's modulus of steel = 2×10^{11} N/m^2.

Solution: Consider a shell of radius x and $x + dx$

Strain on the layer of radius $x = \dfrac{2 \quad x \quad l}{l}$

Stress on this layer = $Y \cdot \dfrac{2 \quad x \quad l}{l}$

Energy per unit volume at the layer = $\dfrac{1}{2}\ Y\ \dfrac{2 \quad x \quad l}{l}\ \dfrac{2 \quad x \quad l}{l}$

$$= \frac{1}{2}\ Y\ \ \frac{x}{R}\ \ 1\ \ ^{2}\ ;\ 2\pi R = l$$

Volume of the shell = $2\pi x \cdot dx \cdot h$

Energy in the shell $dU = \dfrac{1}{2}\ \ \dfrac{Y}{R^2}\ (x - R)^2 \cdot 2\pi x dx \cdot h$

$$U = \quad dU \ = \frac{\pi Y h}{R^2} \int_{R-\frac{\delta}{2}}^{R+\frac{\delta}{2}} (x - R)^2\, x\, dx$$

Let $x - R = p$

$$U = \frac{\pi Y h}{R^2} \int_{-\frac{\delta}{2}}^{+\frac{\delta}{2}} p^2\ (R + p)\, dp$$

$$= \frac{2\pi Y h}{R^2} \left[\int_0^{\frac{\delta}{2}} p^2 R\, dp + \int_0^{\frac{\delta}{2}} p^3 dp \right]$$

$$= \frac{2 \quad Y h \ ^{3}}{24R}\ \ 1\ \ \frac{3}{8R}$$

For small value of δ we have

$$U = \frac{^{2} Y h \ ^{3}}{6l}$$

On substituting the values we have

$$U = \frac{1}{6} \times \frac{3.14^2}{3} \times 2 \times 10^{11} \times 5 \times 10^{-2} \times 1.5^3 \times 10^{-9} = 18.49 \text{ J}$$

6.32. A steel rod of length $l = 1$ m and radius $r = 8$ mm is fixed at one end while the other end is twisted through an angle $\phi = 3°$. Calculate the elastic deformation energy. Modulus of rigidity of steel = 8.1×10^{10} N/m^2.

Solution: Twisting couple per unit angle of twist is given by

$$C = \frac{r^4}{2l}; \eta = \text{modulus of rigidity}$$

$$\text{Torque } T = \frac{\pi \eta r^4 \cdot \theta}{2l}$$

$$\text{Work done} = W = \int_0^{\phi} T \cdot d\theta = \frac{\pi \eta r^4}{2l} \int_0^{\phi} \theta \cdot d\theta$$

$$= \frac{\pi \eta r^4 \cdot \phi^2}{4l}$$

On substituting the values we have

$$W = 3.14 \times 8.1 \times 10^{10} \times (0.8 \times 10^{-2})^4 \cdot \frac{3.14^2}{3600}$$

$$= 0.7 \text{ J}$$

6.33. When a steady current flows through a moving coil dead beat galvanometer its frame is acted upon by a twisting moment equal to 2×10^{-13} N/m. The work done in twisting the frame through a small angle θ is 8.5×10^{-16} J. Over what distance will the reflection from the mirror of the galvanometer shift on a scale placed at a distance of 1 m from the galvanometer?

Solution: Torque $= C\theta = 2 \times 10^{-13}$

$$\text{Work done} = \frac{1}{2} C \cdot \theta^2 = 8.5 \times 10^{-16}$$

$$\therefore \theta = \frac{2 \quad 8.5 \quad 10^{16}}{2 \quad 10^{13}} = 8.5 \times 10^{-3}$$

The reflection from the mirror of the galvanometer will be obtained at an angle 2θ given by

$2\theta = \frac{l}{L}$; l = distance moved on the scale placed at a distance L from the galvanometer.

$\therefore l = 2\theta \cdot L$

On putting the values we have

$l = 2 \times 8.5 \times 10^{-3} \times 1$

$= 17 \times 10^{-3}$ m

$= 17$ mm

6.34. A load suspended from a steel rope is lowered from a ship to measure the depth of a sea. Neglecting the weight of the load as compared with that of the rope, find the maximum depth that can be measured by this method. Density of sea water = 1000 kg/m^3, Breaking stress of steel = 7.85×10^8 N/m^2, density of steel = 7700 kg/m^3.

Solution: Net stretching force = $Mg - \rho' hgA$; ρ' = density of sea water

h = depth of sea

A = area of cross-section

Mass of rope $M = A\rho h$; ρ = density of steel

Net stretching force = $Ahg\,(\rho - \rho')$

$$\text{Breaking stress } T = \frac{\text{Net stretching force}}{A}$$

$$= hg\,(\rho - \rho')$$

or $$h = \frac{T}{g}$$

On putting the values we have

$$h = \frac{7.85 \quad 10^8}{9.8 \quad (7700 \quad 1000)}$$

$$= 11.955 \text{ km.}$$

6.35. A rubber hose 50 cm long and with an internal diameter of 1 cm is stretched until its length becomes greater by 10 cm. Find the internal diameter of the stretched hose. Poisson's ratio of the rubber $\sigma = 0.5$.

Solution: Poisson's ratio $\sigma = \dfrac{\text{lateral strain}}{\text{longitudinal strain}} = \dfrac{dD/D}{dL/L}$

$$\sigma = \frac{L}{D} \cdot \frac{dD}{dL}$$

$$\therefore dD = \frac{\sigma \cdot D \cdot dL}{L}$$

On putting the values of various terms we have

$$dD = \frac{0.5 \quad 10^{\,2} \quad 10^{\,1}}{0.5} = 10^{-3} \text{ m} = 1 \text{ mm}$$

$\therefore$ Final diameter = $D - dD = (10 - 1)$ mm = 9 mm

6.36. The minor of a galvanometer is suspended on a wire $L = 10$ cm long and $d = 0.01$ mm in diameter. Find the twisting moment which corresponds to a deviation of the reflection of $l = 1$ mm on a scale removed from the mirror by $D = 1$ m. The shear modulus of the material of the wire $\eta = 4 \times 10^{11}$ dyne/cm^2.

Solution: Torque $T = \dfrac{r^4}{2L}$

here $\theta = \frac{1}{2} \times 10^{-3} = 5 \times 10^{-4}$

$r = 0.005 \text{ mm} = 5 \times 10^{-6} \text{ m}$

$\eta = 4 \times 10^{11} \times \frac{10^{5}}{10^{4}} = 4 \times 10^{10} \text{ N/m}^2$

On substituting the values

$$T = \frac{3.14 \quad 4 \quad 10^{10} \quad 5^4 \quad 10^{24} \quad 5 \quad 10^{4}}{2 \quad 0.1}$$

$$= 1.96 \times 10^{-13} \text{ N/m}$$

6.37. A hollow tube of length $2L$ open at both ends is rigidly fastened in the middle to the wire of length l and radius r and is suspended vertically from a support as shown in the Fig. 6.12. Two hollow and two solid metal cylinders of equal length and diameter can be filled into the tube such that, put end to end they just fill it completely. Mass of each hollow cylinder is m_1 and that of each solid cylinder m_2. When the solid cylinders are put into the inner position and the hollow ones in the outer position the period of torsional vibration is T_1. When the solid and the hollow cylinders are interchanged in position the period of torsional vibration is T_2. Find the modulus of rigidity of the material of the wire.

Solution:

$$T_1 = 2\pi \cdot \sqrt{\frac{I_1}{C}}$$

and $$T_2 = 2\pi \cdot \sqrt{\frac{I_2}{C}}$$

where I_1 and I_2 are the moment of inertia of the loaded tube about the wire as axis and C is the twisting couple per unit angle of twist.

$$T_2^2 \quad T_1^2 = \frac{4}{C}^{2} (I_2 - I_1)$$

$$I_2 = I_1 + 2 (m_2 - m_1) \quad \frac{9L^2}{16} \quad \frac{L^2}{16}$$

$$= I_1 + 2 (m_2 - m_1) \frac{L^2}{2}$$

or $$I_2 - I_1 = (m_2 - m_1) L^2$$

$$\therefore \quad T_2^2 \quad T_1^2 = \frac{4}{C}^{2} \cdot (m_2 - m_1) L^2$$

or $$T_2^2 \quad T_1^2 = \frac{4\pi^2 \cdot 2l}{\pi \eta r^4} (m_2 - m_1) L^2$$

$$\therefore \quad \eta = \frac{8\pi l \cdot (m_2 - m_1) L^2}{\left(T_2^2 - T_1^2\right) r^4}$$

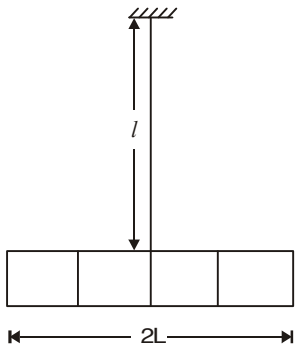


Fig. 6.12

6.38. A thin uniform metallic rod of length 0.5 m and radius 0.1 m rotates with an angular velocity 400 rad/sec in a horizontal plane about a vertical axis passing through one of its ends. Calculate tension in the rod and the elongation of the rod. The density of material of the rod is 10^4 kg/m^3 and the Young's modulus is 2×10^{11} N/m^2. (Roorkee 1995)

Solution: Consider an element of length dx situated at a distance x from the axis of rotation (see Fig. 6.13).

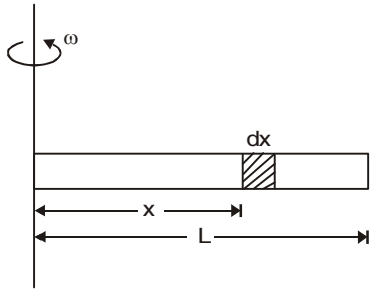


Fig. 6.13

Centripetal force dT on the element is given by

$dT = \rho A dx \cdot x \cdot \omega^2$; A = area of cross-section of the rod.

Tension in the rod at a distance x from the axis of rotation will be due to the mass of the rod between x and L

$$T = \int dT = \rho A \omega^2 \int_x^L x\,dx = \frac{1}{2} \rho A \omega^2 [L^2 - x^2]$$

On substituting the values we have

$$T = \frac{1}{2} \times 10^4 \times \pi \times 10^{-2} \times 400^2 [0.25 - x^2]\ N$$

$$= 8\pi \times 10^6 \left[0.25 - x^2\right] N$$

Let dl be the elongation in the element of length dx situated at a distance x from the axis of rotation then

$$\frac{dl}{dx} = \frac{T}{AY}\text{; Y = Young's modulus}$$

or $$dl = \frac{1}{2} \frac{\rho\omega^2}{Y} \left[L^2 - x^2\right] dx$$

or $$l = \int dl = \frac{\rho\omega^2}{2Y} \int_0^L \left(L^2 - x^2\right) dx = \frac{1}{3} \frac{\rho\omega^2 L^3}{\Upsilon}$$

On putting the values we have

$$l = \frac{\frac{1}{3} \times 10^4 \times (400)^2 \times (0.5)^3}{2 \times 10^{11}} = \frac{1}{3} \times 10^{-3} \text{m}$$

6.39. Two blocks of mass m_1 and m_2 connected through a light string of area of cross-section A arranged as shown in the Fig. 6.14. The block of mass m_1 is pulled to the left by a force $F = m_2 g/3$. The Young's modulus of the material of string is Y. In the absence of the frictional force find the strain developed in the string.

Solution: The equation of motion of the blocks is given by

$$F - T = m_1 a$$

and $$T - m_2 g = m_2 a$$

or $$T - m_2 g = \frac{m_2}{m_1}(F - T)$$

or $$T\left(1 + \frac{m_2}{m_1}\right) = \frac{m_2}{m_1} F + m_2 g$$

$$T\left(\frac{m_1 + m_2}{m_1}\right) = \frac{m_2}{m_1}\left(\frac{m_2 g}{3}\right) + m_2 g$$

$$T = \frac{m_2 g}{3}\left(\frac{m_2 + 3m_1}{m_1 + m_2}\right)$$

$$\therefore \quad \text{Stress} = \frac{T}{A} = \frac{m_2 g}{3A}\left(\frac{m_2 + 3m_1}{m_1 + m_2}\right)$$

$$\text{Strain} = \frac{\text{Stress}}{Y} = \frac{m_2 g}{3AY}\left(\frac{m_2 + 3m_1}{m_1 + m_2}\right)$$

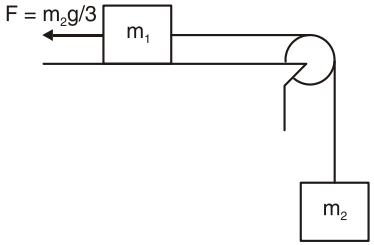


Fig. 6.14

6.40. A steel bar 1 cm square in cross-section is supported by two knife edges 1 m apart. Calculate the depression of the centre of the bar when a load of 2 Kg is applied at that point. Young's modulus of steel $Y = 2 \times 10^{11}$ N/m^2.

Solution:

$$Y = \frac{W \cdot L^3}{4 y b d^3};$$

W = weight suspended
L = distance between the knife edges
y = depression produced
b = breadth of the bar
d = depth of the bar

$$\therefore \quad y = \frac{W L^3}{4 \cdot Y \cdot b d^3}$$

On substituting the values we have

$$y = \frac{2 \quad 9.8 \quad 1}{4 \quad 2 \quad 10^{11} \quad 10^{-2} \quad 10^{-6}}$$

$$= 2.45 \text{ mm}$$

6.41. A ring of radius 0.1 m is made out of a thin metallic wire of area of cross-section 10^{-6} m^2. The ring has a uniform charge of π Coulomb. Find the change in the radius of the ring when a charge of 10^{-8} Coulomb is placed at centre of the ring. Young's modulus of the metal is 2×10^{11} N/m^2.

Solution: Consider an element of length Δl. Downward force F is given by (see Fig. 6.15).

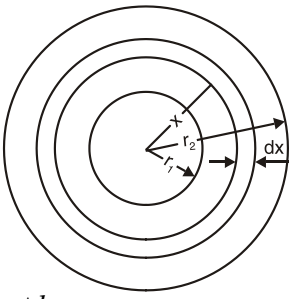


Fig. 6.15

$$F = 2T \cdot \sin \frac{}{2} = T \cdot \Delta\theta = T \cdot \frac{l}{R};$$

For small angles $\sin \frac{}{2} = \frac{}{2}$

$$\therefore T = \frac{F \cdot R}{\Delta l}$$

Charge on the element of length $\Delta l = \frac{}{2\ R} \times \Delta l$

$$= \frac{\Delta l}{2 \times 0.1} = 5\,\Delta l$$

Force on the element $= F = \frac{1}{4\pi\epsilon_0} \cdot \frac{5 \cdot \Delta l \times 10^{-8}}{(0.1)^2}$

$$= 9 \times 10^9 \times 5\,\Delta l \times 10^{-6}$$

$$= 4.5 \times 10^4\,\Delta l$$

$$\therefore T = \frac{4.5 \times 10^4\,\Delta l \times 0.1}{\Delta l} = 4.5 \times 10^3 \text{ N}$$

Change in circumference = strain × circumference

$$= \frac{T}{A \cdot Y} \times 2\pi R$$

$$\therefore \text{Change in radius } \Delta r = \frac{\text{Change in circumference}}{2}$$

$$\Delta r = \frac{T R}{A \cdot Y}$$

On putting the various values we have

$$\Delta r = \frac{4.5\ \ 10^3\ \ 0.1}{10^{\ 6}\ \ 2\ \ 10^{11}} = 2.25 \times 10^{-3} \text{ m}$$

6.42. A ring of mass m and outer radii r_2 is firmly fixed on a shaft of radius r_1. The shaft rotates about its own axis with constant angular acceleration α. Find the torque in the ring as a function of distance from the axis of rotation.

Solution: Consider a ring of width dx and radius x.

Turning force = $S \times 2\pi x dx$; S is the shear stress at a distance x

$$S = \cdot\frac{x}{r_2};\ \sigma = \text{maximum shear stress}$$

$$\therefore \text{Turning force} = \frac{x}{r_2} \cdot 2\pi x dx$$

$$\therefore \text{Torque on the element } d\tau = \frac{2\pi\sigma}{r_2} \cdot x^2 dx \cdot x$$

$$= \frac{2\pi\sigma}{r_2} \cdot x^3 dx$$

$$\therefore \tau = \quad d \quad = \frac{2\pi\sigma}{r_2} \int_r^{r_2} x^3 dx$$

$$\tau = \frac{2}{4r_2} \quad r_2^4 \quad r^4$$

$$\sigma = \frac{ma}{r_2^2 \quad r_1^2} = \frac{m \quad \cdot r_2}{r_2^2 \quad r_1^2}; \quad a = \text{linear acceleration}$$

α = angular acceleration

$$\therefore \tau = \frac{\pi}{2r_2} \cdot \frac{m\alpha \cdot r_2}{\pi\left(r_2^2 - r_1^2\right)} \cdot \left(r_2^4 - r^4\right)$$

$$\tau = \frac{m \quad r_2^4 \quad r^4}{2 \quad r_2^2 \quad r_1^2}$$

6.43. Find the maximum pressure produced when water freezes. ρ_{ice} = 0.92 gm/cc, Young's modulus = 2.8×10^{10} N/m^2, Poisson's ratio σ = 0.3.

Solution: Relative change in volume is given by

$$\frac{V}{V} = \frac{3 \; 1 \quad 2}{Y} \cdot p \; ; \; p = \text{pressure}$$

$$\text{Now} \quad \Delta V = \frac{m}{\text{ice}} - \frac{m}{\text{water}} = \frac{m\left(\rho_{\text{water}} - \rho_{\text{ice}}\right)}{\rho_{\text{water}} \cdot \rho_{\text{ice}}}$$

$$\frac{V}{V} = \frac{\text{water} \qquad \text{ice}}{\text{water}} = \frac{3 \; 1 \quad 2}{Y} \cdot p$$

$$\therefore p = \frac{Y\left(\rho_{\text{water}} - \rho_{\text{ice}}\right)}{\rho_{\text{water}} \cdot 3\left(1 - 2\sigma\right)}$$

$$p = \frac{2.8 \times 10^{10} \times \left(1 - 0.92\right)}{1 \times 3\left(1 - 0.6\right)}$$

$$= \frac{2.8 \quad 10^{10} \quad 0.08}{3 \quad 0.4}$$

$$= 1.867 \times 10^9 \text{ N/m}^2$$

6.44. Calculate the % change in volume of a solid glass sphere if it is placed in the vacuum than in air. Atmospheric pressure = 75 cm of mercury. Compressibility of glass β = 0.025 GPa^{-1}.

Solution:

$$\beta = -\frac{dv}{v\,dp}$$

$$\therefore \frac{dv}{v} \times 100 = \beta \cdot dp \times 100$$

$$= 0.025 \times 10^{-9} \times 0.75 \times 13.6 \times 10^3 \times 9.8 \times 100$$

$$= 2.5 \times 10^{-4}$$

6.45. A ball of mass $m = 0.5$ kg is attached to an iron wire of length 25 cm and radius 0.5 mm. At what maximum number of revolutions per sec can the wire rotate uniformly in a vertical plane so that the wire does not break. $\sigma_{iron} = 2.94 \times 10^8$ N/m^2.

Solution: The equation of motion is given by

$$T - mg = ml\,\omega^2$$

$$A\ \frac{T}{A}\ \frac{mg}{A} = ml\omega^2;\ A = \text{area of cross-section of the wire}$$

$$25\pi \times 10^{-8}\ \ 2.94\ \ 10^8\ \ \frac{0.5\ \ 9.8\ \ 10^8}{25} = 0.5 \times 0.25 \cdot \omega^2$$

$$\omega = \sqrt{\frac{230.79\ \ 4.9}{0.5\ \ 0.25}}$$

$$= 42.5$$

$$n = \frac{4.25}{6.28} = 6.77 \text{ rps}$$

6.46. N rods of torsional rigidities $C_1, C_2 \ldots C_N$ are joined end to end. Find the torsional rigidity of the combination.

Solution: Let us fix the upper end of the combination by a rigid support and rotate the lower end. Let ϕ be the angle of twist at the lowest end. Let ϕ_1, ϕ_2, ϕ_3 . . . etc. be the twists at the first second and the successive junctions respectively. The location of the junction is with respect to the lower end.

$$C_0 = C_1\,(\phi - \phi_1) = C_2\,(\phi_1 - \phi_2) \ldots = C\,\phi$$

C = torsional rigidity of the combination

$$\text{Now}\ \ \phi - \phi_1 = \frac{C_0}{C_1},\ \phi_1 - \phi_2 = \frac{C_0}{C_2} \ldots \phi = \frac{C_0}{C}$$

$$\phi = \frac{C_0}{C} = (\phi - \phi_1) + (\phi_1 - \phi_2) \ldots = \frac{C_0}{C_1} + \frac{C_0}{C_2} + \frac{C_0}{C_3} + \ldots \frac{C_0}{C_N}$$

$$\text{or} \qquad \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots \frac{1}{C_N}$$

6.47. Find the torsional rigidity of a tapered wire of length L having end radii as r_1 and r_2. The modulus of rigidity of the material of wire is η.

Solution: Consider a slice of thickness dx at a distance x from the apex O. (See Fig. 6.16)

From the geometry we have

$$\frac{r_2}{l_2} = \frac{r}{x} = \frac{r_1}{l_1}$$

Torsional rigidity of the slice $d\tau = \dfrac{r^4}{2\,dx}$

$$\frac{1}{d} = \frac{2\,dx}{r^4}$$

$$\frac{1}{d} = \frac{1}{-} = \int_{l_1}^{l_2} \frac{2\,dx}{\pi\eta \dfrac{r_1^4}{l_1^4} \cdot x^4}$$

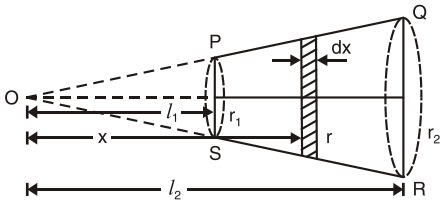


Fig. 6.16

$$= -\frac{2l_1^4}{3\pi\eta r_1^4}\left|\frac{1}{x^3}\right|_{l_1}^{l_2}$$

$$= \frac{2l_1^4}{3\quad r_1^4}\ \frac{l_2^3\quad l_1^3}{l_1^3\ l_2^3}$$

$$= \frac{2l_1}{3\quad r_1^4}\ \frac{l_2\quad l_1\quad l_2^2\quad l_1 l_2\quad l_1^2}{l_2^3}$$

$$= \frac{2L}{3\quad r_1^4}\ \frac{l_1}{l_2}\ \frac{l_1^2}{l_2^2}\ \frac{l_1^3}{l_2^3}$$

$$= \frac{2L}{3\quad r_1^4}\ \frac{r_1}{r_2}\ \frac{r_1}{r_2}^2\ \frac{r_1}{r_2}^3$$

$$= \frac{2L}{3\quad r_1^3 r_2^3}\ r_2^2\quad r_1 r_2\quad r_1^2$$

or $$\tau = \frac{3\quad r_1^3 r_2^3}{2L\ r_1^2\quad r_1 r_2\quad r_2^2}$$

6.48. A cylindrical rod of length L mass m and area of cross-section A is suspended from the ceiling of a room. Find the elastic deformation energy of the rod.

Solution: Consider an element of length dx situated at a distance x from the lower end.

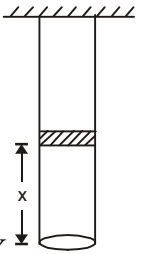


Fig. 6.17

$$\text{Force} = \left(\frac{m}{L}\cdot x\right)g$$

$$\text{Stress} = \frac{mxg}{LA}$$

$$\text{Strain} = \frac{\text{Stress}}{Y} = \frac{mgx}{LAY}\,;\ Y = \text{Young's modulus}$$

$$\text{Elastic energy/volume} = \frac{1}{2}\ \text{stress} \times \text{strain} = \frac{1}{2}\ \text{strain}^2 \times Y$$

$$= \frac{1}{2}\ \frac{m^2g^2x^2}{L^2A^2Y}$$

dU = Energy associated with the element $= \dfrac{1}{2}\dfrac{m^2g^2x^2A\cdot dx}{L^2A^2Y}$

$$U = \int dU = \frac{\frac{1}{2}m^2g^2}{L^2AY}\cdot\int_0^L x^2dx$$

$$= \frac{m^2g^2L}{6AY}$$

6.49. A wire of diameter 0.32 mm elongates by 1 mm when stretched by a force of 0.33 kgf and twists through 1 rad when a torque of 145×10^{-7} N/m is applied. Find the value of Poisson's ratio.

Solution:

$$Y = \frac{F\cdot L}{A\cdot l} = \frac{0.33\quad 9.8\quad L}{0.16^{\ 2}\quad 10^{\ 6}\quad 10^{\ 3}}$$

$$\tau = \frac{\pi\eta r^4}{2L}$$

$$\therefore\ \eta = \frac{\tau\cdot 2L}{\pi r^4} = \frac{2\quad 145\quad 10^{\ 7}\quad L}{(0.16)^4\quad 10^{\ 12}}$$

$$\frac{Y}{} = 2.85 = 2 + 2\sigma$$

$$\sigma = 0.43$$

4.50. A metal bar soldered to the lower end of a wire as shown in Fig. 6.18 performs torsional oscillations. How the period of oscillation will change by reducing (i) the length of the wire by 50% (ii) the length of the bar by 50% (iii) the length of the wire and bar both by 50%.

Solution:

$T = 2\pi\cdot\sqrt{\dfrac{I}{C}}$; I = moment of inertia of the bar about an axis passing through the wire.

C = Twisting couple per unit angle of twist.

$$I = \frac{mL^2}{12},\ C = \frac{r^4}{2l}$$

$$\text{Now } T = 2\pi \cdot \sqrt{\frac{mL^2}{12} \times \frac{2l}{\pi \eta r^4}}$$

$$= 2\pi \cdot \sqrt{\frac{mL^2 l}{6\pi \eta r^4}}$$

(i) $$T_1 = 2\pi \cdot \sqrt{\frac{mL^2 \cdot l/2}{6\pi \eta r^4}} = \frac{T}{\sqrt{2}}$$

(ii) $$T_2 = 2\pi \cdot \sqrt{\frac{m \cdot L^2/4 \cdot l}{6\pi \eta r^4}} = \frac{T}{2}$$

(iii) $$T_3 = 2\pi \cdot \sqrt{\frac{m \cdot L^2/4 \cdot l/2}{6\pi \eta r^4}} = \frac{T}{2\sqrt{2}}$$

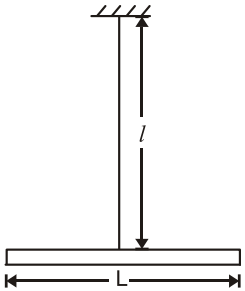


Fig. 6.18

6.51. Two cylindrical bars of length l and radius r are attached through a steel wire AB. A loop of thread is attached at the ends C and E of the two bars. The system is suspended through the two threads AA' and BB' as shown in Fig. 6.19. When the thread is burnt out the two rods starts oscillating simple harmonically. Find the Young's modulus of the wire AB.

Solution: Let θ be the angular deflection of the two bars

Bending moment on each rod $M = \frac{YAK^2}{R}$

A = area of cross-section, K = radius of gyration, R = radius of curvature.

$$\frac{l}{R} = 2\theta;\ M = \frac{2YAK^2}{l}$$

$$\text{Now } I\frac{d^2\theta}{dt^2} = -\frac{2YAK^2}{l} \cdot\ ;$$

I = moment of inertia of the bar.

$$\omega = \frac{2}{T} = \sqrt{\frac{2YAK^2}{Il}}$$

$$T = 2\pi \cdot \sqrt{\frac{Il}{2YAK^2}}$$

$$= 2\pi \cdot \sqrt{\frac{I \cdot l}{2Y \cdot \pi r^2 \cdot r^2/4}}$$

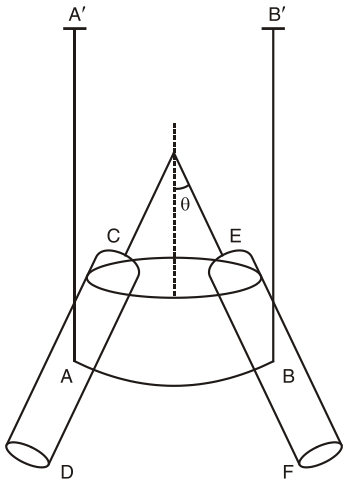


Fig. 6.19

$$T^2 = \frac{8\ ^2 I l}{Y r^4}$$

$$\therefore Y = \frac{8\ I l}{T^2 r^4}$$

6.52. A highly rigid cubical block A of small mass m and side L is fixed rigidly on another cubical block B of the same dimension and of low modulus of rigidity η such that the lower surface of A completely covers the upper surface of B. The lower face of B is rigidly held on a horizontal surface. A small force is applied perpendicular to one of the faces of A. Find the period with which A will execute oscillations when the force is withdrawn.

Solution:

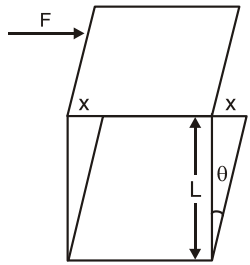


Fig. 6.20

$$\eta = \frac{F}{A};\ \theta = \frac{x}{L} \text{ (see Fig. 6.20)}$$

$$F = \frac{Ax}{L} = \eta L x;\ A = L^2$$

$$m \cdot \text{acceleration} = -\eta L \cdot x$$

$$\omega = \frac{2}{T} = \sqrt{\frac{L}{m}}$$

$$T = 2\pi \cdot \sqrt{\frac{m}{\eta L}}$$

6.53. Two cylinders A and B of radii r and $2r$ are soldered coaxially. The free end of A is clamped and the free end of B is twisted by an angle θ. Calculate the twist at the junction taking the materials of the two cylinders to be same and of equal lengths.

Solution: Let T be the couple applied at the free end of B. Let ϕ be the twist at the junction. The relative twist between the ends of the rod $A = \phi$ and the relative twist between the ends of the rod $B = \theta - \phi$.

$$T = \frac{r^4}{2l} = \frac{\pi\eta \cdot (2r)^4\ (\theta - \phi)}{2l}$$

$$17\ \phi = 16\ \theta$$

$$\therefore \phi = \frac{16\ \theta}{17}$$

6.54. Two solid cylinders of same material having lengths l and $2l$ and radii r and $2r$ are joined coaxially. When a couple is applied between the free ends, the shorter cylinder shows a tiwst of 30°. Find the twist of the longer cylinder.

Solution:

$$T = \frac{r^4}{2l} = \frac{\pi\eta \cdot (2r)^4 \cdot \phi}{2 \times 2l}$$

$$\theta = 8\phi$$

$$\therefore \phi = \frac{}{8} = \frac{30}{8} = 3.75°$$

6.55. A thin walled circular tube of mean radius 10 cm and thickness 0.05 cm is melted and recast into a solid cylinder of the same length. Compare their torsional rigidities.

Solution:

$$\text{Inner radius} = r_1 = 10 - \frac{0.05}{2} = 9.975 \times 10^{-2} \text{ m}$$

$$\text{Outer radius} = r_2 = 10 + \frac{0.05}{2} = 10.025 \times 10^{-2} \text{ m}$$

$$r_2^2 \quad r_1^2 \quad l = \pi r^2 \rho l$$

or

$$r = \sqrt{r_2^2 \quad r_1^2}$$

$$\frac{C_1}{C_2} = \pi\eta \frac{\left(r_2^4 - r_1^4\right)}{2l} \cdot \frac{2l}{\pi\eta r^4} = \frac{r_2^2 \quad r_1^2}{r^2}$$

$$= \frac{r_2^2 \quad r_1^2}{r_2^2 \quad r_1^2} = \frac{100.5 \quad 99.5}{100.5 \quad 99.5} = 200$$

5.56. A suspension thread consists of a wire of length l and radius r carrying below it a wire of length $l/10$ and radius $10\,r$. Each time the material is the same. The top end is clamped and the lower end is twisted about a vertical axis. Calculate (i) the torsional rigidity of the whole suspension. (ii) the ratio of the angles through which the lower ends of each part of the suspension are twisted.

Solution:

(i) $$\frac{1}{-} = \frac{2l}{r^4} \quad \frac{2l}{r^4 \quad 10 \quad 10^4} = 2l \quad \frac{1 \quad 10^{\ 5}}{r^4}$$

$$t = \frac{r^4}{2l \quad 1 \quad 10^{\ 5}}$$

(ii) $$T = \frac{\pi\eta r^4\theta}{2l} = \frac{\pi\eta r^4 \times 10^5 \phi}{2l}$$

$$\therefore - = 10^5$$

6.57. A hollow cylinder of length l and inner and outer radii r_1 and r_2 is fixed rigidly at one end. The other end of the cylinder is rotated through

θ. Find (i) the torque developed (ii) the amount of work done (iii) compare the modulus of rigidity of a hollow and solid shaft.

Solution:

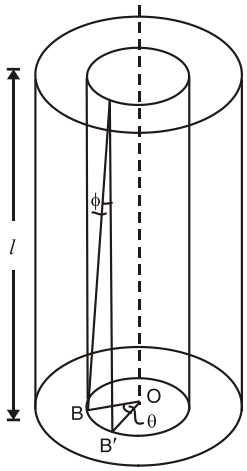


Fig. 6.21

(i) $BB' = x\,\theta = l\,\phi$ (see Fig. 6.21)

or $\phi = \dfrac{x}{l}$

$$\eta = \frac{\text{shearing stress } f}{} = \frac{f \cdot l}{x\theta}$$

$$f = \frac{x}{l}$$

Consider a shell of radius x and $x + dx$

Area of the shell $= 2\pi\, x dx$

Shearing force tangential to the surface of the cylinder $= \dfrac{\eta x \theta}{l} \cdot 2\pi x dx$

Torque acting on the cylinder

$$d\tau = \frac{2\pi\eta\theta}{l} \cdot x^2 dx \cdot x$$

Net torque $= d = \dfrac{2\pi\eta\theta}{l}\int_{r_1}^{r_2} x^3 dx$

$$= \frac{\pi\eta\theta\left(r_2^4 - r_1^4\right)}{2l}$$

(ii) Work done $= d = \int \dfrac{\pi\eta\left(r_2^4 - r_1^4\right)}{2l} \cdot \theta \cdot d\theta = \dfrac{\pi\eta\left(r_2^4 - r_1^4\right)\theta^2}{4l}$

(iii) Torsional rigidity $C' = \dfrac{r_2^4 \quad r_1^4}{2l}$ for hollow cylinder

and $C = \dfrac{r^4}{2l}$ for solid cylinder

$$\frac{C}{C} = \frac{r_2^4 \quad r_1^4}{r^4} = \frac{\left(r_2^2 - r_1^2\right)\left(r_2^2 + r_1^2\right)}{r^4}$$

If masses of both the cylinders are same then

$$r_2^2 \quad r_1^2 \; l = \pi r^2 \, l \, \rho$$

$$\therefore \; r_2^2 \quad r_1^2 = r^2$$

Now $$\frac{C}{C} = \frac{r_2^2 \quad r_1^2 \quad 2r_1^2}{r^2}$$

$$= 1 + \frac{2r_1^2}{r^2}$$

$$\therefore C' > C$$

7

SIMPLE HARMONIC MOTION

7.1. Show that a particle moving on the circumference of a circle executes simple harmonic motion.

Solution: Let a particle be at A (see Fig. 7.1). It starts moving on the circumference of the circle with an angular velocity ω and reaches B after time t. Its displacements along the x and y axes is given by

$$x = a\cos\omega t$$

$$y = a\sin\omega t$$

where a is the radius of the circle.

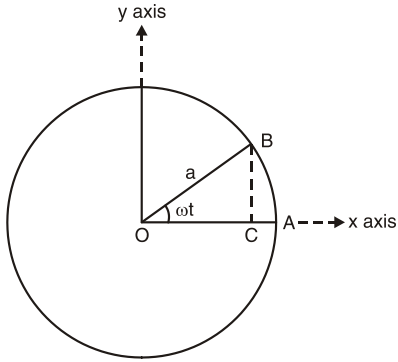


Fig. 7.1

The acceleration along the x and y-axes is given by

$$\frac{d^2x}{dt^2} = -\omega^2 x \text{ and } \frac{d^2y}{dt^2} = -\omega^2 y$$

The acceleration is proportional to displacement along the two axes; hence the particle executes simple harmonic motion.

$$\text{Now } \omega = \frac{2}{T} = \sqrt{\frac{\text{acceleration}}{\text{displacement}}}$$

$$\text{or } T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

7.2. A particle is executing simple harmonic motion. Its amplitude is 4 cm and time period 3 *s*. What is the time it will take to travel from its position of maximum displacement to one corresponding to half its amplitude?

Solution: The equation of motion of a particle executing simple harmonic motion is $x = a\sin\omega t$

$$\text{then } \frac{dx}{dt} = a\omega\cos\omega t = \omega\sqrt{a^2 - x^2}$$

$$\text{or } \frac{dx}{\sqrt{a^2 - x^2}} = \omega \cdot dt$$

Integrating the above equation within the limits given, the required time is as follows:

$$\int_{2}^{4} \frac{dx}{\sqrt{4^2 - x^2}} = \frac{2\pi}{3} \int_{0}^{t} dt$$

or $$\left| \sin^{-1} \frac{x}{4} \right|_{2}^{4} = \frac{2\pi}{3} \cdot t$$

or $$\sin^{-1} 1 - \sin^{-1} \frac{1}{2} = \frac{2}{3} \cdot t$$

or $$\frac{}{2} - \frac{}{6} = \frac{2}{3} \cdot t$$

or $$\frac{}{3} = \frac{2}{3} \cdot t$$

or $$t = 0.5\, s$$

7.3. Two linear simple harmonic motions of different amplitudes and same frequency are imposed on a particle along the x and y axes respectively. Find out the resultant path followed by the particle if the initial phase difference between them is 0, π and $\pi/2$ respectively.

Solution: Let the equation of motion of a particle along x and y axes be

$$x = a \sin \omega t$$

$$y = b \sin (\omega t + \phi)$$

where ϕ is the initial phase.

Now $$\frac{y}{b} = \sin \omega t \cos \phi + \cos \omega t \sin \phi$$

or $$\frac{y}{b} = \frac{x}{a} \cos \phi + \sqrt{1 - \frac{x^2}{a^2}} \cdot \sin$$

or $$\frac{y}{b} - \frac{x}{a} \cos \quad^{2} = 1 - \frac{x^2}{a^2} \sin^2$$

or $$\frac{y^2}{b^2} - \frac{x^2}{a^2} - \frac{2xy}{ab} \cos = \sin^2$$

(i) When $\phi = 0$ we have

$\frac{y}{b} - \frac{x}{a} = 0$ or $y = \frac{b}{a} x$, i.e., the particle will describe a straight line in the first and third quadrants.

(ii) When $\phi = \pi$, we have

$\frac{y}{b} + \frac{x}{a} = 0$ or $y = -\frac{b}{a} x$, i.e., the particle will describe a straight line in the second and fourth quadrants.

(iii) When $\phi = \pi/2$, we have

$$\frac{x^2}{a^2} \quad \frac{y^2}{b^2} \quad 1 \text{ i.e., the particle will describe an ellipse.}$$

When $a = b$, the above equation is transformed into the equation of the circle $x^2 + y^2 = a^2$

7.4. Two linear simple harmonic motions of equal amplitudes and frequencies ω and 2ω are imposed on a particle along the axes of x and y respectively. If the initial phase difference between them is $\pi/2$, then find the resultant path followed by the particle. (Roorkee 1985)

Solution: Let the equation of the particle along the x and y axes be

$$x = a\cos\omega t \text{ and } y = a\cos \quad 2 \quad t \quad \frac{}{2}$$

Now
$$y = -a\sin 2\omega t$$
$$= -2a\sin\omega t\cos\omega t$$
$$= -2a\sqrt{1-\frac{x^2}{a^2}}\cdot\frac{x}{a}$$

or
$$y^2 = 4x^2 \quad 1 \quad \frac{x^2}{a^2}$$

7.5. A particle executes a simple harmonic motion with time period $T_1 = 3s$ under a constraining force F_1, with time period $T_2 = 4s$ under a similar force F_2. What will be its time period when the two forces acts together?

Solution: Let m be the mass of the particle. Suppose the first force F_1 causes an acceleration a_1 and the second force F_2 and acceleration a_2 in it. Also, assume that the sum of these forces produces an acceleration a in the particle.

Then, according to Newton's second law

$$a_1 = \frac{F_1}{m}; a_2 = \frac{F_2}{m} \text{ and } a = \frac{F_1 \quad F_2}{m}$$
$$\therefore a = a_1 + a_2$$

Since the particle executes simple harmonic motion,

$$a_1 = - \quad {}^2_1 x, a_2 = - \quad {}^2_2 x \text{ and } a = -\omega^2 x$$

where $\omega_1 = \frac{2}{T_1}$, $\omega_2 = \frac{2}{T_2}$ and $\omega = \frac{2}{T}$ and x is the displacement.

Now
$$\omega^2 = \quad {}^2_1 \quad {}^2_2$$

or
$$\frac{1}{T^2} = \frac{1}{T_1^2} \quad \frac{1}{T_2^2}$$

or
$$T = \frac{T_1 T_2}{\sqrt{T_1^2 \quad T_2^2}}$$

On substituting the values of T_1 and T_2, we have

$$T = \frac{3 \quad 4}{\sqrt{3^2 \quad 4^2}} = \frac{12}{5} = 2.4 \text{ s}$$

7.6. A block is resting on a piston which is moving vertically in simple harmonic motion of period 1.0 s. At what amplitude of motion will the block and piston separate? What is the maximum velocity of the piston at this amplitude? (Roorkee 1985)

Solution: Let us assume that the piston holding the block is executing simple harmonic motion along the y-axis. The equation of motion is given by

$$y = a \sin \omega t$$

$$\frac{dy}{dt} = \text{velocity} = a\,\omega \cos \omega t$$

and $$\frac{d^2y}{dt^2} = \text{acceleration} = -\omega^2 y; \quad \left(\frac{d^2y}{dt^2}\right)_{\max} = -\omega^2 a$$

The maximum value of acceleration at which the block separates should be equal to g, i.e.,

$$\omega^2 a = g$$

or $$a = \frac{g}{4^{\ 2}} \cdot T^2 = \frac{g}{4^{\ 2}} = \frac{9.8}{4 \quad (3.14)^2} = 0.248 \text{ m}$$

The maximum velocity of the particle is equal to

$$\omega a = \frac{2}{T} \cdot a = 2 \times 3.14 \times 0.248 = 1.56 \text{ m/s}$$

7.7. A particle starting from the origin executes simple harmonic motion along x axis. Its velocity at any instant t is given by $v_x = 22 \cos \pi t/2$ cm/sec. Calculate the total distance covered by the particle in time $t = 4.5$ sec.

Solution:

$$\frac{2}{T} = \frac{}{2}$$

$$\therefore T = 4 \text{ sec}$$

$$v_x = \frac{dx}{dt} = 22 \cos \frac{}{2} \cdot t$$

$$x = 2 \times \frac{22}{} \cdot \sin \frac{}{2} t$$

or $$x = 14 \sin \frac{}{2} \cdot t \text{ cm}$$

Distance travelled in 4 sec = 4 × 14 = 56 cm

Distance travelled in remaining 0.5 sec = $x = 14 \sin \frac{}{2} \times 0.5$

$$= \frac{14}{\sqrt{2}} = 7\sqrt{2} = 9.898 \text{ cm}$$

$\therefore$ Total distance covered = 56 + 9.9 = 65.9 cm

= 0.659 m.

7.8. A particle participates simultaneously in two simple harmonic oscillations in the same direction; $x_1 = \frac{35}{} \cos \frac{\pi t}{2}$ cm and $x_2 = \frac{35}{} \cos \pi t$ cm. Calculate the maximum velocity of the particle.

Solution:

$$\frac{dx_1}{dt} = -\frac{}{2}\ \frac{35}{\pi} \cdot \sin\frac{\pi}{2}t,\ \frac{dx_2}{dt} = -\pi \times \frac{35}{\pi} \cdot \sin \pi t$$

$$\therefore\ v = -\frac{35}{2}\left[\sin\frac{\pi}{2}t + 4\sin\frac{\pi}{2}t \cdot \cos\frac{\pi}{2}t\right]$$

Put $\frac{t}{2} = \theta$

$$v = -\frac{35}{2}\ [\sin\theta + 4\sin\theta \cdot \cos\theta]$$

$$\frac{dv}{d} = 0 = -\frac{35}{2}\ [\cos\theta + 4\cos^2\theta - 4\sin^2\theta]$$

$$\cos\theta + 4\cos^2\theta - 4\ (1 - \cos^2\theta) = 0$$

$$8\cos^2\theta + \cos\theta - 4 = 0$$

$$\cos\theta = \frac{1\quad \sqrt{1\quad 128}}{16} = 0.6474$$

$$\sin\theta = 0.762$$

$$v_{max} = \frac{35}{2}\ [0.762 + 4 \times 0.762 \times 0.6474]$$

$$= 47.86\ \text{cm/sec}$$

7.9. A spring of force constant K suspended from the ceiling of an elevator contains a ball of mass m at its other end. At an instant $t = 0$ the elevator is at rest and at the origin. It starts going up with an acceleration $a = \alpha t$. Where α is constant. Find the law of motion of the ball relative to the elevator.

Solution: The ball will execute simple harmonic motion with frequency

$$\omega = \sqrt{\frac{K}{m}}$$

Let the acceleration be given by

$$\frac{d^2y}{dt^2} = A\omega^2 \sin\omega t$$

On integration

$$\frac{dy}{dt} = -A\omega\cos\omega t + C$$

at $t = 0, v = 0 \therefore C = A\omega$

$$\therefore \frac{dy}{dt} = A\omega(1 - \cos \omega t)$$

On integration we have

$$y = A[\omega t - \sin \omega t] + C'$$

at $t = 0, y = 0 \therefore C' = 0$

$$\therefore y = A[\omega t - \sin \omega t]$$

Now $a = \alpha t$

$$\frac{da}{dt} = \alpha = A\omega^3 \cos \omega t = A\omega^3$$

$$\therefore A = \frac{}{3}$$

$$\therefore y = \frac{}{3}[\omega t - \sin \omega t]$$

7.10. A particle of mass m located at the origin and having an initial velocity v_0 along y axis moves in the xy plane due to the force $F = a(\dot{y}i - \dot{x}j)$. Find (i) the law of motion of the particle (ii) the trajectory of the particle.

Solution:

(i) $m\ddot{x} = a\dot{y}$ and $m\ddot{y} = -a\dot{x}$

$\ddot{x} = \frac{a}{m}\dot{y}$ $\dot{y}$ and $\ddot{y}$ $\frac{a}{m}\dot{x}$ $\dot{x}$

In view of the boundary condition $x = y = 0$ at $t = 0$ and $\dot{y} = v_0$ at $t = 0$.

Let the solution of the above equation be

$$x = A(1 - \cos \omega t) \text{ and } y = B \sin \omega t$$

$$\dot{y} = B\omega \cos \omega t$$

at $t = 0$, $v_0 = B\omega$

$$\therefore B = \frac{v_0}{}$$

$$\therefore y = \frac{v_0}{} \cdot \sin \omega t$$

Now $\ddot{x} = A\omega^2 \cos \omega t = \omega \cdot v_0 \cos \omega t$

$$\therefore A = \frac{v_o}{}$$

$$\therefore x = \frac{v_o}{}(1 - \cos \omega t)$$

(ii) $x \frac{v_0}{}^2 \quad y^2 = \frac{v_0}{}^2$

This is the equation of circle with centre $\frac{v_0}{}, 0$

7.11. A plank having a body of mass m on it starts going up and its displacement from the initial position is given by the relation. $y = a(1 - \cos \omega t)$ where a is constant. Find (i) the force exerted by the body on the plank at an instant t (ii) the amplitude of oscillation of the plank at which the body starts falling behind the plank (iii) the amplitude of oscillation of the plank at which the body jumps upto a height h with respect to its initial position.

Solution:

(i) $y = a(1 - \cos \omega t)$

$$v = \frac{dy}{dt} = a\omega \sin \omega t$$

$$\frac{d^2y}{dt^2} = a\omega^2 \cos \omega t$$

Force due to motion = $ma\,\omega^2 \cos \omega t$

Force exerted by the body = $mg + ma\,\omega^2 \cos \omega t$

$$= mg\left[1 + \frac{a\omega^2}{g} \cdot \cos \omega t\right]$$

(ii) $g = a_{min} \cdot \omega^2$

$$\therefore a_{min} = \frac{g}{2}$$

(iii) Maximum velocity $v = A\omega$; A = amplitude

$$A^2\omega^2 = v^2 = 2gh$$

$$\therefore A = \sqrt{\frac{2gh}{2}}$$

Maximum amplitude $= A - a_{min} = \sqrt{\frac{2gh}{2}} \quad \frac{g}{2}$

$$= \left(\omega \cdot \sqrt{\frac{2h}{g}} - 1\right)\frac{g}{\omega^2}$$

7.12. A ball of mass m is suspended by a massless spring of force constant K. The ball is set free without any push. Find (i) the displacement of the ball from its equilibrium position as a function of time (ii) the maximum value of the tension produced in the spring.

Solution:

(i) The equation of motion of the ball is given by

$$\frac{md^2x}{dt^2} = -Kx$$

$$\omega = \sqrt{\frac{K}{m}}$$

Let the displacement of ball be represented by the equation

$$y = A(1 - \cos \omega t)$$

$$\frac{dy}{dt} = A\omega \sin \omega t$$

$$\frac{d^2y}{dt^2} = A\omega^2 \cos \omega t$$

at $t = 0$ $\quad \frac{d^2y}{dt^2} = g$

$$\therefore g = A\omega^2$$

or $$A = \frac{g}{\omega^2} = \frac{mg}{K}$$

$$\therefore y = \frac{mg}{K}(1 - \cos \omega t)$$

(ii) $$T - mg = ma$$

$$T = m(g + a)$$

$$T_{max} = 2mg$$

7.13. A particle of mass m having an initial displacement r_0 along x axis and initial velocity v_0 along y axis moves under the action of a force $F = -am(ix + jy)$ where a is a positive constant. Find the path of the particle.

Solution: The acceleration along x axis is given by

$$\frac{d^2x}{dt^2} = -ax$$

Let $x = x_0 \cos \omega t$ be the solution of the equation

$$\frac{d^2x}{dt^2} = -\omega^2 \cdot x = -a\,x$$

$$\therefore \omega = \sqrt{a}$$

$$\therefore x = x_0 \cos \sqrt{a} \cdot t$$

at $t = 0$, $\quad x = r_0 \therefore x_0 = r_0$

$$\therefore x = r_0 \cos \sqrt{a} \cdot t$$

The acceleration along y axis is given by

$$\frac{d^2y}{dt^2} = -ay$$

Let $y = y_0 \sin \omega t$ be the solution of the equation

then $$\frac{dy}{dt} = y_0 \omega \cos \omega t$$

at $t = 0$, $\frac{dy}{dt} = v_0$

$$v_0 = y_0 \omega$$

or $$y = \frac{v_0}{\sqrt{a}} \sin\left(\sqrt{a} \cdot t\right)$$

The path of particle is given by

$$\frac{x}{r_0}^2 \quad a \quad \frac{y}{v_0}^2 = 1$$

7.14. A particle of mass m is executing simple harmonic motion about its equilibrium position with an amplitude a. During oscillation when it is at a distance x from the equilibrium position the particle receives a blow in the direction of motion and as a consequence of that its velocity becomes 1.5 times the velocity at that point. Find the new amplitude of the particle.

Solution: The velocity in simple harmonic motion is given by

$$v = \omega . \sqrt{a^2 - x^2}$$

As a result of receiving the blow the velocity becomes $1.5v$.

or $$1.5v = \omega . \sqrt{A^2 - x^2}$$; A is the new amplitude of motion.

or $$1.5 . \sqrt{a^2 - x^2} = \sqrt{A^2 - x^2}$$

$$2.25\, a^2 - 2.25\, x^2 = A^2 - x^2$$

or $$A = \sqrt{2.25\, a^2 - 1.25\, x^2}$$

7.15. A particle of mass m performs simple harmonic motion with an amplitude a. The angular frequency of oscillation is ω. In the state of oscillation when it is at a distance x from the mean position an impulse I is given to the particle in the same direction. Find the new amplitude of oscillation.

Solution: The velocity of a particle executing simple harmonic motion is given by

$$v = \omega \cdot \sqrt{a^2 - x^2}$$

Now $$I = mv'$$

$$\therefore v' = \frac{I}{m}$$

or $$v + v' = v + \frac{I}{m} = \omega \cdot \sqrt{A^2 - x^2}$$

$$\frac{v}{\omega} + \frac{I}{m\omega} = \sqrt{A^2 - x^2}$$

or $$\left[\sqrt{a^2 - x^2} + \frac{I}{m\omega}\right]^2 = A^2 - x^2$$

or $$A = \sqrt{x^2 + \left[\frac{I}{m\omega} + \sqrt{a^2 - x^2}\right]^2}$$

7.16. A particle of mass m starts moving from a point where $x = x_0$ with an initial velocity u along a straight line under the action of an attraction force $m\omega^2 x$ toward a point O lying on the line where x is the distance from O. Find the displacement.

Solution: Here acceleration and displacement are inter related by the equation

$$\text{Acceleration} = -\omega^2 \cdot x$$

This is the case of simple harmonic motion.

Let the solution be

$$x = A \sin \omega t + B \cos \omega t$$

at $t = 0$, $x = x_0$ and $\dfrac{dx}{dt} = u$ (given)

Now $B = x_0$

$$\frac{dx}{dt} = A\,\omega \cos \omega t - B \cos \sin \omega t$$

$$u = A\,\omega$$

$$\therefore A = \frac{u}{\omega}$$

$$x = x_0 \cos \omega t + \frac{u}{\omega} \sin \omega t$$

7.17. Describe the simple harmonic motion when the displacements are 0, $\dfrac{\sqrt{3}}{2}$ cm and $-\dfrac{3}{2}$ cm at $t_1 = 0$, $t_2 = 2$ sec and $t_3 = 3$ sec.

Solution: Let the equation of the simple harmonic motion be

$$x = a \sin (\omega t + \phi)$$

$$o = a \sin (\omega + \phi) \qquad \therefore \phi = -\omega$$

$$\frac{\sqrt{3}}{2} = a \sin (2\omega + \phi)$$

$$= a \sin \omega$$

Similarly $$-\frac{3}{2} = a \sin (3\omega + \phi)$$

$$= a \sin 2\omega$$

$$-\frac{3}{2} = 2 \cos \omega \cdot a \sin \omega$$

$$= 2 \cos \omega \cdot \frac{\sqrt{3}}{2}$$

$$\therefore \cos \omega = -\frac{\sqrt{3}}{2}$$

or $$\omega = \frac{5\pi}{6}$$

Now $$\frac{\sqrt{3}}{2} = a \sin \frac{5\pi}{6} \quad \therefore a = \sqrt{3} \text{ cm}$$

$$\therefore x = \sqrt{3} \sin \left(\frac{5\pi}{6} \cdot t - \omega \right)$$

$$= \sqrt{3} \sin \frac{5\pi}{6} (t - 1) \text{ cm}$$

7.18. A particle moves simple harmonically along a straight line. It starts from equilibrium without any initial velocity and travells a distance l_1 in the first second and l_2 in the second second in the same direction. Find the amplitude of oscillation.

Solution: Let the equation of motion of the particle be

$$x = a(1 - \cos \omega t); \; a = \text{amplitude}$$

In view of the conditions given

$$l_1 = a(1 - \cos \omega)$$

$$l_1 + l_2 = a(1 - \cos 2\omega) = 2a \sin^2 \omega$$

$$\left(1 - \frac{l_1}{a}\right)^2 + \frac{l_1 + l_2}{2a} = 1$$

$$1 + \frac{l_1^2}{a^2} - \frac{2l_1}{a} + \frac{l_1 + l_2}{2a} = 1$$

$$\frac{l_1^2}{a} - \frac{3l_1 + l_2}{2} = 0$$

or $$a = \frac{2l_1^2}{3l_1 - l_2}$$

7.19. A particle oscillates simple harmonically along x axis. At times t, $2t$ and $3t$ it is located at a distance a, b and c respectively from the equilibrium position. Find the period of oscillation.

Solution: In the light of equation $x = A \sin \omega t$ we have

$$a = A \sin \omega t$$

$$b = A \sin 2\omega t$$

$$c = A \sin 3\omega t$$

Now $$a + c = A[\sin \omega t + \sin 3\omega t]$$

$$= 2A \sin 2\omega t \cos \omega t$$

$$= 2b \cos \omega t$$

$$\therefore \omega t = \cos^{-1} \left(\frac{a + c}{2b} \right)$$

$$\frac{2\pi t}{T} = \cos^{-1} \left(\frac{a + c}{2b} \right)$$

or $$T = \frac{2\pi t}{\cos^{-1}\left(\frac{a+c}{2b}\right)}$$

7.20. A particle oscillating simple harmonically has velocities v_1, v_2 and v_3 respectively when it is at a distance of x_1, x_2 and x_3 from the equilibrium position. Find its period of oscillation.

Solution:

$$v_1^2 = \omega^2\left(a^2 - x_1^2\right),\ v_2^2 = \omega^2\left(a^2 - x_2^2\right) \text{ and } v_3^2 = \omega^2\left(a^2 - x_3^2\right)$$

$$\begin{vmatrix} v_1^2 & v_2^2 & v_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = \omega^2\left(a^2 - x_1^2\right)(x_2 - x_3) + \omega^2\left(a^2 - x_2^2\right)(x_3 - x_1)$$

$$+ \omega^2\left(a^2 - x_3^2\right)(x_1 - x_2)$$

$$= \omega^2\left[a^2(x_2 - x_3) + a^2(x_3 - x_1) + a^2(x_1 - x_2) - x_1^2 x_2\right.$$

$$\left. + x_1^2 x_3 - x_2^2 x_3 + x_2^2 x_1 - x_3^2 x_1 + x_3^2 x_2\right]$$

$$= \omega^2\left[(x_1 - x_2)(x_3 - x_1)(x_2 - x_3)\right]$$

$$T = 2\pi \cdot \sqrt{\frac{(x_1 - x_2)(x_3 - x_1)(x_2 - x_3)}{\begin{vmatrix} v_1^2 & v_2^2 & v_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}}}$$

7.21. A particle of mass m is executing simple harmonic motion along x axis under the action of a force $F = -kx$ with a period of 16 sec. In the course of motion it crosses the equilibrium position at $t = 2$ sec and acquire a velocity of 4 m/sec at $t = 4$ sec. Find (1) the equation of motion (ii) the amplitude of oscillation.

Solution: $m \cdot$ acceleration $= -kx$

$$\therefore \omega = \sqrt{\frac{k}{m}}$$

Let the equation of motion be $x = a \sin \omega(t-2)$ where a = amplitude of motion.

Now $$\frac{dx}{dt} = \omega a \cos \omega (t-2)$$

$$4 = a\,\omega \cos 2\omega$$

$$\therefore a = \frac{4}{\omega \cos 2\omega}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{16} = \frac{\pi}{8}$$

$$\therefore a = \frac{4}{\frac{\pi}{8} \cdot \frac{1}{\sqrt{2}}}$$

$$= \frac{32\sqrt{2}}{\pi}$$

$$x = \frac{32\sqrt{2}}{\pi} \sin \frac{\pi}{8} (t-2)$$

7.22. A particle of mass m is located in a unidimensional potential where the potential energy of the particle depends on the coordinate x as $U(x) = \frac{a}{x^2} - \frac{b}{x}$. Find the period of small ocillations that the particle executes about the equilibrium position.

Solution: The particle executes motion about the position of minimum energy.

$$\frac{dU}{dx} = -\frac{2a}{x^3} + \frac{b}{x^2} = 0 \quad \text{or } x = \frac{2a}{b}$$

Now $$U = \frac{a}{\left(\frac{2a}{b} + p\right)^2} - \frac{b}{\left(\frac{2a}{b} + p\right)}; p = \text{displacement}$$

$$F = m \times \text{acceleration} = -\frac{dU}{dp}$$

$$= \frac{2a}{\frac{8a^3}{b^3}} \left(1 - \frac{3b}{2a}p\right) - \frac{b}{\frac{4a^2}{b^2}} \left(1 - \frac{2b}{2a}p\right)$$

$$m \times \text{acceleration} = -\frac{b^4}{8a^3} \cdot p$$

$$\therefore \text{acceleration} = -\frac{b^4}{8ma^3} \cdot \text{displacement}$$

$$\omega = \frac{2\pi}{T} = \frac{b^2}{2a \cdot \sqrt{2ma}}$$

or $$T = \frac{4\pi a}{b^2}\sqrt{2ma}$$

7.23. Two particles executes simple harmonic motion of same amplitude and frequency along a straight line. They pass one another, when going in opposite direction each time their displacement is half of their amplitude. Find the phase difference between them.

Solution: The equation of motion of the two particles is given by

$$y_1 = a \sin \omega t \text{ and } y_2 = a \sin (\omega t + \phi)$$

Now $\frac{a}{2} = a \sin \omega t$

$\therefore \omega t = 30°$

$$\frac{1}{2} = \sin \omega t \cos \phi + \cos \omega t \sin \phi$$

$$\frac{1}{2} = \frac{1}{2} \cos \phi + \frac{\sqrt{3}}{2} \sin \phi$$

$$1 = \cos \phi + \sqrt{3} \sqrt{\left(1 - \cos^2 \phi\right)}$$

$$(1 - \cos \phi)^2 = 3 (1 - \cos^2 \phi)$$

$$1 - 2 \cos \phi + \cos^2 \phi = 3 - 3 \cos^2 \phi$$

$$4 \cos^2 \phi - 2 \cos \phi - 2 = 0$$

$$4 \cos^2 \phi - 4 \cos \phi + 2 \cos \phi - 2 = 0$$

$$4 \cos \phi (\cos \phi - 1) + 2 (\cos \phi - 1) = 0$$

$$(4 \cos \phi + 2) (\cos \phi - 1) = 0$$

$$\cos \phi = 1 \text{ or } \phi = 0$$

$$\cos \phi = -\frac{1}{2} \text{ or } \phi = 120°$$

7.24. Find the average value of the potential and kinetic energy of a particle executing simple harmonic motion. When (i) the averaging is done with respect to position over one period (ii) the averaging is done over one period.

Solution:

(i) Let the force acting on the particle be $F = -Kx$

Now $m \times$ acceleration $= -K \cdot$ displacement

$$\therefore \omega = \sqrt{\frac{K}{m}}$$

Let the equation of motion of the particle be

$x = A \cos \omega t$; A = amplitude of oscillation

U = Potential energy $= \frac{1}{2} Kx^2$

Average value of potential energy $= \langle U_{av} \rangle = \frac{1}{2A}\int_{-A}^{A} \frac{1}{2} Kx^2 dx$

$= \frac{KA^2}{6}$

Total energy of the particle $= \frac{1}{2} KA^2$

$\therefore$ Average value of kinetic energy $= \langle K \cdot E_{av} \rangle$

$= \frac{1}{2A}\int_{-A}^{A} \frac{1}{2} K\left(A^2 - x^2\right) dx = \frac{KA^2}{3}$

(ii) $\langle U_{av} \rangle = \frac{1}{T}\int_0^T \frac{1}{2} KA^2 \cos^2 \omega t \cdot dt$

$= \frac{1}{2}\frac{KA^2}{2T}\int_0^T (1 + \cos 2\omega t)\, dt$

$= \frac{KA^2}{4}$

$\langle K \cdot E_{av} \rangle = \frac{1}{T}\int_0^T \frac{1}{2} KA^2 \sin^2 \omega t \cdot dt$

$= \frac{KA^2}{4T}\int_0^T (1 - \cos 2\omega t)\, dt = \frac{KA^2}{4}$

7.25. A weightless wooden stick whose one end is fixed at O is kept horizontal (Fig. 7.2). A $m = 0.5$ kg load is applied at the other end produces a depression of $\delta = 1$ cm. The stick is further depressed a little and released. Find the period of oscillation.

Solution: Let c be the twisting couple per unit depression then in equilibrium, we have

$c \cdot \delta = mgl$; l = length of the stick

Let the stick be further depressed by y then torque is given by

$mgl - c(\delta + y) = -cy$

The equation of motion is given by

$$-c \cdot y = m \cdot l^2 \cdot \frac{d^2}{dt^2}\left(\frac{y}{l}\right)$$

or $$ml \cdot \frac{d^2 y}{dt^2} = -cy$$

$$\frac{d^2 y}{dt^2} = -\frac{g}{\delta} \cdot y$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{g}{\delta}}$$

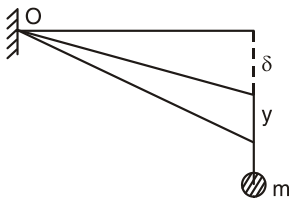


Fig. 7.2

$$\therefore T = 2\pi.\sqrt{\frac{\delta}{g}}$$

On substituting the values of g and δ we have

$$T = 2\times 3.14\cdot\sqrt{\frac{10^{-2}}{9.8}}$$

$$= \frac{2\times 3.14}{31.30}$$

$$= 0.2 \text{ sec}$$

7.26. A point particle of mass 0.1 kg is executing simple harmonic motion of amplitude of 0.1 m. When the particle passes through the mean position, its kinetic energy is 8×10^{-3} Joule. Obtain the equation of motion of this particle if the initial phase of oscillation is 45°. (Roorkee 1991)

Solution: Let the equation of simple harmonic motion be

$y = a \sin(\omega t + \theta)$

Velocity $\frac{dy}{dt} = a\,\omega \cos(\omega t + \theta)$

When the particle passes through the mean position its velocity is maximum and is equal to ωa.

Now $\frac{1}{2}\, m\cdot\omega^2 a^2 = 8\times 10^{-3}$ J

$\frac{1}{2} \times 0.1 \times \omega^2 \times 0.1^2 = 8\times 10^{-3}$

$\omega = \pm 4$

$\therefore$ Required equation is $y = 0.1 \sin(\pm 4t + \pi/4)$ meter

7.27. A cylindrical piece of wood of mass m and area of cross-section A is floating erect in a liquid density ρ (see Fig. 7.3). It is pressed down slightly and then released. Show that its motion is simple harmonic, and calculate the time period of this motion.

Solution: Let the cylinder be pressed vertically through a depth y. The volume of liquid displaced by it is given by

$V = Ay$

Upward thrust = weight of liquid displaced $= A\rho g y$

This force is acting vertically upward, i.e., opposite to the displacement y.

The equation of motion of the cylinder is given by

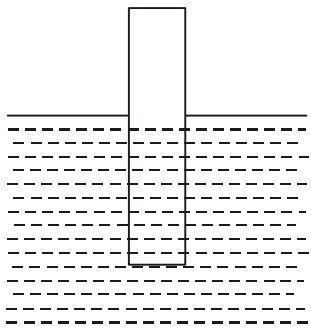


Fig. 7.3

$$m\frac{d^2y}{dt^2} = -A\rho g y$$

or $$\frac{d^2y}{dt^2} = -\frac{A\rho g}{m}\cdot y$$

This acceleration $\frac{d^2y}{dt^2}$ is proportional to the displacement y. The motion of cylinder is therefore simple harmonic.

Now $$\omega = \sqrt{\frac{A\rho g}{m}}$$

$$\therefore T = 2\pi\sqrt{\frac{m}{A\rho g}}$$

7.28. The graduated cylindrical stem of a lactometer has an area of cross-section $A = 2 \times 10^{-3}$ m^2. Its mass m including the mass of the lead shots confined in a spherical space at its bottom is 100 g. It is immersed in water of density $\rho = 10^3$ kg/m^3 to at a slightly greater depth than actually needed for equilibrium, and then released. Show that it will execute simple harmonic motion. Calculate its time period.

Solution: Let the lactometer be pressed to a depth greater than that required for equilibrium in the vertical direction. Volume of water displaced = Ay.

Upward thrust = $A\rho g y$

This force is acting vertically upward, i.e., opposite to the displacement y.

The equation of motion of the lactometer is given by

$$m\frac{d^2y}{dt^2} = -A\rho g y$$

or $$\frac{d^2y}{dt^2} = -\frac{A\rho g}{m}\cdot y$$

The acceleration is proportional to the displacement y. The motion is therefore simple harmonic.

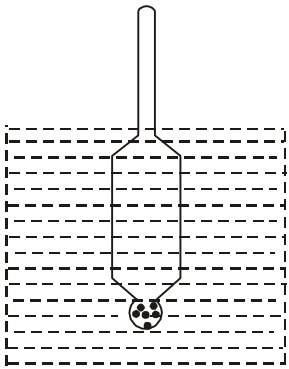


Fig. 7.4

Now $$\omega = \frac{2\pi}{T} = \sqrt{\frac{A\rho g}{m}}$$

$$\therefore T = 2\pi\sqrt{\frac{m}{A\rho g}}$$

On substituting the values of the various terms occurring in the above equation, we have

$$T = 2\pi\sqrt{\frac{0.1}{2\times10^{-3}\times10^{3}\times9.8}}$$

$$= \frac{2\pi}{14} = \frac{\pi}{7}\text{ s}$$

$$= 0.45\text{ s}$$

7.29. A string of length $l = 1$ m containing a small particle of mass $m = 4$ g at its middle point O is fixed vertically between points A and B (Fig. 7.5). The string is stretched by a tension $T = 9.86 \times 10^{-3}$ N. The particle at O is slightly pulled to one side at right angles to the string and let go. Show that the particle will execute simple harmonic motion. Find its time period.

Solution: The particle of mass m was initially at O. It is displaced up to C by a distance x. The force acting on the particle in the horizontal direction is given by $F = 2T\cos\theta = 2T\cdot\dfrac{2x}{l}$

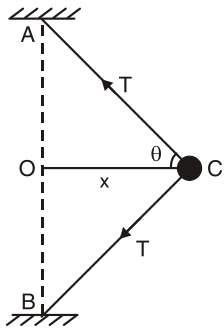


Fig. 7.5

This force is acting along CO and will bring the particle to its initial position O. Also, it acts opposite to the displacement x. The equation of motion of the particle is given by

$$\frac{md^2x}{dt^2} = -\frac{4T}{l}x$$

or $$\frac{d^2x}{dt^2} = -\frac{4T}{ml}x$$

The acceleration $\dfrac{d^2x}{dt^2}$ is proportional to the displacement x. The motion of the particle is therefore simple harmonic.

Now $$\omega = \frac{2\pi}{t} = \sqrt{\frac{4T}{ml}}$$

$$\therefore t = \pi\sqrt{\frac{ml}{T}}$$

On substituting the values of the terms occurring in the above equation we have

$$t = 3.14\sqrt{\frac{4\times10^{-3}\times1}{9.86\times10^{-3}}}$$

$$= \frac{2\times3.14}{3.14} = 2\text{ s}$$

7.30. Two non-viscous, incompressible and immiscible liquids of densities ρ and 1.5 ρ are poured into two limbs of a circular tube of radius R and small cross-section kept fixed in a vertical plane as shown in the Fig. 7.6. Each liquid occupies one fourth the circumference of the tube. (a) Find the angle θ that the radius vector to the interface make with the vertical in equilibrium position. (b) If the whole liquid is given a small displacement from the equilibrium position, show that the resulting oscillations are simple harmonic. Find the time period of these oscillations.

(IIT 1991)

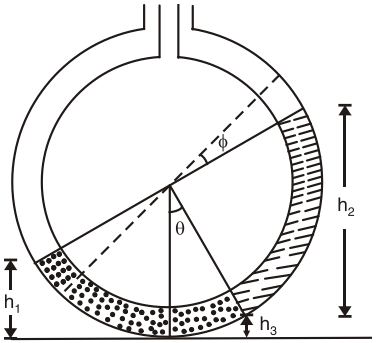


Fig. 7.6

Solution:

(a) In equilibrium we have

$$\rho_1 h_1 g = \rho_2 h_2 g + \rho_1 h_3 g$$

$$1.5\,\rho R(1-\sin\theta) = \rho R(\sin\theta+\cos\theta) + 1.5\,\rho R(1-\cos\theta)$$

or $$2.5\sin\theta = 0.5\cos\theta$$

$$\theta = \tan^{-1}\left(\frac{1}{5}\right) = 11.3°$$

(b) Displacement of the liquid is given by

$y = R\phi$

Change in height of liquid column = $h = R[\sin(\theta+\phi) - \sin\theta]$

$h = R\{[\sin\theta\cos\phi - \cos\theta\sin\phi] - \sin\theta]$

$= R\phi\cos\theta$; here $\cos\phi = 1$ and $\sin\phi = \phi$

Restoring force $= -2.5\,\rho A R g\cos\theta \cdot \dfrac{y}{R}$; A = area of cross-section

$$= -2.5\,\rho A g\cos\theta \cdot y$$

$$2.5\,\rho A g\cos\theta = k \text{ (force constant)}$$

$$F = m \times \text{acceleration} = -ky$$

$$\omega = \sqrt{\frac{K}{m}} = \frac{2\pi}{T}$$

$$\therefore T = 2\pi\sqrt{\frac{m}{2.5\,\rho A g\cos\theta}}$$

$$m = A\cdot\frac{2\pi R}{4}\,[1.5\,\rho + \rho] = 1.25\,\pi R\rho$$

$$T = 2\pi\sqrt{\frac{\pi R}{2g\cos\theta}}$$

$$\cos 11.3^\circ = 0.98$$

$$T = 2\pi\sqrt{\frac{\pi R}{2\times 9.8\times 0.98}} = 2.54\sqrt{R}$$

7.31. One end of a long metallic wire of length L is tied to the ceiling. The other end is tied to a massless spring of spring constant K. A mass m hangs freely from the free end of the spring. The area of cross-section and the Young's modulus of the wire are A and Y respectively. Find the time period with which mass m will oscillate if it is slightly pulled down and released. (IIT 1993)

Solution: $F = K_C y$

Where K_C is the total force constant and y is total elongation

$$y = y_1 + y_2$$

y_1 = elongation of the spring

y_2 = elongation of the wire

$$y_1 = \frac{F}{K}$$

$$Y = \frac{FL}{Ay_2}$$

$$\therefore y_2 = \frac{FL}{AY}$$

$$y = \frac{F}{K} + \frac{FL}{AY}$$

$$= \left(\frac{AY + KL}{AKY}\right)F$$

$$F = \left(\frac{AKY}{AY + KL}\right)y$$

$$m \times \text{accelerationn} = -\left(\frac{AKY}{AY + KL}\right)y$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{AKY}{m(AY + KL)}}$$

$$T = 2\pi\sqrt{\frac{m(AY + KL)}{AKY}}$$

7.32. Two identical balls A and B each of mass 0.1 kg are attached to two identical massless springs. The spring-mass system is constrained to move inside a rigid smooth pipe bent in the form of a circle as shown in Fig. 7.7. The pipe is fixed in a horizontal plane. The centres of the balls can move in a circle of radius 0.06 m. Each spring has a natural length of 0.06 πm and force constant 0.1 N/m. Initially both the balls are displaced by an

angle $\theta = \pi/6$ radiation with respect to diameter PQ of the circle and released from rest. (a) Calculate the frequency of oscillation of the Ball B. (b) What is the total energy of the system? (c) Find the speed of ball A when A and B are at the two ends of the diameter PQ. (IIT 1993)

Solution:

(a) The reduced mass m of the system is given by

$$m = \frac{0.1 \times 0.1}{0.1 + 0.1} = 0.05 \text{ kg}$$

Combined force constant

$$K = K_1 + K_2 = 0.1 + 0.1 = 0.2$$

Frequency of oscillation

$$n = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{0.02}{0.05}} = \frac{1}{\pi}\text{ Hz}$$

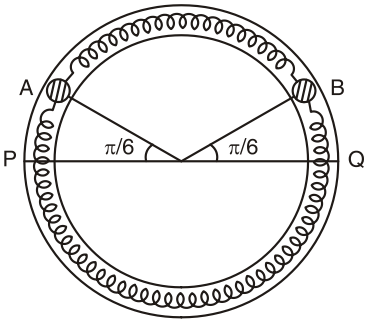


Fig. 7.7

(b) $E = \frac{1}{2} K_1 y^2 + \frac{1}{2} K_2 y^2$

where $y = 2R\theta = 2R\pi/6 = 2 \times 0.06 \times \pi/6 = 0.02\ \pi$m

Energy $E = \frac{1}{2} [0.1 + 0.1] \times (0.02\ \pi)^2$

$$= 0.1 \times (0.02\ \pi)^2$$

$$= 3.94 \times 10^{-4} \text{ J}$$

(c) $\frac{1}{2} mv^2 + \frac{1}{2} mv^2 = 3.94 \times 10^{-4}$

$$mv^2 = 3.94 \times 10^{-4}$$

$$v = \sqrt{\frac{3.94}{0.1}} \times 10^{-2}$$

$$= 6.277 \times 10^{-2} \text{ m/s}$$

7.33. A solid cylinder of mass M is attached to a horizontal massless spring of force constant K so as to roll without slipping along a horizontal surface as shown in the Fig. 7.8. The spring is stretched by x and released show that the centre of mass of the cylinder executes simple harmonic motion of period $T = 2\pi\sqrt{\frac{3M}{2K}}$.

Solution:

Total energy $= \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 + \frac{1}{2} Kx^2$

$$= \text{Constant}$$

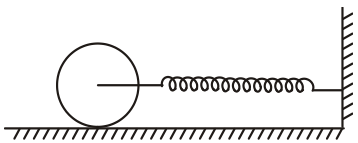


Fig. 7.8

$\frac{3}{4} Mv^2 + \frac{1}{2} Kx^2 = \text{constant}; \frac{1}{2} I\omega^2 = \frac{1}{2} \cdot \frac{1}{2} Mr^2\omega^2 = \frac{1}{4} Mv^2;$

r = radius of the cylinder

On differentiation

$$\frac{3}{2} Mv \cdot \frac{dv}{dt} = -kx \cdot v$$

$$\text{acceleration} = \frac{dv}{dt} = -\frac{2K}{3M} \cdot \text{displacement}$$

Hence the centre of mass will execute S.H.M.

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{2K}{3M}}$$

$$\therefore T = 2\pi \sqrt{\frac{3M}{2K}}$$

7.34. An ideal gas is enclosed in a vertical cylindrical container and supports a freely moving piston of mass M. The piston and the cylinder have the same area of cross-section A. Atmospheric pressure is P and when the piston is in equilibrium, the volume of the gas is V. The piston is now displaced slightly from the equilibrium position and left. Assuming that the system is completely isolated from its surroundings, show that the piston executes simple harmonic motion and find the frequency of oscillation. (IIT 1981)

Solution: P and V are, respectively, the pressure and volume of the gas enclosed in the cylinder-piston system under conditions of equilibrium (Fig. 7.9). Let the piston be pressed through a depth y. This will cause an increase of pressure and decrease in volume. Let the new pressure and volume $P + dP$ and $V - dV$ respectively. The process is adiabatic since the system is completely isolated.

In an adiabatic process,

$PV^{\gamma} = \text{constant}$

On differentiating the above, we have

$$dP \cdot V^{\gamma} + \gamma P \cdot V^{\gamma - 1} \cdot dV = 0$$

$$\text{or} \quad dP = -\frac{\gamma P}{V} dV = -\frac{\gamma P}{V} \cdot A \cdot y$$

Restoring force $= F = dP \cdot A$

$$\therefore F = -\frac{\gamma P}{V} A^2 y$$

$$\text{or} \quad \frac{d^2y}{dt^2} = -\frac{\gamma P}{MV} A^2 y$$

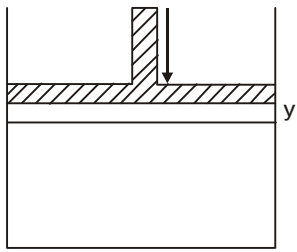


Fig. 7.9

The acceleration $\frac{d^2y}{dt^2}$ is proportional to displacement y. Therefore the motion is simple harmonic.

$$\omega = 2\pi n = \sqrt{\frac{\gamma PA^2}{MV}}$$

$$\therefore n = \frac{1}{2\pi}\sqrt{\frac{\gamma PA^2}{MV}}$$

7.35. Fig. 7.10 shows a large glass bottle B fitted air-tight with a glass tube at the top. The glass tube has a uniform bore in which a steel ball of mass m fits perfectly. If the ball is dropped into the tube it begins to oscillate up and down. Find the period of this oscillation.

Solution: Let A be the area of cross-section of the glass tube and v be the volume of the bottle. Under the conditions of equilibrium the pressure inside the bottle when a ball of mass m is dropped is given by

$$P = P_0 + \frac{mg}{A}$$

where P_0 is the atmospheric pressure. If now the ball moves downwards through a vertical distance y the pressure inside the bottle increases to $P + dP$. The equation of motion of the ball is given by

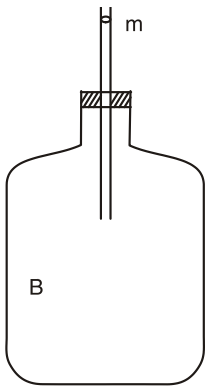


Fig. 7.10

$$\frac{md^2y}{dt^2} = -AdP$$

The system is completely isolated from the surroundings. Therefore, the process is adiabatic, in which the pressure P and volume v are related by

Pv^{γ} = constant

On differentiating this equation, we have

$$dP \cdot v^{\gamma} + \gamma P \cdot v^{\gamma-1} \cdot dv = 0$$

or $$dP = -\gamma P \frac{dv}{v}$$

On substituting the value of dP in the equation of motion, we have

$$\frac{md^2y}{dt^2} = -A\gamma P \frac{dv}{v}$$

or $$\frac{d^2y}{dt^2} = -\frac{\gamma P}{mv} A^2 \cdot y;\ dv = A \cdot y$$

The acceleration $\frac{d^2y}{dt^2}$ is proportional to the displacement y; hence the motion is simple harmonic.

Now $\omega = \sqrt{\frac{\gamma PA^2}{mv}}$

$\therefore T = 2\pi \sqrt{\frac{mv}{\gamma PA^2}}$

7.36. An enclosure of length $2l$ containing a gas has a piston of mass m and area of cross-section A at its middle. The piston divides the enclosure into two compartments having equal pressure P and volume V. Keeping the temperature of the gas constant the piston is slightly displaced to the right and released (Fig. 7.11). Show that it will execute simple harmonic motion and calculate the frequency of its oscillation.

Solution: The volume of the gas in each compartment is $V = Al$ and the pressure of the gas in each compartment = P. On displacing the piston to the right through a distance x, let the pressures in the first and second compartments be P_1 and P_2 respectively.

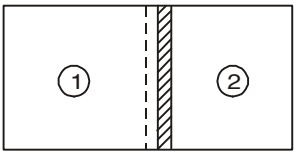


Fig. 7.11

According to Boyle's law,

$P_1V_1 = P_2V_2 = PV$

or $P_1 (l + x) A = P_2 (l - x) A = PlA$

Force F acting on the piston $= (P_2 - P_1) A$

$$= \frac{PlA}{(l-x)} - \frac{PlA}{(l+x)} = \frac{2PlA \cdot x}{l^2 - x^2}$$

$$= \frac{2PVx}{l^2 - x^2}$$

This force will bring the piston to its original position. If we assume that x is very small in comparison l, then we have

$$F = \frac{2PVx}{l^2}$$

Under this condition, the equation of motion of the piston is given by

$$\frac{md^2x}{dt^2} = -\frac{2PV}{l^2}x$$

or $$\frac{d^2x}{dt^2} = -\frac{2PV}{ml^2}x$$

The acceleration $\frac{d^2x}{dt^2}$ is proportional to the displacement x since $\frac{PV}{ml^2}$ is constant. Therefore the piston executes simple harmonic motion.

Now $\omega = 2\pi n = \sqrt{\frac{2PV}{ml^2}}$

$\therefore n = \frac{1}{2\pi} \sqrt{\frac{2PV}{ml^2}}$

7.37. A cylinder of length $2l$ enclosed at both ends contains a gas and a piston of mass m and area of cross-section A at its middle (Fig. 7.12). The piston divides the cylinder into two compartments having equal pressure P and volume V. The piston is slightly displaced to the left suddenly and released. Show that it will execute simple harmonic oscillation. Find the frequency of oscillation assuming the processes in the gas to be an adiabatic.

Solution: Volume of the gas $V = A \cdot l$

Pressure of the gas $= P$

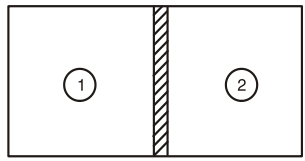


Fig. 7.12

On displacing the piston to the left horizontally through a distance x, let P_1 and P_2 be the pressures in the first and second compartment. As the process is an adiabatic, we have

$$P_1 [(l-x)\,A]^{\gamma} = P_2 [(l+x)\,A]^{\gamma} = P\,(lA)^{\gamma}$$

or $$P_1 (l-x)^{\gamma} = P_2 (l+x)^{\gamma} = Pl^{\gamma}$$

Force F acting on the piston $= (P_1 - P_2)\,A = PA\left[\left(\frac{l}{l-x}\right)^{\gamma} - \left(\frac{l}{l+x}\right)^{\gamma}\right]$

$$= PA\left[(1-x/l)^{-\gamma} - (1+x/l)^{-\gamma}\right] \cong \frac{2P\gamma Ax}{l}$$

This force will bring the piston to its equilibrium position. The equation of motion of the piston is given by

$$\frac{md^2x}{dt^2} = -2P\,\frac{\gamma Ax}{l}$$

or $$\frac{d^2x}{dt^2} = -\frac{2P\gamma A}{ml}\cdot x$$

$$= \frac{-2P\gamma A^2}{mV}\cdot x$$

The acceleration $\frac{d^2x}{dt^2}$ is proportional to the displacement x since $\frac{2P\gamma A^2}{mV}$ is constant. Therefore the piston will execute simple harmonic motion.

Now $$\omega = \sqrt{\frac{2P\gamma A^2}{mV}}$$

$$\therefore\ n = \frac{A}{2\pi}\sqrt{\frac{2P\gamma}{mV}}$$

7.38. A point of mass M is suspended by a weightless wire of length L and area of cross-section A. If Y is the Young's modulus of the material of the wire, then determine the frequency of simple harmonic motion of the mass in the vertical direction.

Solution: The suspended mass will elongate the wire by a length l.

According to Hooke's law,

$$Y = \frac{Mg}{Al} L$$

or $$Mg = \frac{YAl}{L}$$

The weight Mg will be equal to the tension in the wire. Therefore

$$T = Mg = \frac{YAl}{L}$$

On pulling the mass M vertically through a distance y, the tension will increase. The tension at that instant is given by

$$T_1 = \frac{AY}{L}(l + y)$$

Net force acting on the mass $= T - T_1 = -\frac{AY}{L} y$

The equation of motion of the mass is given by

$$M \frac{d^2y}{dt^2} = -\frac{AY}{L} \cdot y$$

or $$\frac{d^2y}{dt^2} = -\frac{AY}{ML} \cdot y$$

The acceleration $\frac{d^2y}{dt^2}$ is proportional to displacement y. Therefore the mass will execute simple harmonic motion.

Now $$\omega = \sqrt{\frac{YA}{ML}}$$

or $$n = \frac{1}{2\pi} \sqrt{\frac{YA}{ML}}$$

7.39. Show that a particle dropped into a tunnel dug along the diameter of the earth (see Fig. 7.13) will execute simple harmonic motion. Find its time period.

Solution: Let a particle of mass m be dropped through the tunnel and let it be at a distance y from the centre of the earth. For the purpose of gravitation only the mass of the unshaded portion of the earth will be

effective. The mas of this portion is $M' = \frac{4}{3}\pi y^3 \cdot \rho$ where ρ is the mean density of the earth.

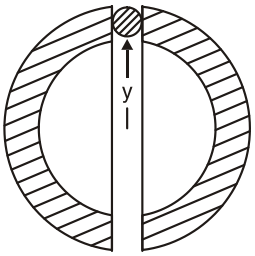


Fig. 7.13

The force of attraction between m and M' is given by

$F = -\frac{GM'm}{y^2}$ (the force is directed towards the centre of the earth)

or $F = -G \cdot \frac{4}{3}\pi \rho\, m\, y$

The equation of motion of the particle is given by

$$m\frac{d^2y}{dt^2} = -G \cdot \frac{4}{3}\pi\rho\, m \cdot y$$

or $$\frac{d^2y}{dt^2} = -G\frac{4}{3}\pi\rho y$$

The acceleration $\frac{d^2y}{dt^2}$ is proportional to the displacement y since $(4/3)\pi\rho\, G$ is constant.

Therefore the particle will execute simple harmonic motion.

Now $$\omega = \frac{2\pi}{T} = 2\sqrt{\frac{G\pi\rho}{3}}$$

or $$T = \sqrt{\frac{3\pi}{\rho G}}$$

On substituting the mean density of earth $\rho = 5.51 \times 10^3$ kg/m^3 and $G = 6.67 \times 10^{-11}$ Nm^2/kg^2 in the above, $T = 84.2$ min.

7.40. Show that a particle dropped into a tunnel AB dug through the earth (Fig. 7.14) will execute simple harmonic motion. Find its time period.

Solution: Let m be the mass of particle and O be the centre of the earth. Force acting on the particle towards the centre of the earth in mg. Due to the force mg cos θ the particle will move towards B. In the course of motion its velocity will go on increasing and it will have the highest value at C. As it moves from C towards B its velocity will continuously decrease and at B its velocity will be zero. At this stage the component force mg cos θ will act in the direction BA and the particle will start moving up.

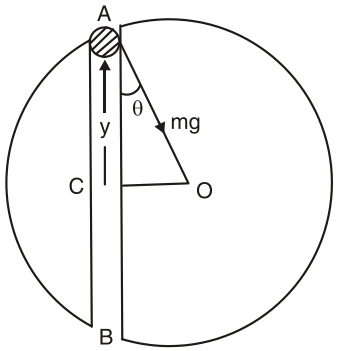


Fig. 7.14

Thus restoring force $= -mg\cos\theta = -\dfrac{mgy}{R_e}$; R_e = radius of the earth

The equation of motion of the particle is $\dfrac{md^2y}{dt^2} = -\dfrac{mg}{R_e}\cdot y$

or $\dfrac{d^2y}{dt^2} = -\dfrac{g}{R_e}\cdot y$

The acceleration $\dfrac{d^2y}{dt^2}$ is proportional to the displacement. Therefore the particle will execute simple harmonic motion.

Now $\omega = \dfrac{2\pi}{T} = \sqrt{\dfrac{g}{R_e}}$

or $T = 2\pi = \sqrt{\dfrac{R_e}{g}}$

On substituting $R_e = 64 \times 10^5$ m; $T = 84.4$ min

7.41. A small rectangular board is placed horizontally on two cylinders spinning about their own axes in the opposite sense (Fig. 7.15a). The axes are separated by a distance $l = 30$ cm. The coefficient of friction between the board and cylinder is $k = 0.15$. Show that the board will perform simple harmonic motion. Find its time period.

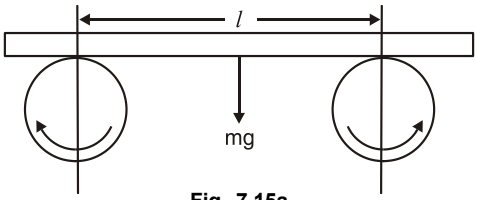


Fig. 7.15a

Solution: Let P_1 and P_2 be the pressures of the board on the respective cylinders (Fig. 7.15b). The friction forces acting on the board are given by

$F_1 = kP_1$ and $F_2 = k \cdot P_2$

On displacing the board to the left through a small distance x, the two pressures P_1 and P_2 will be represented as

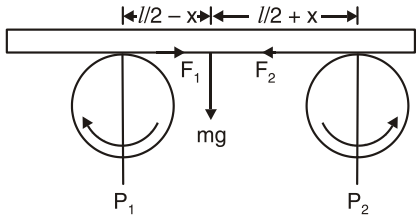


Fig. 7.15b

$$P_1 = \frac{(l/2 + x)mg}{l} \text{ and } P_2 = \frac{(l/2 - x)mg}{l}$$

where mg = weight of the board.

The resultant force acting on the board is given by

$$F = F_1 - F_2 = k\,(P_1 - P_2) = \frac{2k \cdot mgx}{l}$$

It is evident from the Fig. 7.15b that the resultant force will be opposite to the displacement x.

The equation of motion of the board is given by

$$\frac{md^2x}{dt^2} = -\frac{2k \cdot g}{l} \cdot x \cdot m$$

or $$\frac{d^2x}{dt^2} = -\frac{2kg}{l} \cdot x$$

The acceleration $\frac{d^2x}{dt^2}$ is proportional to the displacement x. Therefore the board will execute simple harmonic motion.

Now $$\omega = \frac{2\pi}{T} = \sqrt{\frac{2kg}{l}}$$

or $$T = 2\pi \sqrt{\frac{l}{2kg}}$$

$$= \pi \sqrt{\frac{2l}{kg}}$$

On substituting the values of l, k and g, we get

$$T = 3.14 \sqrt{\frac{2 \times 0.3}{0.15 \times 9.8}}$$

$$= 2.006 \text{ s}$$

$$= 2 \text{ s}$$

7.42. The right arm B of a uniform-bore U-tube of cross-sectional area $A = 0.6$ cm^2 makes an angle $\theta = 60°$ with the vertical. Mercury of mass $m = 306$ g is introduced in it. If the mercury in one limb is depressed and then released, show that it will perform simple harmonic oscillations. Find its time period.

Solution: On pressing the mercury in column A through a depth y the mercury in column B rises vertically through a distance $y \cos\theta$. In column B there is an additional mercury column of length $y\,(1 + \cos\theta)$. When the pressed mercury level is set free the entire mercury column starts oscillating.

Restoring force $= -Ay\,(1 + \cos\theta)\,\rho\,g$

where A = area of cross-section of the tube

ρ = density of mercury

The equation of motion is

$$\frac{md^2y}{dt^2} = -A\rho g(1+\cos\theta)\,y$$

or $$\frac{d^2y}{dt^2} = -\frac{A\rho g}{m}(1+\cos\theta)\,y$$

The acceleration $\frac{d^2y}{dt^2}$ is therefore proportional to the displacement. The motion is therefore simple harmonic.

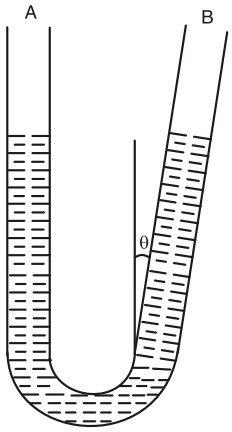


Fig. 7.16

Now $$\omega = \sqrt{\frac{A\rho g(1+\cos\theta)}{m}}$$

or $$T = 2\pi\sqrt{\frac{m}{A\rho g(1+\cos\theta)}}$$

On substituting the values of various terms,

$$T = 2\times 3.14\sqrt{\frac{306}{0.6\times 13.6\times 980\,(1+1/2)}} \quad \text{(all in the CGS system)}$$

$$= 1.003 \text{ s}$$

$$= 1 \text{ s}$$

7.43. Two identical and massless threads of length l = 120 cm hold a thin and uniform rod of mass m = 2.0 kg horizontally as shown in Fig. 7.17a. On rotating the rod through a small angle about a vertical axis passing through its midpoint, it is observed that the threads are deviated through an angle $\phi = 9°$. When the rod is released from this position it starts oscillating. Find (i) the time period of oscillation, and (ii) the rod's oscillation energy.

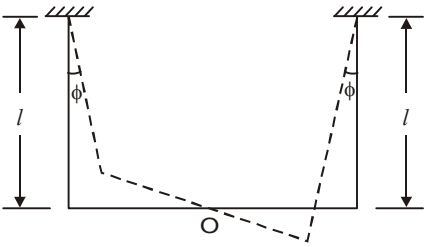


Fig. 7.17a

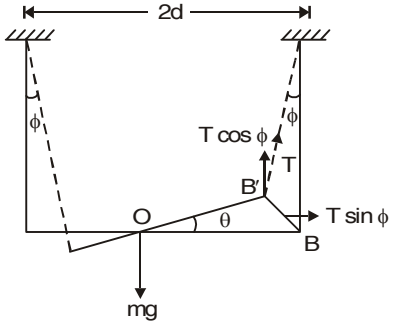


Fig. 7.17b

Solution:

(i) Let us concentrate on Fig. 7.17b. Let T be the tension in the thread. The vertical component of tension $2T\cos\phi$ is balanced by the weight of the rod and the horizontal component $T\sin\phi$ will provide a couple.

Now $2T\cos\phi = mg$; when ϕ is small $\cos\phi = 1$

$$\therefore\ T = \frac{mg}{2}$$

$$\text{Restoring couple} = T\sin\phi \cdot 2d;\ BO = d$$

$$= T\cdot\phi\cdot 2d;\ \sin\phi = \phi$$

From the figure, $\phi = \dfrac{BB'}{l}$ and $\theta = \dfrac{BB'}{d}$

$$\therefore \phi = \frac{\theta\cdot d}{l}$$

In the light of this, restoring couple $= T\cdot 2d\cdot\dfrac{\theta\cdot d}{l} = mg\cdot\dfrac{d^2\theta}{l}$

The equation of motion of the rod $= \dfrac{I\,d^2\theta}{dt^2} = -mg\,\dfrac{d^2\theta}{l}$

where $I = md^2/3$ is the moment of inertia of the rod about an axis passing through O.

Substituting this value of I in the equation of motion,

$$\frac{md^2}{3}\frac{d^2\theta}{dt^2} = -mg\,\frac{d^2\theta}{l}$$

or $$\frac{d^2\theta}{dt^2} = -\frac{3g}{l}\cdot\theta$$

The angular acceleration $\dfrac{d^2\theta}{dt^2}$ is proportional to the angle described θ. Therefore it is a case of simple harmonic motion.

Now $$\omega = \frac{2\pi}{T} = \sqrt{\frac{3g}{l}} \text{ or } T = 2\pi\sqrt{\frac{l}{3g}}$$

On substituting the value of $l = 120$ cm,

$$T = 2\times 3.14\sqrt{\frac{120}{3\times 980}} = 1.269\text{ s} = 1.27\text{ s}$$

(ii) Work done $= \dfrac{1}{2}\,C\theta$ where C is the restorting couple

$$= \frac{1}{2}\,mg\,\frac{d^2\theta^2}{l} = \frac{1}{2}mg\,\frac{d^2}{l}\,\frac{\phi^2 l^2}{d^2} = \frac{1}{2}mg\cdot l\phi^2$$

$$= \frac{1}{2}\times 2.0\times 9.8\times 1.20\times\left(\frac{9\pi}{180}\right)^2$$

$$= 0.2898\text{ J} \approx 0.29\text{ J}$$

7.44. If the inertial mass m_i and the gravitational mass m_g of the bob of a simple pendulum of length l are not the same what will be the time period of the pendulum ?

Solution: Let us consider a simple pendulum of length l. The mass of the bob m_g. Let the bob be displaced through an angle θ (Fig. 7.18). On resolving its weight into two mutually perpendicular components we have $m_g \cdot g \cos\theta$ parallel to the thread and directed opposite to the tension. The other component $m_g \cdot g \sin\theta$ is perpendicular to the thread. It is this force which will bring the pendulum to its original position.

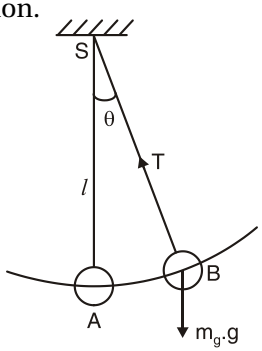


Fig. 7.18

Thus the restoring force is given by

$$F = m_g g \sin\theta$$

$$= -m_g g \frac{x}{l} \text{ where } AB = x$$

The equation of motion of the bob is

$$m_i \frac{d^2x}{dt^2} = -m_g \cdot \frac{g}{l} \cdot x$$

or $$\frac{d^2x}{dt^2} = -\frac{m_g}{m_i} \cdot \frac{g}{l} \cdot x$$

The acceleration $\frac{d^2x}{dt^2}$ is proportional to the displacement x. Therefore the motion is simple harmonic one.

Now $$\omega = \frac{2\pi}{T} = \sqrt{\frac{m_g}{m_i} \cdot \frac{g}{l}}$$

or $$T = 2\pi \sqrt{\frac{m_i}{m_g} \cdot \frac{l}{g}}$$

7.45. What will be the time period of a simple pendulum of length $l = 1$ m which hung against a wall inclined at angle $\theta = 45°$ to the horizontal.

Solution: Let l be the length of the pendulum and m the mass of the bob. Let it be displaced to a position A (Fig. 7.19).

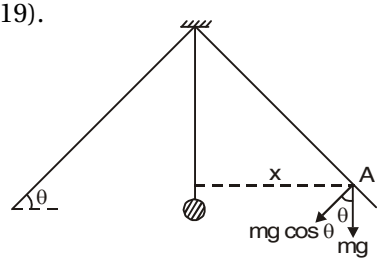


Fig. 7.19

Restoring force = $mg \sin\theta \cos\theta$

$$= -mg \sin\theta \cdot \frac{x}{l}$$

The equation of motion of the bob is given by

$$\frac{md^2x}{dt^2} = -\frac{mg \sin\theta \cdot x}{l}$$

or $$\frac{d^2x}{dt^2} = -\frac{g\sin\theta}{l}\cdot x$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{g\sin\theta}{l}}$$

$$\therefore\ T = 2\pi\sqrt{\frac{l}{g\sin\theta}}$$

On substituting the values of l, θ and g, we have

$$T = 2\times 3.14\sqrt{\frac{1\times\sqrt{2}}{9.8}} = 2\times 3.14\times\sqrt{\frac{1.41}{9.8}}$$

$$= 2.38\text{ s}$$

7.46. A simple pendulum of length $l = 1$ m is properly placed on a small cart rolling down a frictionless plane inclined at an angle $\theta = 30°$ with the horizontal. Find out the time period of oscillation of the pendulum.

Solution: As the cart rolls down the inclined plane, the force $mg\sin\theta$ provides a translational motion to the pendulum, whereas the force $mg\cos\theta$ provides an oscillatory motion to it.

Restoring force = $mg\cos\theta\cdot\sin\theta$

$= -mg\cos\theta\cdot\frac{x}{l}$, where x is the displacement of the pendulum

or $$m\cdot\frac{d^2x}{dt^2} = -mg\cos\theta\cdot\frac{x}{l}$$

or $$\frac{d^2x}{dt^2} = -\frac{g\cos\theta}{l}\cdot x$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{g\cos\theta}{l}}$$

$$\therefore\ T = 2\pi\sqrt{\frac{l}{g\cos\theta}}$$

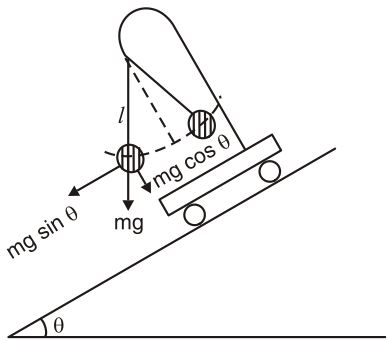


Fig. 7.20

On substituting the values, we get

$$T = 2\times 3.14\sqrt{\frac{1\times 2}{9.8\times 1.73}} = 2.157 = 2.16\text{ s}$$

7.47. A simple pendulum of length l has its point of suspension S on a wall which is inclined at an angle θ with the vertical. In conditions of equilibrium the bob of the pendulum lies at B. On displacing the bob, the thread of the pendulum makes an angle $\phi > \theta$. If the bob is released from this point it moves along the path AB and collides with the wall elastically at C and returns through the path CBA (Fig. 7.21). Determine the time period of oscillation of the pendulum.

Solution: Time taken by pendulum to traverse the path $AB = T/4 = \frac{\pi}{2}\sqrt{l/g}$

where T is the time period of the conventional simple pendulum.

Length of arc $BC = l\theta$

Length of the arc $AB = l\phi$

For the part BC let the equation of motion of the bob be

$x = a \sin \omega t$

Here x is the displacement and is equal to $l\theta$ and a is the amplitude equal to $l\phi$. On substituting the values of x and a, we have

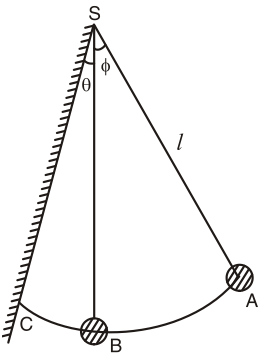


Fig. 7.21

$l\theta = l\phi \sin \omega t$

or $\sin^{-1}\left(\frac{\theta}{\phi}\right) = \frac{2\pi}{T} \cdot t$

$$\therefore t = \frac{T}{2\pi}\sin^{-1}\left(\frac{\theta}{\phi}\right) = \sqrt{\frac{l}{g}}\sin^{-1}\left(\frac{\theta}{\phi}\right)$$

Total time for the path AC is equal to $\frac{\pi}{2}\cdot\sqrt{\frac{l}{g}} + \sqrt{\frac{l}{g}}\sin^{-1}\left(\frac{\theta}{\phi}\right)$

$$\therefore \text{Time period} = 2\sqrt{\frac{l}{g}}\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{\theta}{\phi}\right)\right]$$

7.48. Figure 7.22a shows a simple pendulum suspended at a point S, designed using a thin metallic wire and a hollow metallic sphere of radius $r = 10$ cm filled with liquid. The distance between the point of suspension and the point of oscillation is $l = 100$ cm. The time period of the pendulum is $T_1 = 2$ s. Calculate the time period T_2 of the pendulum when the liquid contained in the sphere freezes.

Solution: When the sphere is filled with liquid the pendulum will execute a translational motion. The equation of motion is well-known and is given by

$$\frac{md^2x}{dt^2} = -mg\sin\theta = -mg\frac{x}{l};\ x = \text{displacement}$$

or $\frac{d^2x}{dt^2} = -\frac{g}{l}\cdot x$

Now $\omega_1 = \frac{2\pi}{T_1} = \sqrt{\frac{g}{l}}$

or $T_1 = 2\pi\sqrt{\frac{l}{g}} = 2$ s

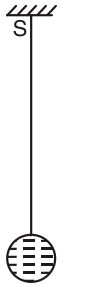


Fig. 7.22a

when the liquid freezes then it will execute rotational motion. The equation of motion is given by

$$\frac{I d^2\theta}{dt^2} = -mgl \sin\theta = -mgl\,\theta \quad \text{(see Fig. 7.22b)}$$

where I is the moment of inertia of the solid sphere about an axis passing through S and is given by

$$I = \frac{2}{5} mr^2 + ml^2 \text{ (using the theorem of parallel axes)}$$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{gl\theta}{l^2 + (2/5)r^2}$$

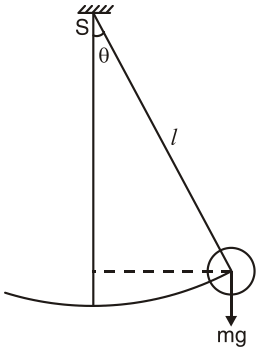


Fig. 7.22b

Now $\omega_2 = \frac{2\pi}{T_2} = \sqrt{\frac{g/l}{1+(2/5)(r/l)^2}}$

or $T_2 = 2\pi\sqrt{\frac{1+(2/5)(r/l)^2}{g/l}}$

On substituting the values of r, g and l, we have

$$T_2 = 2 \times 3.14 \sqrt{\frac{1+(2/5)\times(1/100)}{9.8}}$$

$$= 2 \times 3.14 \times \sqrt{\frac{1+(1/250)}{9.8}}$$

$$= 2 \times 3.14 \times \sqrt{\frac{251}{250 \times 9.8}} = 2.01 \text{ s}$$

7.49. A simple pendulum suspended at a point S is designed using a non-conducting thread and a hollow metallic sphere of mass m = 49 g lying above a large horizontal conducting platform at a height h = 15 cm (see Fig. 7.23). On imparting a charge Q to the sphere the time period of the pendulum is reduced by η = 2.5 times that in the absence of the charge. Calculate the amount of charge imparted to the sphere.

Solution: Let l be the length of the pendulum. Had there been no charge on the ball the time period of the pendulum would be

$$T_1 = 2\pi\sqrt{\frac{ml}{mg}}$$

After a charge Q is imparted to the sphere, the conducting plane will provide an electrical image of the charge, i.e., it will provide a charge $-Q$ at a distance h below the surface of the conducting plane. Thus net downward force acting on the sphere is

$$F = mg + \frac{1}{4\pi\varepsilon_o} \cdot \frac{Q^2}{4h^2}$$

The new time period $T_2 = 2\pi \sqrt{\dfrac{ml}{mg + \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{Q^2}{4h^2}}}$

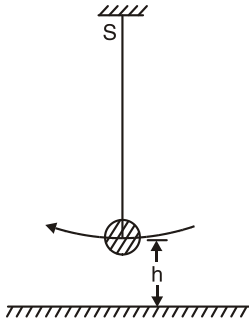


Fig. 7.23

Now $T_1 = \eta \cdot T_2$ (given)

or $2\pi \sqrt{\dfrac{l}{g}} = \eta \cdot 2\pi \sqrt{\dfrac{ml}{mg + \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{Q^2}{4h^2}}}$

or $mg + \dfrac{Q^2}{4\pi\varepsilon_0 \cdot 4h^2} = \eta^2 \cdot mg$

or $Q = 4h \sqrt{\pi\varepsilon_0 mg\left(\eta^2 - 1\right)}$

On substituting the values of the various terms, we have

$$Q = 4 \times 0.15 \sqrt{\frac{22}{7} \times 8.85 \times 10^{-12} \times 49 \times 10^{-3} \times 9.8 \times (6.25 - 1)}$$
$$= 5.02\ \mu C$$

7.50. In the arrangement shown in Fig. 7.24 a disc of mass $M = 500$ g and radius $r = 20$ cm is pivoted at O and is capable of rotating about an axis passing through it. One end of a tightly wound thread is attached to the rim of the disc and the other to a mass M' which is balanced against a point mass $m = 200$ g placed at the rim of the disc at an angle $\theta = 30°$ with the vertical. If the mass M' is pulled down a little and then released the disc starts oscillating. Find out the period of oscillation.

Solution: The two masses m and M' are related to each other by the equation $M'gr = mgr \sin\theta$ or $M' = m \sin\theta$. Let the mass M' be pulled down a little. The mass m moves up through a small angle α with respect to the reference line OB. According to the law of conservation of energy we have

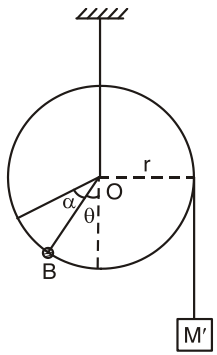


Fig. 7.24

$$\frac{1}{2} mr^2\omega^2 + \frac{1}{2} \cdot \frac{1}{2} Mr^2\omega^2 + \frac{1}{2} M'r^2\omega^2$$
$$= M'gr \sin\alpha - mgr\,[\cos\theta - \cos(\alpha + \theta)]$$

or $\dfrac{1}{2} mr^2\omega^2 + \dfrac{1}{4} Mr^2\omega^2 + \dfrac{1}{2} m \sin\theta \cdot r^2\omega^2 = mgr$

$[(\sin\alpha \cdot \sin\theta - \cos\theta + \cos\alpha \cos\theta - \sin\alpha \sin\theta)]$

or $\dfrac{1}{2} mr^2\omega^2 + \dfrac{1}{4} Mr^2\omega^2 + \dfrac{1}{2} m \sin\theta\, r^2\omega^2 = mgr\,(\cos\alpha \cos\theta - \cos\theta)$

On differentiating the above equation with respect to t, we have

$$\left(mr^2\omega + \frac{1}{2}Mr^2\omega + m\sin\theta\, r^2\omega\right)\frac{d\omega}{dt} = -mgr\sin\alpha\cdot\cos\theta\cdot\omega;\ \frac{d\alpha}{dt} = \omega$$

or $[Mr + 2mr(1+\sin\theta)]\dfrac{d\omega}{dt} = -2mg\cos\theta\cdot\alpha;\ \sin\alpha = \alpha$ when α is small.

The angular acceleration $d\omega/dt$ is proportional to the angle described (α). Therefore, this is a case of simple harmonic motion.

Now $$\omega_0 = \frac{2\pi}{T} = \sqrt{\frac{2mg\cos\theta}{Mr + 2mr(1+\sin\theta)}}$$

or $$T = 2\pi\sqrt{\frac{Mr + 2mr(1+\sin\theta)}{2mg\cos\theta}}$$

On substituting the values of the various terms we have

$$T = 2\times 3.14\sqrt{\frac{0.5\times 0.2 + 2\times 0.2\times 0.2\times 1.5}{2\times 0.2\times 9.8\times\sqrt{3}/2}}$$

$$= 2\times 3.14\sqrt{\frac{0.5+0.6}{9.8\times 1.73}}$$

$$= 2\times 3.14\sqrt{\frac{1.1}{9.8+1.73}}$$

$$= 1.6\text{ s}$$

7.51. A block of mass m = 200 g is attached with two springs of force constant $k_1 = 1.06$ N/m and $k_2 = 0.9$ N/m, as shown in Fig. 7.25. On displacing the block towards the right and set free, it starts oscillating. Find the period of oscillation, neglecting friction.

Solution: Let the block of mass m be displaced towards the right through a distance x. The restoring force developed in the two springs is given by

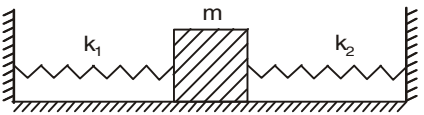


Fig. 7.25

$F_1 = -k_1 x$ and $F_2 = -k_2 x$

Net restoring force $= F = F_1 + F_2 = -(k_1 + k_2)\,x$

The equation of motion of the block is given by

$$\frac{md^2x}{dt^2} = -(k_1 + k_2)\,x$$

or $$\frac{d^2x}{dt^2} = -\frac{(k_1 + k_2)\,x}{m}$$

The acceleration $\frac{d^2x}{dt^2}$ is proportional to the displacement x. Therefore the block will execute simple harmonic motion.

Now $\omega = \frac{2\pi}{T} = \sqrt{\frac{k_1 + k_2}{m}}$

or $T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$

On substituting the values we have

$$T = 2 \times 3.14 \sqrt{\frac{0.2}{1.06 + 0.9}} = 2 \times 3.14 \sqrt{\frac{1}{9.8}} = 2.0 \text{ s}$$

7.52. A block of mass $m = 200$ g falls from a height of $h = 9.8$ cm on the pan of a spring balance (Fig. 7.26). The mass of the pan and the spring is negligible. The block gets stuck to the pan and starts oscillating simple harmonically in the vertical direction. Find the amplitude and energy of oscillations. Force constant of the spring of the balance $k = 1960$ N/m.

Solution: Compression x' in spring with weight $mg = \frac{mg}{k}$

After travelling a distance h the block of mass m touches the pan of the spring balance, sticks to it, and brings about a compression of x in the spring.

In view of the conservation of energy, we have

$$mg(h + x) = \frac{1}{2} kx^2$$

or $$x^2 - \frac{2mg}{k} \cdot x - \frac{2mgh}{k} = 0$$

or $$x = \frac{mg}{k} + \frac{1}{2}\sqrt{\frac{4m^2g^2}{k^2} + \frac{8mgh}{k}}$$

or $$x = \frac{mg}{k} + \frac{mg}{k}\sqrt{1 + \frac{2hk}{mg}}$$

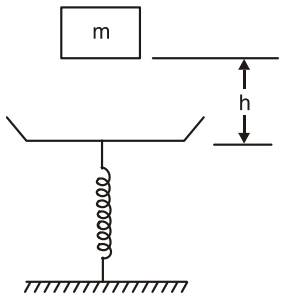


Fig. 7.26

Amplitude of oscillation $A = x - x'$

$$= \frac{mg}{k} + \frac{mg}{k}\sqrt{1 + \frac{2hk}{mg}} - \frac{mg}{k}$$

$$= \frac{mg}{k}\sqrt{1 + \frac{2hk}{mg}}$$

On substituting the values of h, m, g and k, we have

$$A = \frac{0.2 \times 9.8}{1960} \sqrt{1 + \frac{2 \times 9.8 \times 10^{-2} \times 1960}{0.2 \times 9.8}}$$

$$= 10^{-3} \times \sqrt{197} = 1.4 \text{ cm}$$

Energy of oscillation $E = mgh + \frac{1}{2} k x'^2$

$$= mgh + \frac{1}{2} k \left(\frac{mg}{k} \right)^2$$

$$= mgh + \frac{m^2 g^2}{2k}$$

$$= mg \left(h + \frac{mg}{2k} \right)$$

On substituting the values of various terms, we get

$$E = 0.2 \times 9.8 \left(9.8 \times 10^{-2} + \frac{0.2 \times 9.8}{2 \times 1960} \right)$$

$$= 19.3 \times 10^{-2} \text{ J}$$

7.53. One mole of a Vander Waal's gas whose constant b is known is enclosed in a vertical cylinder and supports a freely moving piston of mass M. The piston and the cylinder have the same area of cross-section S. Calculate the period of oscillations of small amplitude under the assumption that the compression and rarefaction takes place at a constant temperature $T = 2T_C$ and the gas volume under equilibrium is equal to $3b$.

Solution:

$$\left(P + \frac{a}{9b^2} \right) (3b - b) = \frac{R \times 2 \times 8a}{27Rb}$$

$$\left(P + \frac{a}{9b^2} \right) = \frac{8a}{27b^2}$$

or $P = \frac{5a}{27b^2} = 5P_C$ where P_C = critical pressure

Further $\left(P + \frac{a}{v^2} \right) (v - b) = RT$

$$Pv + \frac{a}{v} - Pb - \frac{ab}{v^2} = RT$$

On differentiating we have

$$Pdv + vdp - \frac{a}{v^2} dv - bdp + \frac{2ab}{v^3} dv = 0$$

$$(v-b)\,dp = -\left[P - \frac{a}{v^2} + \frac{2ab}{v^3}\right]dv$$

$$2b\,dp = -\left[\frac{5a}{27b^2} - \frac{a}{9b^2} + \frac{2ab}{27b^3}\right]dv$$

$$= -\frac{4a}{27b^2}\,dv$$

$$dp = -\frac{2a}{27b^3}\,dv = -\frac{2a}{27b^3}\cdot S\cdot y\,;\ y = \text{vertical displacement.}$$

$$\text{Force} = M\times \text{acceleration} = dp\cdot S = -\frac{2aS^2y}{27b^3}$$

Under equilibrium $\frac{5aS}{27b^2} = Mg$

$$\therefore \text{ acceleration} = -\frac{2Sg\cdot y}{5b}$$

acceleration is proportional to displacement

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{2Sg}{5b}}$$

or $$T = 2\pi\sqrt{\frac{5b}{2Sg}}$$

7.54. A sphere of mass $m = 5$ kg is suspended from the dome of a hall by a light inextensible string of length $g/4$ m. The sphere has been moved along x axis upto a distance $x = A = 30$ cm. and a momentum of $P = 1.00$ kg m/sec was imparted to it along y axis. Neglecting friction find the path described by the centre of the sphere.

Solution: The sphere will execute simple harmonic motion of angular frequency $\omega = \sqrt{\frac{g}{l}} = 2$ rad/sec in the xy plane.

Let the equation of motion of the sphere be

$x = A\sin\omega t$ and $y = B\cos\omega t$ where $A = 30$ cm

Now $\frac{dy}{dt}$ = velocity along y axis $= \frac{1}{5} = 0.2$ m/sec $= 20$ cm/sec

$$\frac{dy}{dt} = B\,\omega = 20 \text{ cm}$$

$$\therefore B = \frac{20}{\omega} = 10$$

or $$\frac{x^2}{30^2} + \frac{y^2}{10^2} = 1$$

The path described will be an ellipse.

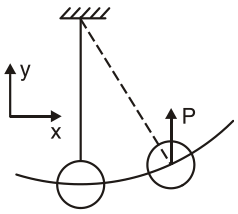


Fig. 7.27

7.55. A solid circular thin disc of mass m and radius r is suspended in the vertical plane about the axis A_1A_2 passing through its cord. For what value of x (see Fig. 7.28) the frequency of small oscillation of the disc about this axis will have the maximum value.

Solution: The equation of motion of the disc is given by

$$\left(mx^2 + \frac{mr^2}{4}\right)\alpha = -mgx \cdot \sin\theta = -mgx \cdot \theta$$

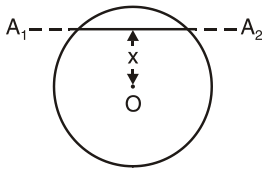


Fig. 7.28

$$\omega = 2\pi n = 2\sqrt{\frac{gx}{r^2 + 4x^2}}$$

$$\text{or} \quad n = \frac{1}{\pi}\sqrt{\frac{gx}{r^2 + 4x^2}}$$

$$\text{Now} \quad \frac{dn}{dx} = 0 = \left[\frac{x^{1/2} \times -\frac{1}{2} \times 8x}{\left(r^2 + 4x^2\right)^{3/2}} + \frac{1}{2 \cdot \sqrt{x}\left(r^2 + 4x^2\right)^{1/2}}\right]\frac{\sqrt{g}}{\pi}$$

$$\text{or} \quad -8x^2 + r^2 + 4x^2 = 0$$

$$\text{or} \quad x = \pm\frac{r}{2}$$

7.56. A simple pendulum as shown in the Fig. 7.29 oscillates simple harmonically. In the state of motion the angle ϕ varies as $\phi = \phi_m \cos \omega t$. Find the tension in the string at an instant t.

Solution: When the pendulum moves from its equilibrium position A it acquires a position B at an instant t.

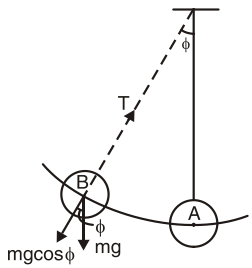


Fig. 7.29

The equation of motion at B is given by

$$T_B - mg\cos\phi = \frac{mv_B^2}{l}; \; l = \text{length of the string.}$$

$$\text{Now } v_B = l\omega = \frac{ld\phi}{dt} = -l\omega\phi_m \sin\omega t$$

$$\therefore \; T_B = mg\cos[\phi_m \cos\omega t] + ml\omega^2 \phi_m^2 \sin^2\omega t$$

7.57. Find the equation of motion of a simple pendulum of length l when the point of suspension O moves along a horizontal line according to the law $x = a\cos\phi t$.

Solution: The restoring force $F_1 = mg\sin\theta = mg\theta$. [For small oscillations]

Torque $= mg\theta \cdot l$

Force F_2 due to displacement of the point of suspension

$= -a\phi^2 m\cos\phi t$

Torque due to force $F_2 = -a\,\phi^2\,l\,m\cos\phi\,t$

The equation of motion is given by

$ml^2 \cdot \alpha = -mgl \cdot \theta - a\,\phi^2\,ml \cdot \cos\phi\,t$

or $\alpha = -\dfrac{g}{l}\cdot\theta - \dfrac{a\phi^2}{l}\cdot\cos\phi t$

or $\alpha + \omega_0^2\theta + A\cos\phi t = 0$

where $\omega_0 = \sqrt{\dfrac{g}{l}}$ and $A = \dfrac{a\phi^2}{l}$

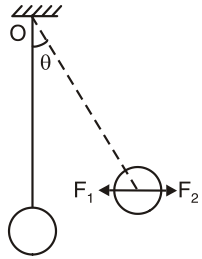


Fig. 7.30

7.58. A mass m is attached at the end of a cantilever beam of length l. Neglecting the mass of the beam find the natural frequency of oscillation.

Solution: The mass placed at the end of the cantilever will produce a static deflection Δ (see Fig. 7.31) given by

$$\Delta = \frac{mgl^3}{3YI}$$

Where Y is the Young's modulus of the material of the beam and I is the geometrical moment of inertia of the beam about the neutral axis.

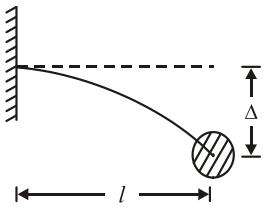


Fig. 7.31

$$\omega = \sqrt{\frac{g}{\Delta}} = 2\pi n$$

or $$n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}} = \frac{1}{2\pi}\sqrt{\frac{3YI}{ml^3}}\ \text{Hz}$$

7.59. A conical object of mass m floats with its axis vertical and vertex downwards in a tank of water as in Fig. 7.32. In the conditions of equilibrium the length of the axis of the cone immersed in water is h. When the object is depressed a little downwards and released it starts oscillating. Neglecting force due to friction find the period of oscillation.

Solution: In the conditions of equilibrium, we have

$$mg = \frac{1}{3}\pi r'^2 h\rho g;$$

m = mass of the conical object

ρ = density of water

Let it be depressed through y then

$$m \times \text{acceleration} = \frac{1}{3}\pi r'^2 h\rho \times \text{acceleration}$$
$$= -\pi r'^2 y\rho g$$

$$\therefore \text{acceleration} = -\frac{3g}{h}\cdot\text{displacement}$$

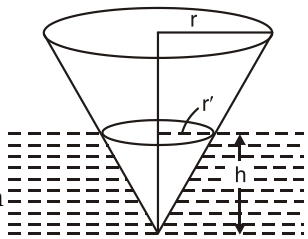


Fig. 7.32

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{3g}{h}}$$

or $$T = 2\pi \sqrt{\frac{h}{3g}}$$

7.60. A uniform solid sphere of radius r and mass m is suspended vertically downwards from a point P on its surface (see Fig. 7.33). Find (i) the period of small oscillations in the plane (ii) the equivalent length of a simple pendulum.

Solution:

(i) Moment of inertia of the solid sphere about P is given by

$$I = \frac{2}{5} mr^2 + mr^2 = \frac{7}{5} mr^2$$

The equation of motion is given by

$$\frac{7}{5} mr^2 \alpha = -mgr \sin\theta = -mgr\theta$$

$$\omega = \sqrt{\frac{5g}{7r}}$$

or $$\omega = \frac{2\pi}{T} = \sqrt{\frac{5g}{7r}}$$

or $$T = 2\pi \sqrt{\frac{7r}{5g}}$$

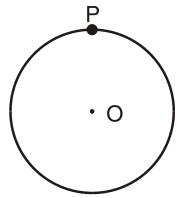


Fig. 7.33

(ii) On comparing it with the equation $T = 2\pi\sqrt{\frac{l}{g}}$ the equivalent length is equal to $\frac{7r}{5}$

7.61. What must be the length of a rod suspended from one end so that it will be a second pendulum on making small vibrations in a plane.

Solution: Consider a rod of mass m and length l suspended from one end in a vertical plane. If it is rotated through a small angle θ from the vertical then equation of motion is given by

$$\frac{1}{3} ml^2\alpha = -mg \cdot \frac{l}{2} \sin\theta = -mg \frac{l}{2} \theta$$

$$\omega = \sqrt{\frac{3g}{2l}} = \frac{2\pi}{T} = \pi \; ; \; T = 2 \text{ sec}$$

$$2\pi^2 l = 3g$$

or $$l = \frac{3g}{2\pi^2}$$

On putting the values

$$l = \frac{3 \times 9.8}{2 \times 3.14^2}$$
$$= 1.49 \text{ m}$$

7.62. In a lapse of time of 1 sec. the amplitude of oscillation reduces to half of its initial value. Calculate the time during which the amplitude diminishes to 1/10th of its initial value.

Solution:

$$A = A_o e^{-\alpha t}$$
$$\frac{A_o}{2} = A_o e^{-\alpha}$$
or $$e^{\alpha} = 2$$
or $$\alpha = 2.303 \log_{10}{}^2$$
Now $$\frac{A_o}{10} = A_o e^{-\tau \cdot \alpha}; \tau = \text{required time}$$
$$e^{\tau\alpha} = 10$$
$$\tau\alpha = 2.303 \log_{10}{}^{10}$$
$$\tau = \frac{2.303}{2.303 \log_{10}{}^2}$$
$$= \frac{1}{0.3010} = 3.3 \text{ sec}$$

7.63. One end thin and uniform bar of mass M and length l is attached to a bob of mass m. The radius of the bob is very small. The other end of the bob is fixed at O (see Fig. 7.34). Find the period of small oscillations in the vertical plane.

Solution: Moment of inertia I about $O = ml^2 + \frac{1}{3} Ml^2 = (M + 3m) \frac{l^2}{3}$

Distance of centre of mass from O $= \frac{M \cdot \frac{l}{2} + ml}{M + m}$

$$= \frac{(M + 2m) \cdot l}{2(M + m)}$$

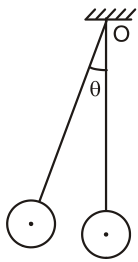


Fig. 7.34

The equation of motion is given by

$$(M + 3m) \frac{l^2}{3} \cdot \alpha = -(M + m)\, g \cdot \frac{(M + 2m)\, l \cdot \theta}{2(M + m)}$$

$$\Omega = \frac{2\pi}{T} = \sqrt{\frac{3(M + 2m)\, g}{2(M + 3m)\, l}}$$

or $T = 2\pi\sqrt{\dfrac{2(M+3m)l}{3(M+2m)g}}$

7.64. A cube of side a and mass m is suspended vertically from one of its edges as shown in Fig. 7.35. Find (i) the period of small oscillations (ii) the length of equivalent simple pendulum.

Solution:

(i) Moment of inertia of the cube about $O = \dfrac{2}{3}\ ma^2$

Equation of motion

$$\frac{2}{3}ma^2 \cdot \alpha = -mg \cdot \frac{a}{\sqrt{2}} \cdot \sin\theta\ ;\ \theta = \text{angle from the vertical}$$

$$\alpha = -\frac{3g}{2\sqrt{2}a} \cdot \theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{3g}{2\sqrt{2}a}}$$

or $$T = 2\pi\sqrt{\frac{2\sqrt{2}a}{3g}}$$

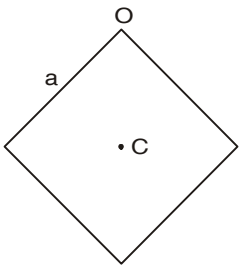


Fig. 7.35

(ii) Length of equivalent simple pendulum $= \dfrac{2\sqrt{2}a}{3}$

7.65. A V tube having a uniform bore of cross-section S is kept vertical as shown in the Fig. 7.36. The two arms are making an angle ϕ with the vertical. The tube is filled with water and the total length of water column is equal to l. If the water level in one of the two arms is displaced a little from its original position the entire column starts oscillating. Find the period of oscillation.

Solution: Let the water column be depressed through y in one arm and let l_1 and l_2 be the length of water column in the two arms at that time then

$(l_1 \sec\phi + l_2 \sec\phi)\ S\rho \cdot \text{acceleration}$

$= -2yS\rho g$

or $l \sec\phi \cdot \text{acceleration} = -2g \cdot \text{displacement}$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{2g}{l\sec\phi}}$$

or $$T = 2\pi\sqrt{\frac{l\sec\phi}{2g}}$$

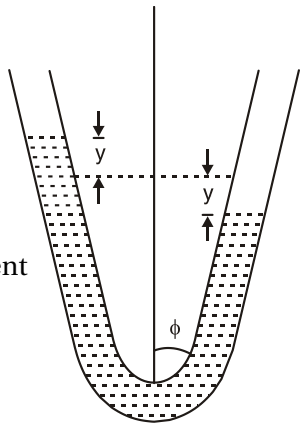


Fig. 7.36

7.66. A disc of mass $M = 200$ g is pivoted at O and is free to rotate about a horizontal axis passing through it (Fig. 7.37). A mass $m = 20$ g and a spring of force constant $k = 1960$ N/m are connected by a string passing over the disc. The string does not slide over it. If the mass m is pulled down a little and set free the disc starts oscillating. Find out the frequency of oscillation.

Solution: On compressing the spring through a distance x, and energy $(1/2)\,kx^2$ is stored in it. The source of this energy in the present arrangement is the kinetic energy of translation of mass m plus the kinetic energy of rotation of the disc, i.e.,

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}kx^2$$

where I is the moment of inertia of the disc about an axis passing through O.

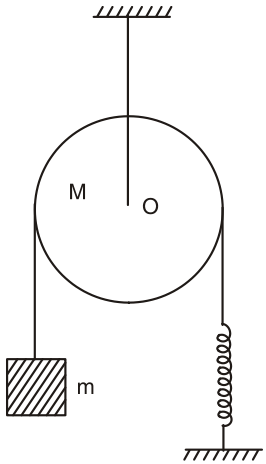


Fig. 7.37

or $$\frac{1}{2}mv^2 + \frac{1}{2}\cdot\frac{1}{2}MR^2\omega^2 = \frac{1}{2}kx^2$$

$$I = \frac{1}{2}MR^2$$

where R is the radius of the disc

or $$\left(m + \frac{M}{2}\right)v^2 = kx^2$$

On differentiating the above with respect to t, we have

$$\left(m + \frac{M}{2}\right)\cdot 2v\cdot\frac{dv}{dt} = 2kx\cdot\frac{dx}{dt}$$

or $\dfrac{dv}{dt} = -\dfrac{kx}{m + M/2}$; $\dfrac{dx}{dt} = v$. The negative sign has been introduced because the acceleration and displacement are in opposite directions.

Now $$\omega_0 = \sqrt{\frac{k}{m + M/2}}$$

or $$n = \frac{1}{2\pi}\sqrt{\frac{k}{m + M/2}}$$

On substituting the values of k, m and M we have

$$n = \frac{1}{2\times 3.14}\sqrt{\frac{1960}{(20+100)\times 10^{-3}}}$$

$$= \frac{10}{2\times 3.14}\sqrt{\frac{490}{3}}$$

$$= 20.35 \text{ c/s}$$

7.67. A solid cylinder of mass M performs oscillations due to the action of two springs whose combined force constant is k. Find the period of oscillations in the absence of sliding.

Solution: On slightly rotating the cylinder about an axis passing through O and perpendicular to the plane of Fig. 7.38, an energy $\frac{1}{2} kx^2$ is stored in the spring. Owing to conservation of energy

$$\frac{1}{2} I \omega^2 = \frac{1}{2} kx^2$$

where I is the moment of inertia of the cylinder about an axis passing through O.

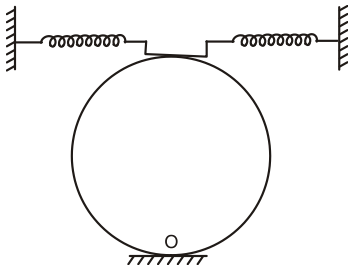


Fig. 7.38

$I = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2$ in view of theorem of parallel axes. Here R is the radius of the cylinder.

Now $\frac{1}{2} \times \frac{3}{2} MR^2 \omega^2 = \frac{1}{2} kx^2$

$$\frac{3}{16} Mv^2 = \frac{1}{2} kx^2; \; v = 2R\omega$$

On differentiating the above with respect to t, we have

$$\frac{3}{8} M \cdot v \cdot \frac{dv}{dt} = kx \cdot \frac{dx}{dt}; \; \frac{dx}{dt} = v$$

or $\frac{dv}{dt} = -\frac{8k}{3M} x$ since acceleration and displacement are in opposite directions.

Now $\omega_0 = \frac{2\pi}{T} = \sqrt{\frac{8k}{3M}}$

or $T = \pi \sqrt{\frac{3M}{2k}}$

7.68. Two solid spheres A and B of mass M_1 = 1.5 kg and M_2 = 2.5 kg connected by a spring of force constant k = 60 N/m are slipped on a rod over which they can slide without friction (Fig. 7.39). When an initial velocity V = 20 cm/s is imparted to sphere A through external means, the two spheres start oscillating. Find (i) the angular frequency of oscillations, and (ii) the amplitude and energy of oscillation.

Solution:

(i) The two masses can be replaced by their reduced mass

$$\mu = \frac{M_1 M_2}{M_1 + M_2}$$

When a velocity is imparted to mass A there happens to be a compression x in the spring. According to the law of conservation of energy,

$$\frac{1}{2}\mu v^2 = \frac{1}{2}kx^2$$

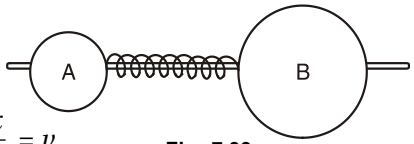


Fig. 7.39

or $\mu v \frac{dv}{dt} = kx \frac{dx}{dt} = k\,xv;\ \frac{dx}{dt} = v$

or acceleration $\frac{dv}{dt} = -\frac{k}{\mu}x$

$$\therefore\ \omega = \sqrt{\frac{k}{\mu}} = \sqrt{60 \times \frac{16}{15}} = 8 \text{ rad/s}$$

(ii) amplitude $a = \frac{V}{\omega} = \frac{20}{8} = 2.5$ cm

$$\text{Energy } E = \frac{1}{2}\mu V^2 = \frac{1}{2} \times \frac{15}{16} \times 20 \times 20 \times 10^{-4}$$
$$= 18.75 \text{ m J}$$

7.69. A sphere of mass $m = 200$ g fixed between two springs of total force constant $k = 28.8$ N/m can slide over a frictionless circular rod PQ placed horizontally. The system rotates with an angular velocity of $\omega = 8$ rad/s about a vertical axis OO' (Fig. 7.40). Show that the mass will execute simple harmonic motion, and find the period of oscillation. At what value of ω the mass will not oscillate ?

Solution: In the state of rotation of the system as a whole there are two forces which act on the sphere:

(i) elastic and (ii) centripetal.

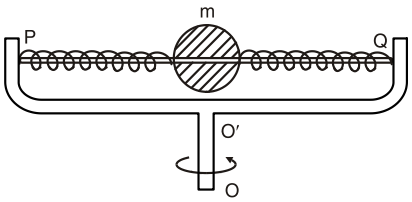


Fig. 7.40

The equation of motion of the sphere is

$$\frac{md^2x}{dt^2} = -kx + m\omega^2 x;\ \text{(here } x \text{ stands for displacement)}$$

or $\frac{d^2x}{dt^2} = -\left(\frac{k}{m} - \omega^2\right)x$

The acceleration $\frac{d^2x}{dt^2}$ is proportional to the displacement, and therefore the sphere will execute simple harmonic motion.

Now $\omega_0 = \sqrt{\frac{k}{m} - \omega^2}$

$$T = \frac{2\pi}{\sqrt{k/m - \omega^2}} = \frac{2 \times 3.14}{\sqrt{(28.8/0.2) - 64}} = \frac{2 \times 3.14}{\sqrt{80}} = 0.70 \text{ s}$$

The oscillation with stop when $T = \infty$. This is possible when

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{28.8}{0.2}} = 12 \text{ rad/sec}$$

7.70. A cylindrical solid block of mass m = 500 g attached to a spring of force constant k = 30 N/m slides on a horizontal rod fixed on a wall (Fig. 7.41). The coefficient of friction μ between the block and the rod is 0.1. If the block is pressed up to 20 cm from its equilibrium position and set free, it starts oscillating. Calculate the number of swings the block will complete before coming to rest, assuming that in one swing it will move from its equilibrium position to a distance equal to its amplitude or vice versa. $g = 10 \text{ m/s}^2$.

Solution: This is a case of damped simple harmonic motion. Due to damping the amplitude will no longer be constant. Let us assume that the block moves from one state of amplitude a_0 to another of amplitude a_1. Owing to the conservation of energy, we then have

$$\frac{1}{2}ka_0^2 - \mu mg\, a_0 = \frac{1}{2}ka_1^2 + \mu mg\, a_1$$

or $$\frac{1}{2}k\left(a_1^2 - a_0^2\right) = -\mu mg\,(a_0 + a_1)$$

or $$a_1 - a_0 = -\frac{2\mu mg}{k}$$

or $$a_1 = a_0 - \frac{2\mu mg}{k}$$

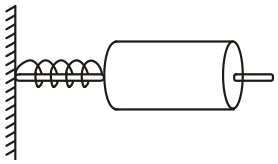


Fig. 7.41

Since the amplitude will be in progression, it can be identified by following a general expression.

$$a_n = a_0 - \frac{2n\mu mg}{k}$$

The block will cease to move when $a_n = 0$

i.e., $$n = \frac{ka_0}{2\mu mg}$$

The number of swings $N = 2n - 1$ since during oscillations all amplitudes except the first will pass through twice the number of swings. Owing to the friction the amplitude will reduce at a faster rate and finally the oscillations will cease.

$$\therefore N = \frac{ka_0}{\mu mg} - 1$$

On substituting the values of k, a_0, μ, m and g, we have

$$N = \frac{30 \times 0.2}{0.1 \times 0.5 \times 10} - 1 = 11$$

7.71. A thin uniform rod of mass m and length l with one end fixed at a point S is kept vertical with the help of two springs whose combined force constant is k (Fig. 7.42a). The rod is displaced a little to the right and set free. It starts oscillating simple harmonically. Find the frequency of oscillation.

Solution: In view of Fig. 7.42b we have $\frac{x}{l} = \theta$ where x is the distance through which compression occurs during motion of the rod.

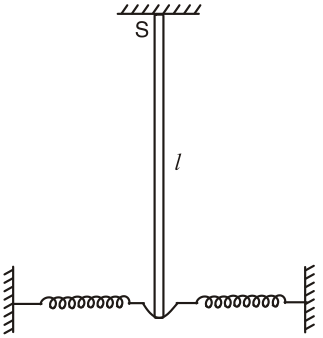


Fig. 7.42a

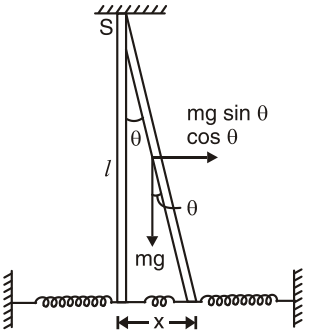


Fig. 7.42b

Now $\frac{Id^2\theta}{dt^2} = -mg \sin\theta \cos\theta \cdot \frac{l}{2} - kxl$; I is the moment of inertia of the rod about an axis passing through one end.

or $\frac{ml^2}{3} \cdot \frac{d^2\theta}{dt^2} = -\left(mg\frac{l}{2} \cdot \theta + kl^2\theta\right)$; $\cos\theta = 1$; $\sin\theta = \theta$ when θ is small.

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{3g}{2l}\left(1 + \frac{2kl}{mg}\right)\theta$$

Angular acceleration $\frac{d^2\theta}{dt^2}$ is proportional to the angle described θ. Therefore, this is a case of simple harmonic motion.

Now $\omega = 2\pi n = \frac{3g}{2l}\left(1 + \frac{2kl}{mg}\right)^{1/2}$ or $n = \frac{3g}{4\pi l}\left(1 + \frac{2kl}{mg}\right)^{1/2}$

7.72. A spring of force constant K, length l and mass m is suspended from the point O as shown in the figure 7.43. A mass M is attached at the free end and set in motion. Find the period of oscillation.

Solution: Let a be amplitude and ω be the angular velocity. Consider an element of length dx at a distance x from O.

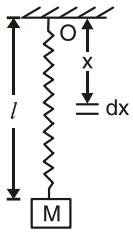


Fig. 7.43

Velocity of mass M in the mid position = ωa

Velocity of the element in the mid position = $\frac{x}{l}(\omega a)$

Kinetic energy of the element in the mid position

$$= \frac{1}{2}\left(\frac{m}{l}\cdot dx\right)\left(\frac{x}{l}\cdot \omega a\right)^2$$

Total kinetic energy of the spring in the mid position

$$= \frac{1}{2}\frac{m}{l^3}\omega^2 a^2 \int_0^l x^2 dx$$

$$= \frac{1}{6}m\omega^2 a^2 = \frac{mv^2}{6}$$

Total kinetic energy of the system = $\frac{1}{2}Mv^2 + \frac{1}{6}mv^2 = \frac{1}{2}(M + m/3)v^2$

This energy is balanced by the potential energy stored in the spring

$$\frac{1}{2}(M + m/3)v^2 = \frac{1}{2}Ky^2$$

$$(M + m/3)\, v \cdot A = K \cdot y \cdot v;\ A = \text{acceleration}$$

$$A = -\frac{K}{(M + m/3)} \cdot \text{displacement}$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{K}{M + m/3}}$$

$$T = 2\pi\sqrt{\frac{M + m/3}{K}}$$

7.73. A thin bar of mass m and length l is hinged at O. The bar is supported in a horizontal position of equilibrium by a spring of force constant K attached between O' and A (see Fig. 7.44). The end of the bar A is displaced downwards and then released. It executes oscillatory rotation about the axis through O. Find the period of oscillation.

Solution: Let θ be the angular displacement. The upward reaction force exerted by the spring on the end A is equal to.

$$F = \frac{mg}{2} + Kl\theta$$

The equation of motion is given by

$$\frac{1}{3}ml^2 \cdot \alpha = mg \cdot \frac{l}{2} - \left(\frac{mg}{2} + Kl\theta\right) \cdot l$$

or $\frac{1}{3} ml^2 \cdot \alpha = -K l^2 \theta$

or $\alpha = -\frac{3K}{m} \cdot \theta$

angular acceleration = $-\omega^2$ angle described

$$\therefore \omega = \frac{2\pi}{T} = \sqrt{\frac{3K}{m}}$$

$$T = 2\pi \sqrt{\frac{m}{3K}}$$

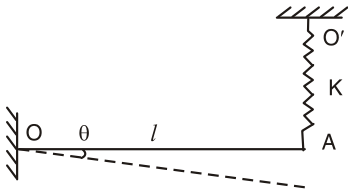


Fig. 7.44

Alternative Solution:

Extension in the spring = $l\theta$

According to conservation of energy

$\frac{1}{2} \cdot \frac{1}{3} ml^2 \cdot \omega^2 + \frac{1}{2} K l^2 \theta^2 =$ Constant

On differentiation we have

$$\frac{1}{3} ml^2 \cdot \omega \cdot \alpha = -K l^2 \theta \cdot \omega$$

or $\alpha = -\frac{3K}{m} \cdot \theta$

$$\therefore \omega = \sqrt{\frac{3K}{m}}$$

or $T = 2\pi \sqrt{\frac{m}{3K}}$

7.74. A thin bar of mass *m*, length *l* hinged at *O* carries a small sphere of mass *M* as shown in the Fig. 7.45. The bar is supported in a horizontal position by a spring of force constant *K* attached between the middle point of the bar *A* and *O'*. The bar is displaced downwards and then released. It executes simple harmonic motion. Find the period of oscillation.

Solution: Let θ be the angular displacement of the bar. The extension in the spring is equal to $\frac{l}{2}\theta$.

Net moment of inertia of the system

$$I = \left(\frac{ml^2}{3} + Ml^2 \right)$$

Taking the moment about *O* we have

$$\left(\frac{ml^2}{3} + Ml^2 \right) \alpha = -K \cdot \frac{l\theta}{2} \cdot \frac{l}{2}$$

$$\frac{(3M+m)}{3}\alpha = -\frac{K}{4}\theta$$

$$\alpha = -\frac{3K}{4(3M+m)}\cdot\theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{3K}{4(3M+m)}}$$

or $$T = 4\pi\sqrt{\frac{(3M+m)}{3K}}$$

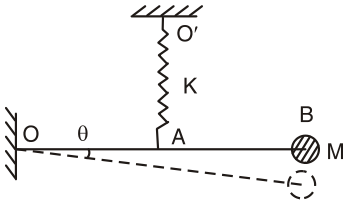


Fig. 7.45

7.75. A thin bar *AB* of length *l* and mass *m* is hinged at *A*. It is kept horizontal with the help of two springs of force constant K_1 and K_2 attached to the bar at *C* and *D* lying at a respective distance of l_1 and l_2 from *A*. The bar is displaced downwards and released. Find the period of harmonic oscillations.

Solution: Let θ be the angular displacement of the bar. The elongation in the two springs are $l_1\theta$ and $l_2\theta$ (see Fig. 7.46).

Taking the moment about *A* we have

$$\frac{ml^2}{3}\cdot\alpha = -[K_1 l_1\,\theta\cdot l_1 + K_2\, l_2\,\theta\cdot l_2]$$

or $$\alpha = -\frac{3\left[K_1 l_1^2 + K_2 l_2^2\right]}{ml^2}\cdot\theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{3\left[K_1 l_1^2 + K_2 l_2^2\right]}{ml^2}}$$

or $$T = 2\pi\sqrt{\frac{ml^2}{3\left[K_1 l_1^2 + K_2 l_2^2\right]}}$$

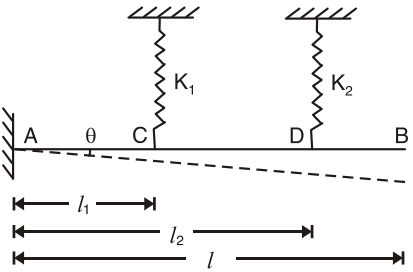


Fig. 7.46

7.76. A *T*-shaped thin bar of mass *m* is hinged at *C* and is kept horizontal with the help of a spring of force constant *K* attached to it at *O* as shown in the Fig. 7.47. The end of *A* of the bar is slightly depressed downward and released. Find the period of rotational oscillations of small amplitude.

Solution: Moment of inertia of section *AB* of the bar about $D = \frac{m}{3}\frac{l^2}{12}$

Moment of inertia of *AB* about $C = \frac{m}{3}\frac{l^2}{12} + \frac{m}{3}\cdot 4l^2$

Moment of inertia of the horizontal bar *CD* about $C = \frac{2m}{3}\cdot\frac{4l^2}{3}$

Total moment of inertia of the bar about $C = \frac{m}{3}\left[\frac{l^2}{12} + 4l^2 + \frac{8l^2}{3}\right]$

$$= \frac{81ml^2}{36}$$

Let an angular displacement θ be given to the bar. Extension of the spring = $l\theta$.

Taking moment about C

$$\frac{81ml^2}{36} \cdot \alpha = -Kl\theta \cdot l$$

$$\alpha = -\frac{36K}{81m} \cdot \theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{36K}{81m}} = \frac{2}{3}\sqrt{\frac{K}{m}}$$

$$T = 3\pi\sqrt{\frac{m}{K}}$$

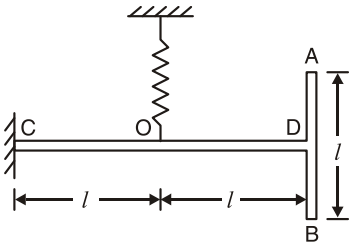


Fig. 7.47

7.77. A small disc of radius $r/2 = 5$ cm and mass $m = 3$ kg is supported at its centre by a thin light rigid rod of length $2r$. The other end of the rod is fixed at O. One end of the two springs each of force constant $K = 1900$ N/m is attached to the centre of the rod and the other to points A and B of a rigid support. The system oscillates as a pendulum. Find (i) the frequency of oscillation of the system when the rod and the disc are joined together with a pin L (ii) the frequency of oscillation of the system when the rod and the disc are not joined (iii) the ratio of the frequency of oscillation in the two conditions.

Solution: Let the rod be displaced through an angle θ from the vertical then according to conservation of energy we have

$$mg \cdot 2r(1 - \cos\theta) + \frac{1}{2}Kr^2\theta^2 + \frac{1}{2}\left[\frac{1}{2}\frac{mr^2}{4} + m \cdot 4r^2\right]\omega^2 + \frac{1}{2}Kr^2\theta^2$$

$$= \text{Constant} \quad \text{(see Fig. 7.48)}$$

or $mg \cdot 2r(1 - \cos\theta) + Kr^2\theta^2 + \frac{33}{16}mr^2\omega^2 = \text{Constant}$

On differentiation we have

$$mg \cdot 2r \sin\theta \cdot \omega + Kr^2 \cdot 2\theta \cdot \omega = -\frac{33}{16}mr^2 \cdot 2\omega \cdot \alpha$$

for small angles $\sin\theta = \theta$

$$2(mg + Kr)\theta = -\frac{33}{8}mr\alpha$$

$$\alpha = -\frac{16(mg + Kr)\theta}{33mr}$$

$$2\pi n = \sqrt{\frac{16(mg + Kr)}{33mr}}$$

$$n = \frac{1}{\pi}\sqrt{\frac{4(mg + Kr)}{33mr}}$$

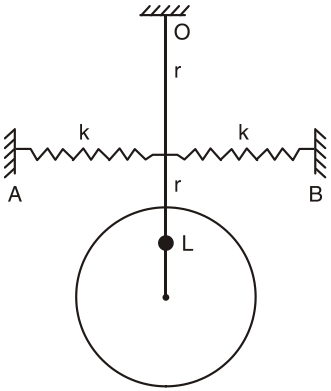


Fig. 7.48

On substituting the values

$$n_1 = \frac{1}{3.14}\left[\frac{4(3\times 9.8 + 1900\times 0.1)}{33\times 3\times 0.1}\right]^{\frac{1}{2}} = 3 \text{ Hz}$$

(ii) In view of conservation of energy we have

$$mg\cdot 2r(1-\cos\theta) + Kr^2\theta^2 + \frac{33}{16}mr^2\omega^2 - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{mr^2}{4}\cdot\omega^2 = \text{Constant}$$

$$mg\cdot 2r(1-\cos\theta) + Kr^2\theta^2 + \frac{32}{16}mr^2\omega^2 = \text{Constant}$$

On differentiation we have

$$mg\cdot 2r\cdot\sin\theta\cdot\omega + Kr^2 2\theta\cdot\omega = -\frac{32}{16}mr^2\cdot 2\omega\alpha$$

$$\alpha = -\frac{16(mg + Kr)\theta}{32mr}$$

$$2\pi n_2 = \sqrt{\frac{16(mg + Kr)}{32mr}}$$

$$n_2 = \frac{1}{\pi}\sqrt{\frac{4(mg + Kr)}{32mr}}$$

On substituting the values we have

$$n_2 = \frac{1}{3.14}\sqrt{\frac{4(3\times 9.8 + 1900\times 0.1)}{32\times 3\times 0.1}}$$

$$= 3.04 \text{ Hz}$$

(iii) $$\frac{n_2}{n_1} = \left(\frac{33}{32}\right)^{\frac{1}{2}}$$

7.78. A rod of mass m and length l carries a mass M at its one end. While the other end is attached to a rigid support O. One end of the two springs of force constants K_1 and K_2 respectively is attached to rod at a distance l_1 and l_2 from O while the other to points A and B as shown in the Fig. 7.49. The rod is displaced to an angle θ from the vertical and then released. Find the frequency of small oscillations of the system.

Solution: Centre of mass from $O = \dfrac{m \cdot \dfrac{l}{2} + M \cdot l}{M + m} = \left(\dfrac{M + \dfrac{m}{2}}{M + m} \right) l$

Let the rod be displaced through an angle θ from the vertical then in view of conservation of energy we have

$$(M + m)\, gl \frac{\left(M + \frac{m}{2}\right)}{(M + m)} (1 - \cos\theta) + \frac{1}{2} K_2 l_2^2 \theta^2 + \frac{1}{2}\left(M + \frac{m}{3}\right) l^2 \omega^2 + \frac{1}{2} K_1 l_1^2 \theta^2 = \text{Constant}$$

On differentiation we have

$$gl\left(M + \frac{m}{2}\right) \sin\theta \cdot \omega + K_2 l_2^2 \theta \cdot \omega + \left(M + \frac{m}{3}\right) l^2 \omega\alpha + K_1 l_1^2 \theta \cdot \omega = 0$$

$$\alpha = -\frac{\left[gl\left(M + \frac{m}{2}\right) + K_1 l_1^2 + K_2 l_2^2\right]\theta}{\left(M + \frac{m}{3}\right) l^2}; \sin\theta = \theta$$

$$2\pi n = \sqrt{\frac{gl\left(M + \frac{m}{2}\right) + K_1 l_1^2 + K_2 l_2^2}{\left(M + \frac{m}{3}\right) l^2}}$$

or $$n = \frac{1}{2\pi} \sqrt{\frac{gl\left(M + \frac{m}{2}\right) + K_1 l_1^2 + K_2 l_2^2}{\left(M + \frac{m}{3}\right) l^2}}$$

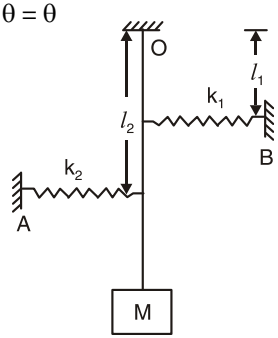


Fig. 7.49

7.79. A uniform thin bar *AB* of mass *M* and length 4*r* is pivoted at its centre *O*. A disc of mass *m* and radius *r* is attached at the end *B* of the bar (see Fig. 7.50). A spring of force constant *K* is attached to the bar at *C* such that *OC* = *r*. The bar is displaced through an angle θ from the vertical and released. Find the frequency of small oscillations.

Solution:

Moment of inertia of the bar about $O = \dfrac{M \cdot (4r)^2}{12} = \dfrac{4}{3} Mr^2$

Moment of inertia of the disc about $O = \dfrac{1}{2} mr^2 + m \cdot (3r)^2$

$$= \frac{19}{2} mr^2$$

Net moment of inertia about $O = \left(\frac{4}{3}M + \frac{19m}{2}\right) r^2 = (8M + 57m)\frac{r^2}{6}$

Let the bar be displaced through small angle θ then the energy equation of the system is given by

$$\frac{1}{2} Kr^2 \theta^2 - mg \cdot 3r (1 - \cos\theta) = \frac{1}{2} (8M + 57m) \frac{r^2}{6} \cdot \omega^2$$

On differentiating the above equation we have

$$Kr^2 \theta \cdot \omega - mg \cdot 3r \sin\theta \cdot \omega = (8M + 57m) \frac{r^2}{6} \omega \cdot \alpha$$

$$(Kr - 3mg)\,\theta = \frac{(8M + 57m)}{6} r\alpha \; ; \sin\theta = \theta$$

$$\alpha = \left[\frac{6(Kr - 3mg)}{8M + 57m}\right] \theta$$

$$\omega = 2\pi n = \left[\frac{6(Kr - 3mg)}{8M + 57m}\right]^{\frac{1}{2}}$$

$$n = \frac{1}{2\pi}\sqrt{\frac{6(Kr - 3mg)}{8M + 57m}}$$

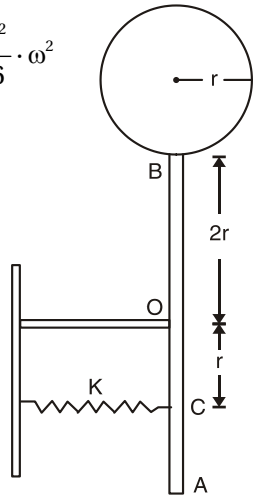


Fig. 7.50

7.80. One end of a thin light rod of length $L = 1$ m is attached to a small ball of mass $m = 500$ gm whereas its other end to a rigid support O. A spring of force constant $K = 700$ N/m is connected between the fixed point O' and the rod at a distance $l = 50$ cm below the point of suspension (see Fig. 7.51). The rod is displaced through a small angle from the vertical and released. Find the frequency of small oscillations.

Solution: In view of conservation of energy we have

$$mgL(1 - \cos\theta) + \frac{1}{2} Kl^2\theta^2 = \frac{1}{2} mv^2$$

On differentiation we have

$$mgL \sin\theta \cdot \omega + Kl^2 \theta \cdot \omega = mv \cdot a = m\omega \cdot L^2 \alpha$$

$$\therefore \alpha = \left(\frac{mgL + Kl^2}{mL^2}\right)\theta \; ; \sin\theta = \theta \text{ for small angles}$$

$$n = \frac{1}{2\pi}\sqrt{\frac{mgL + Kl^2}{mL^2}}$$

On substituting the values of various terms we have

$$n = \frac{1}{2 \times 3.14}\sqrt{\frac{0.5 \times 9.8 \times 1 + 700 \times 0.5^2}{0.5 \times 1}}$$

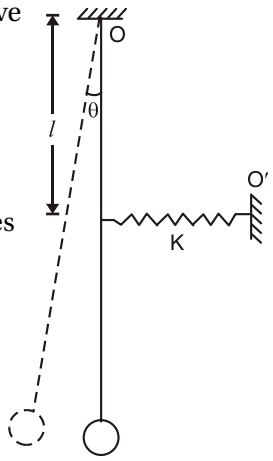


Fig. 7.51

$$= \frac{1}{2 \times 3.14} \sqrt{9.8 + 700 \times 0.5}$$

$$= \frac{1}{2 \times 3.14} \sqrt{359.8}$$

$$= 3 \text{ Hz}$$

7.81. A particle of mass m is attached to three springs of equal force constant K as shown in the Fig. 7.52. The particle is pushed through x against the spring A and released. Find the frequency of small oscillations.

Solution: The displacement for the two springs at right angles to each other is equal to $\frac{x}{\sqrt{2}}$

According to conservation of energy we have

$$\frac{1}{2} mv^2 - \frac{1}{2} K \frac{x^2}{2} - \frac{1}{2} K \frac{x^2}{2} = \frac{1}{2} Kx^2$$

or $\frac{1}{2} mv^2 = Kx^2$

On differentiation we have

$$m \cdot v \cdot a = 2 k x \cdot v$$

$$\therefore a = \frac{2K}{m} \cdot x$$

$$\omega = 2 \pi n = \sqrt{\frac{2K}{m}}$$

$$\therefore n = \frac{1}{2\pi} \sqrt{\frac{2K}{m}}$$

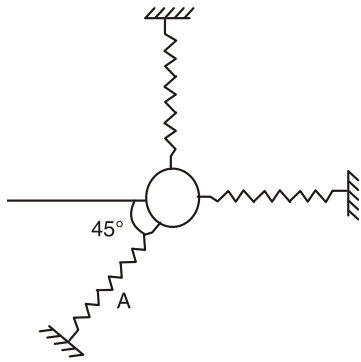


Fig. 7.52

7.82. In the arrangement shown in the Fig. 7.53 pulley of mass $m = 0.5$ kg and radius $r = 10$ cm is supported by a cord and a spring of force constant $K = 69$ N/m. A body of mass $M = 1$ kg is attached to the centre of the pulley. Assuming that the cord does not slip find the frequency of the small vertical oscillations.

Solution: Let the pulley rotates through an angle θ about its centre the energy equation of the system is given by

$$\frac{1}{2} \cdot \frac{1}{2} mr^2\omega^2 + \frac{1}{2} mv^2 + \frac{1}{2} Mv^2 = \frac{1}{2} Kr^2\theta^2$$

$$\frac{1}{2} mr^2\omega^2 + mr^2\omega^2 + Mr^2\omega^2 = K r^2 \theta^2 \; ; \; v = r\omega$$

$$\omega^2 [M + 1.5\, m] = K\theta^2$$

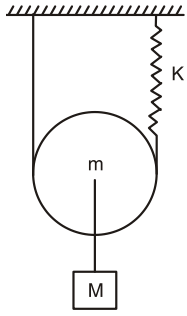


Fig. 7.53

On differentiation we have

$$\omega\,[M + 1.5m]\,\alpha = K \cdot \theta \cdot \omega$$

$$\alpha = \left(\frac{K}{M + 1.5m}\right)\theta$$

$$\pi n = \sqrt{\frac{K}{M + 1.5m}}$$

$$n = \frac{1}{\pi}\sqrt{\frac{K}{M + 1.5m}}$$

On substituting the values of various terms we have

$$n = \frac{1}{3.14}\sqrt{\frac{69}{1 + 1.5 \times 0.5}}$$

$$= 2 \text{ Hz}$$

7.83. A cylinder of mass m and radius r is contrained by a spring of force constant K to roll up and down an inclined plane as shown in the Fig. 7.54. Find the frequency of small oscillations of the system when there is no slipping.

Solution: Let the equilibrium position be at a distance l below the initial position along the slope. Let the cylinder be displaced through x along the plane.

The energy equation is given by

$$\frac{1}{2}mg\left[(l + x) - l\right] = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 + \frac{1}{2}K\left[(l + x)^2 - l^2\right]$$

$$mgx = \frac{mr^2\omega^2}{2} + mv^2 + K\left[2lx + x^2\right]$$

$$mgx = \frac{3}{2}mv^2 + K\left[2lx + x^2\right]$$

On differentiation we have

$$mg \cdot v = 3\,mv \cdot A + K \cdot 2l \cdot v + 2Kx\,v\,;\ A = \text{acceleration}$$

or $$mg = 3\,mA + K \cdot 2l + 2\,Kx$$

$$mg\cos 60 = Kl$$

$$\therefore A = -\frac{2K}{3m} \cdot x$$

$$2\pi n = \sqrt{\frac{2K}{3m}}$$

$$\therefore n = \frac{1}{2\pi}\sqrt{\frac{2K}{3m}} = \frac{1}{\pi}\sqrt{\frac{K}{6m}}$$

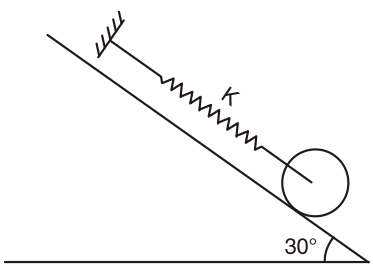


Fig. 7.54

7.84. A uniform solid cylinder of mass m and radius r rolls up and down an inclined plane by a spring of force constant K wraped round the cylinder as shown in the Fig. 7.55. The cylinder is released from the position where the spring is unstretched. Find the frequency of small oscillations assuming that there is no slipping.

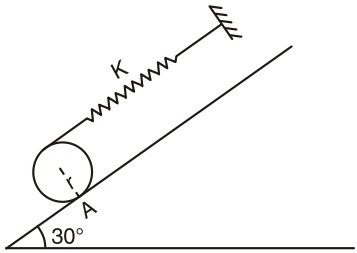


Fig. 7.55

Solution: Let the equilibrium position be at l below the initial position.

The stretched length of the spring = $2l$

Taking moment about A

$K \cdot 2l \cdot 2r = mg \sin 30 \cdot r$;

Spring force = $K \cdot 2l$

$$\therefore\ l = \frac{mg}{8K}$$

velocity $v = \omega \cdot l = \Omega r$

The energy equation of the system is given by

$$mg \cdot \frac{l}{2} = \frac{1}{2}mv^2 + \frac{1}{2}I\Omega^2 + \frac{1}{2}K(2l)^2$$

$$mg \cdot \frac{l}{2} = \frac{1}{2}m \cdot \omega^2 l^2 + \frac{1}{2}\frac{mr^2}{2} \cdot \frac{\omega^2 l^2}{r^2} + 2K \cdot l^2$$

$$mg \cdot \frac{l}{2} = \frac{3}{4}m \cdot \omega^2 l^2 + 2Kl^2$$

$$\frac{mg}{2} = \frac{3}{4}m\omega^2 l + 2K \cdot l = \frac{3}{4}m\omega^2 \cdot \frac{mg}{8K} + 2K \cdot \frac{mg}{8K}$$

or $$\frac{1}{2} = \frac{3m}{32K}\omega^2 + \frac{1}{4}$$

or $$\frac{1}{4} = \frac{3m}{32K}\omega^2$$

or $$\omega = 2\pi n = \sqrt{\frac{8K}{3m}}$$

or $$n = \frac{1}{2\pi}\sqrt{\frac{8K}{3m}}$$

$$= \frac{1}{\pi}\sqrt{\frac{2K}{3m}}$$

7.85. A conducting rod of mass m and length l can side without friction on two smooth parallel rails PQ and RS. A spring of force constant K is fixed between the mid-point of the rod and a rigid support as shown in Fig. 7.56. The ends Q and R of the two rails are connected to a capacitor of

capacity C. The complete arrangement is placed in a magnetic field of induction B. When the rod is slightly displaced to the left and released its starts oscillating. Find the period of oscillation.

Solution: Let the rod moves through dx in time dt then the induced

$$emf = Bl \cdot \frac{dx}{dt} = Bvl$$

Charge q on the capacitor $= Bvlc$

During its motion towards left the energy equation is

$$\frac{1}{2}\frac{q^2}{c} + \frac{1}{2}mv^2 = -\frac{1}{2}Kx^2$$

or $(B^2l^2c + m)\, v^2 = -Kx^2$

On differentiation we have

$$(B^2l^2c + m)a = -Kx$$

$$\therefore \omega_1 = \sqrt{\frac{K}{B^2l^2c + m}}$$

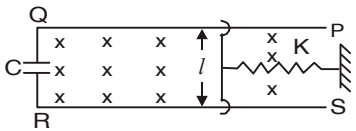


Fig. 7.56

With this angular frequency it will cover only half the period and time t_1 is given by

$$t_1 = \frac{\pi}{\omega_1} = \pi\left(\frac{B^2l^2c + m}{K}\right)^{\frac{1}{2}}$$

During its motion toward right the energy equation is

$$\frac{1}{2}\frac{q^2}{c} = \frac{1}{2}mv^2 + \frac{1}{2}Kx^2$$

or $(m - B^2l^2c)\, v^2 = -Kx^2$

On differentiation we have

$$(m - B^2l^2c)\, a = -Kx$$

$$\therefore \omega_2 = \sqrt{\frac{K}{(m - B^2l^2c)}}$$

Time t_2 for the rest half cycle $= \frac{\pi}{\omega_2} = \pi\left(\frac{m - B^2l^2c}{K}\right)^{\frac{1}{2}}$

$$T = \pi\left[\left(\frac{B^2l^2c + m}{K}\right)^{\frac{1}{2}} + \left(\frac{m - B^2l^2c}{K}\right)^{\frac{1}{2}}\right]$$

7.86. A spring of mass m lies on a smooth table. One end of the spring is fixed to the rigid support and the other end is pulled with a velocity. Calculate the kinetic energy of the spring.

Solution: Consider an element of length dx lying at a distance x from the fixed point O. (see Fig. 7.57)

K.E. of the element $= \frac{1}{2} \cdot \frac{m}{l} \cdot dx \left(\frac{x}{l} \cdot v \right)^2$;

l = length of the spring

Total kinetic energy $E = \frac{m}{2l^3} \cdot v^2 \cdot \int_0^l x^2 dx$

$= \frac{1}{6} mv^2$

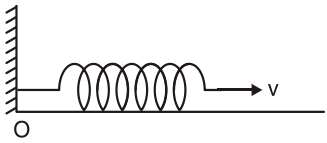


Fig. 7.57

7.87. A cylinder of mass m, radius r and height h, suspended by a spring whose upper end is fixed by a rigid support, is submerged into water. The force constant of the spring is K. In equilibrium the length of the cylinder inside the water is $\frac{h}{2}$. On putting an external mass on the cylinder it has been found that it has submerged to $\frac{2}{3}$ of its height. On removing the external mass the cylinder starts moving vertically without any initial velocity. Find the equation of motion of the cylinder.

Solution: Let the cylinder be displaced by y downwards,

Upward thrust $= \pi r^2 y \rho g$

Force due to spring $= Ky$

Net force $= m \cdot$ acceleration $= -[K + \pi r^2 \rho g] y$

This is the equation of simple harmonic motion.

$\omega = \left(\frac{K + \pi r^2 \rho g}{m} \right)^{\frac{1}{2}}$

amplitude of oscillation $= \frac{2h}{3} - \frac{h}{2} = \frac{h}{6}$

$\therefore y = \frac{h}{6} \cos \omega t$ is the equation of motion of the cylinder

7.88. A cylinder of mass m, radius r and height h, suspended by a spring whose upper end is fixed is submerged in water. In the conditions of equilibrium the cylinder sinks to $\frac{1}{2}$ of its height (see Fig. 7.58). At a certain moment the cylinder was submerged to $\frac{2}{3}$ of its height and then with no initial velocity started to move vertically. Taking the force constant of spring as K, density of water as ρ, resistive force of water as $-\alpha v$, find the equation of motion of the cylinder.

Solution: Let the cylinder be displaced through y downwards, then

$$\frac{m dv}{dt} + \left(K + \pi r^2 \rho g \right) y + \alpha v = 0$$

or $\frac{dv}{dt} + p^2 y + 2\sigma v = 0; p = \sqrt{\frac{K + \pi r^2 \rho g}{m}}, \sigma = \frac{\alpha}{2m}$ (i)

Let the solution of the equation be $y = B_1 e^{at} \sin \omega t + B_2 e^{at} \cos \omega t$

$v = B_1 e^{at} \cdot a \sin \omega t + B_1 e^{at} \cdot \omega \cdot \cos \omega t + B_2 e^{at} \cdot a \cdot \cos \omega t - B_2 e^{at} \omega \sin \omega t$

$$\frac{dv}{dt} = B_1 a^2 e^{at} \sin \omega t + B_1 a \omega e^{at} \cos \omega t + B_1 a \omega e^{at} \cos \omega t - B_1 \omega^2 e^{at} \sin \omega t + B_2 a^2 e^{at} \cos \omega t - B_2 a \omega e^{at} \sin \omega t - B_2 a \omega e^{at} \sin \omega t - B_2 e^{at} \cdot \omega^2 \cos \omega t$$

On substituting the values of y, v and $\frac{dv}{dt}$ in equation (i) and equating the coefficient of $\sin \omega t = 0$ we have

$B_1 [a^2 - \omega^2 + p^2 + 2\sigma a] - B_2 \cdot 2\omega (a + \sigma) = 0$

Since B_1 and B_2 are not equal to zero hence

$a = -\sigma$ and $\omega = \sqrt{p^2 - \sigma^2}$

$At = 0\ y = \frac{h}{6}$ (amplitude of oscillation) and

$v = 0$

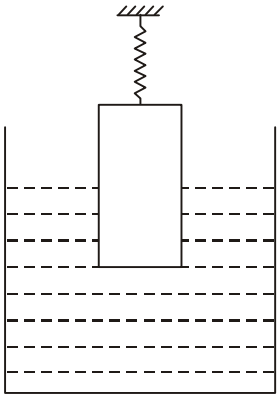


Fig. 7.58

hence $\frac{h}{6} = B_2$ and $B_1 \omega + B_2 a = 0$

$\therefore B_1 = \frac{\sigma}{\omega} \cdot \frac{h}{6}$

$$y = \frac{h}{6}\left[\frac{\sigma}{\omega} e^{-\sigma t} \sin \sqrt{p^2 - \sigma^2} \cdot t + e^{-\sigma t} \cdot \cos \sqrt{p^2 - \sigma^2} \cdot t\right]$$

$$= \frac{h}{6} \cdot \frac{p}{\sqrt{p^2 - \sigma^2}} \cdot e^{-\sigma t}\left[\sin \sqrt{p^2 - \sigma^2} \cdot t \cdot \frac{\sigma}{p} + \cos \sqrt{p^2 - \sigma^2} \cdot t \cdot \frac{\sqrt{p^2 - \sigma^2}}{p}\right]$$

$$= \frac{h}{6} \cdot \frac{p}{\sqrt{p^2 - \sigma^2}} \cdot e^{-\sigma t}\left[\sin\left(\sqrt{p^2 - \sigma^2} \cdot t + \phi\right)\right]; \tan \phi = \frac{\sqrt{p^2 - \sigma^2}}{\sigma}$$

7.89. A body of mass m falls from a height h on a pan hanging from a spring. The body after falling through a height h sticks to the pan. The body together with the pan starts oscillating. Find (i) the amplitude of oscillation (ii) the frequency of oscillation (iii) the total energy of the system.

Solution:

(i) Let the spring moves downward through x_0 then

$$\frac{1}{2} m v^2 = -mg x_0 + \frac{1}{2} K x_0^2$$

or $K x_0^2 - 2\,mgx_0 - mv^2 = 0$

$$x_0 = \frac{2mg \pm \sqrt{4m^2g^2 + 4Kmv^2}}{2K}$$

$$= \frac{mg}{K} \pm \left(\sqrt{1 + \frac{v^2K}{mg^2}}\right)\frac{mg}{K}$$

$$= \frac{mg}{K} \pm \frac{mg}{K}\sqrt{1 + \frac{2Kh}{mg}}\,;\; v = \sqrt{2gh}$$

$$x_0 = \frac{mg}{K}\left[1 + \sqrt{1 + \frac{2Kh}{mg}}\right]$$

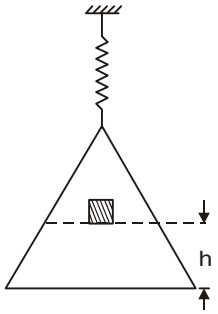


Fig. 7.59

Let the mass together with pan rests at a distance x'_0

then $x'_0 = \frac{mg}{K}$

Amplitude $= x_0 - x'_0 = \frac{mg}{K}\sqrt{1 + \frac{2Kh}{mg}}$

(ii) $\omega = 2\pi n = \sqrt{\frac{K}{m}}$

$$n = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

$$E = \frac{1}{2}m \cdot \frac{m^2g^2}{K^2}\left(1 + \frac{2Kh}{mg}\right)\frac{K}{m} = \frac{1}{2}\frac{m^2g^2}{K}\left(1 + \frac{2Kh}{mg}\right)$$

7.90. A thin bar of mass m and length L is hinged at O. The bar is kept horizontal with the help of a spring of force constant K attached at the end A (see Fig. 7.60). If the end B of the bar is pressed downwards and released it starts rotational oscillation. Find the period of oscillation.

Solution:

Moment of inertia of the bar about $O = \frac{m}{3L}\left[b^3 + (L-b)^3\right]$

Equation of motion

$$\frac{m}{3L}\left[b^3 + (L-b)^3\right]\alpha = -K \cdot b\theta \cdot b$$

$$= -Kb^2\theta$$

$$\omega = \sqrt{\frac{3Kb^2L}{m\left[b^3 + (L-b)^3\right]}}$$

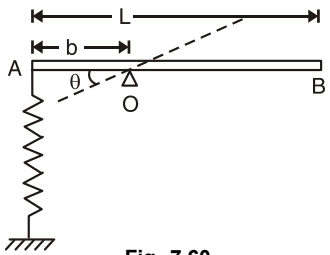


Fig. 7.60

$$T = 2\pi\sqrt{\frac{m\left[b^3 + (L-b)^3\right]}{3Kb^2L}}$$

7.91. A bead of mass m slide on a frictionless wire bent in the form of cycloid with equation

$$x = a(\theta - \sin\theta) \text{ and } y = a(1 + \cos\theta)$$

Find the period of oscillation.

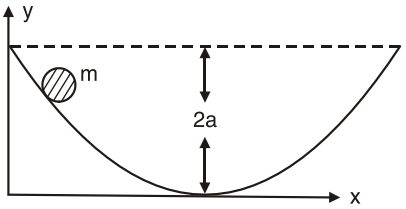


Fig. 7.61

Solution:

$$\text{Kinetic energy} = \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]$$

$$= \frac{1}{2}ma^2\left[(\omega - \cos\theta\cdot\omega)^2 + (-\omega\sin\theta)^2\right]$$

$$= ma^2\omega^2(1 - \cos\theta)$$

Potential energy $= mgy = mga(1 + \cos\theta)$

According to conservation of energy

$ma^2\omega^2(1 - \cos\theta) + mga(1 + \cos\theta) = \text{Constant}$

On differentiation we have

$$ma^2\cdot 2\omega\cdot\ddot{\theta}\ (1-\cos\theta) + ma^2\omega^2\sin\theta\cdot\dot{\theta} - mga\sin\theta\cdot\dot{\theta} = 0$$

$$(1-\cos\theta)\ddot{\theta} + \frac{1}{2}\sin\theta\cdot\dot{\theta}^2 - \frac{g}{2a}\sin\theta = 0$$

Let us substitute $u = \dfrac{\cos\theta}{2}$

$$\frac{du}{dt} = -\frac{1}{2}\cdot\frac{\sin\theta}{2}\cdot\frac{d\theta}{dt}$$

$$\frac{d^2u}{dt^2} = -\frac{1}{2}\left[\frac{1}{2}\cos\frac{\theta}{2}\cdot\left(\frac{d\theta}{dt}\right)^2 + \sin\frac{\theta}{2}\cdot\frac{d^2\theta}{dt^2}\right]$$

$$= -\frac{1}{2}\sin\frac{\theta}{2}\cdot\frac{d^2\theta}{dt^2} - \frac{1}{4}\cos\frac{\theta}{2}\cdot\left(\frac{d\theta}{dt}\right)^2$$

The value of $\frac{d^2u}{dt^2} + \frac{gu}{4a} = 0$

$$-\frac{1}{2}\sin\frac{\theta}{2}\cdot\frac{d^2\theta}{dt^2} - \frac{1}{4}\cos\frac{\theta}{2}\cdot\left(\frac{d\theta}{dt}\right)^2 + \frac{g}{4a}\cdot\cos\frac{\theta}{2}$$

$$\ddot{\theta} + \frac{1}{2}\cot\frac{\theta}{2}\cdot\dot{\theta}^2 - \frac{g}{2a}\cos\frac{\theta}{2} = 0$$

$$\ddot{\theta} + \frac{1}{2}\frac{\sin\theta}{1-\cos\theta}\cdot\dot{\theta}^2 - \frac{g}{2a}\cdot\frac{\sin\theta}{1-\cos\theta} = 0; \text{ Put } \cot\frac{\theta}{2} = \frac{\sin\theta}{1-\cos\theta}$$

or $$(1-\cos\theta)\ddot{\theta} + \frac{1}{2}\sin\theta\cdot\dot{\theta}^2 - \frac{g}{2a}\sin\theta = 0$$

The required equations is therefore

$$\frac{d^2u}{dt^2} + \frac{g}{4a}\cdot u = 0$$

or $$\frac{d^2u}{dt^2} = -\frac{g}{4a}\cdot u$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{g}{4a}}$$

$$T = 4\pi\sqrt{\frac{a}{g}}$$

7.92. A body of mass m is attached to the end A of a massless beam of length l through a spring of force constant K_2. The other end of the beam is hinged at O. The beam is kept horizontal with the help of spring of force constant K_1 (see Fig. 7.62a). If the end A is pressed a little in the downward direction and released it starts oscillating. Find the natural frequency of oscillation.

Solution: Consider a hypothetical spring of force constant K situated at A above the spring K_2. After the end A depressed let the beam makes an

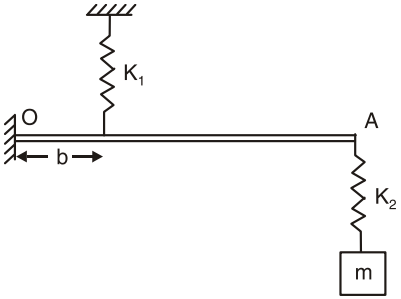


Fig. 7.62a

Fig. 7.62b

angle θ with the horizontal. The forces acting on the springs of force constants K_1 and K are $K_1 b\,\theta$ and $K\,l\,\theta$ respectively. If we take the moment about O in order to know an equivalent torque then

$$K_1 \cdot b\,\theta \cdot b = K \cdot l\,\theta \cdot l$$

or $$K = \frac{b^2 K_1}{l^2}$$

Thus a hypothetical spring of force constant K placed at A will produce the same torque as the spring of force constant K_1.

Now we have the situation as shown in Fig. 7.62b. The two springs are stretched by the same force.

$$\text{Net elongation } y = \frac{F}{K} + \frac{F}{K_2}$$

$$y = F\left[\frac{K_2 l^2 + K_1 b^2}{K_1 K_2 b^2}\right]$$

$$\text{Now} \quad m \times \text{acceleration} = -\frac{K_1 K_2 b^2 \cdot y}{K_2 l^2 + K_1 b^2}$$

$$\text{acceleration} = -\frac{K_1 K_2}{m\left[K_1 + K_2\left(\frac{l}{b}\right)^2\right]} \cdot \text{displacement}$$

$$\omega = 2\pi n = \sqrt{\frac{K_1 K_2}{m\left[K_1 + K_2\left(\frac{l}{b}\right)^2\right]}}$$

or $$n = \frac{1}{2\pi}\sqrt{\frac{K_1 K_2}{m\left[K_1 + K_2\left(\frac{l}{b}\right)^2\right]}}$$

7.93. A massless rod AB, three springs of force constants, K_1, K_2 & K_3 and a block of mass m are arranged as shown in the Fig. 7.63. The end A of the rod is pressed a little and released find the angular frequency of small oscillations.

Solution: Consider a hypothetical spring of force constant K at O. Let the beam be depressed at A so that it forms an angle θ with the horizontal.

Taking the moment about B

$$K_3 \cdot L\,\theta \cdot L = K \cdot \frac{L}{2}\theta \cdot \frac{L}{2}$$

or $K = 4K_3$

This hypothetical spring will now form a series combination with K_2. Equivalent force constant K' is given by

$$K' = \frac{4K_2K_3}{K_2 + 4K_3}$$

The spring of force constant K' will be in parallel combination with K_1. The equivalent force constant K_{eq} is given by

$$K_{eq} = \left(K_1 + \frac{4K_2K_3}{K_2 + 4K_3} \right)$$

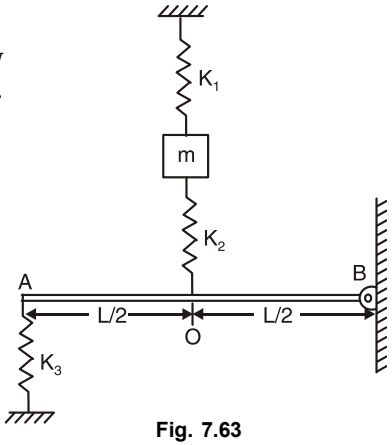


Fig. 7.63

The complete system reduces to a block of mass m attached to a spring of force constant K_{eq}

The equation of motion is given by

$m \cdot \text{acceleration} = -K_{eq} \cdot \text{displacement}$

$$\omega = \sqrt{\frac{K_{eq}}{m}}$$

$$= \sqrt{\frac{1}{m}\left(K_1 + \frac{4K_2K_3}{K_2 + 4K_3} \right)}$$

7.94. A block of mass m connected to a spring of force constant K slides down a frictionless incline wedge of angle θ and mass M. The wedge lies on a frictionless surface. Find the angular frequency of wedge and block system.

Solution: Let l be the position of the block at an instant t and x be the position of the centre of mass.

The acceleration along the plane is given by

$$\frac{d^2l}{dt^2} - \frac{d^2x}{dt^2} \cdot \cos\theta - \frac{Kl}{m}$$

Net force along the horizontal is zero. Net momentum along the horizontal is also zero.

$$(M + m)\frac{dx}{dt} - m\cos\theta \cdot \frac{dl}{dt} = 0$$

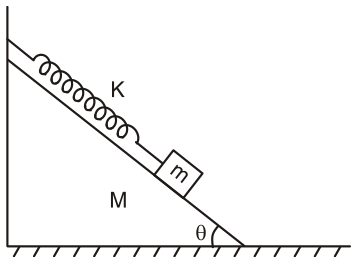


Fig. 7.64

On differentiation

$$(M+m)\frac{d^2x}{dt^2} = m\cos\theta\cdot\frac{d^2l}{dt^2}$$

$$\frac{d^2l}{dt^2} - \frac{m\cdot\cos^2\theta}{(M+m)}\cdot\frac{d^2l}{dt^2} - \frac{Kl}{m} = 0$$

or $$\frac{d^2l}{dt^2}\left[M + m\sin^2\theta\right] = \frac{K(M+m)}{m}\cdot l$$

or $$\frac{d^2l}{dt^2} = \frac{K(M+m)}{m\left[M+m\sin^2\theta\right]}\cdot l$$

$$\omega = 2\pi n = \sqrt{\frac{K(M+m)}{m\left[M+m\sin^2\theta\right]}}$$

7.95. A massless rod, a block of mass m and a spring of force constant K are arranged as shown in the Fig. 7.65. Find the frequency of small oscillations.

Solution: Let the spring be pulled downwards so that the rod makes an angle θ at O then the energy of the system is given by

$$\frac{1}{2}K(l_1\theta)^2 + \frac{1}{2}m\left[\frac{d}{dt}(l_2\theta)\right]^2 = \text{constant}$$

or $$Kl_1^2\theta^2 + m\cdot l_2^2\left(\frac{d\theta}{dt}\right)^2 = \text{constant}$$

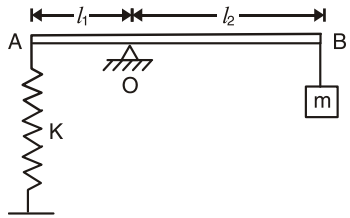


Fig. 7.65

On differentiation we have

$$2Kl_1^2\theta\cdot\omega + 2ml_2^2\cdot\omega\alpha = 0$$

or $$\alpha = -\frac{K}{m}\cdot\frac{l_1^2}{l_2^2}\cdot\theta$$

or $$\omega = \frac{l_1}{l_2}\sqrt{\frac{K}{m}}$$

7.96. A massless bar AB of length L, three springs of force constant K_1, K_2 and K_3 and a block of mass m are arranged as shown in the Fig. 7.66. Find the natural frequency of oscillation of the system.

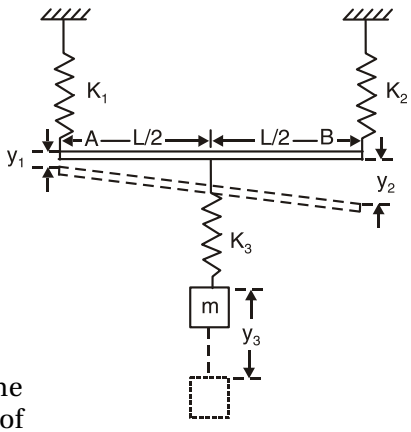


Fig. 7.66

Solution: Let y_1 and y_2 be the displacements at the two ends A and B of the bar. These displacements are due to

extensions of springs of force constant K_1 and K_2 respectively. Let y_3 be the displacement due to the spring of force constant K_3.

$$y_3 = \frac{mg}{K_3},\ y_2 = \frac{mg/2}{K_2} \text{ and } y_1 = \frac{mg/2}{K_1}$$

$$\text{The total static displacement } \Delta = y_3 + \frac{y_1 + y_2}{2}$$

$$= mg\,\frac{\left[4K_1K_2 + K_1K_3 + K_2K_3\right]}{4K_1K_2K_3}$$

$$\text{Frequency of oscillation } n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{4K_1K_2K_3}{4K_1K_2 + K_1K_3 + K_2K_3}}$$

7.97. A massless pulley *P*, two springs of force constants K_1 and K_2 and a block of mass *m* is arranged as shown in the Fig. 7.67. Find the period of small oscillations of mass *m* about the equilibrium position.

Solution: Equation of motion of mass *m*.

$$\frac{md^2x}{dt^2} = -K_1x_1; \qquad x_1 = \text{stretch of spring of force constant } K_1$$

$$T = K_1x_1 \qquad T = \text{tension}$$

$$K_2x_2 = 2T = 2K_1x_1; \quad x_2 = \text{stretch of spring of force constant } K_2$$

$$x_2 = \frac{2K_1}{K_2}\cdot x_1$$

Now $$x = x_1 + 2x_2 = \left(1 + \frac{4K_1}{K_2}\right)x_1$$

$$\frac{md^2x}{dt^2} = -K_1x_1 = -\frac{K_1K_2}{4K_1 + K_2}\cdot x$$

$$\omega = \frac{2\pi}{t} = \sqrt{\frac{K_1K_2}{m\left(4K_1 + K_2\right)}}\ ;\ t = \text{time period}$$

$$\therefore t = 2\pi\sqrt{\frac{m\left(4K_1 + K_2\right)}{K_1K_2}}$$

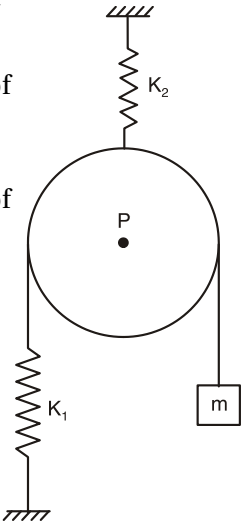


Fig. 7.67

7.98. A steel ball of mass *m* is projected vertically upward with a velocity v_0 from a perfectly elastic horizontal plane. If *h* be the height of the spring from the horizontal surface and *K* be the force constant of the spring, find the period of vertical oscillations of the ball.

Solution: When the ball is going up then we have

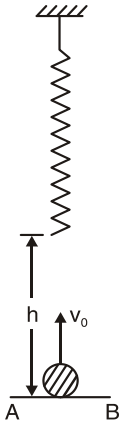


Fig. 7.68

$$\frac{1}{2}mv_0^2 = mgh + mgx + \frac{1}{2}Kx^2;\ x = \text{compression in the spring}$$

$$Kx^2 + 2mgx - m\left(v_0^2 - 2gh\right) = 0$$

$$x = \sqrt{\frac{m^2g^2}{K^2} + \frac{m}{K}\left(v_0^2 - 2gh\right)} - \frac{mg}{K}$$

$$h = v_0 t_1 - \frac{1}{2}g t_1^2$$

or $\quad gt_1^2 - 2v_0 t_1 + 2h = 0$

$$t_1 = \frac{v_0 \pm \sqrt{v_0^2 - 2gh}}{g}$$

$$\frac{1}{2}mv_0^2 = mgh + mgy + \frac{1}{2}Ky^2 + \frac{1}{2}mv^2;$$

v = velocity of the ball at a compression y

$$v_0^2 - 2gh - 2gh - \frac{K}{m}y^2 = v^2$$

$$\sqrt{\frac{K}{m}}\left[\frac{m}{K}\left(v_0^2 - 2gh\right) + \frac{m^2g^2}{K^2} - \left(\frac{mg}{K} + y\right)^2\right]^{-\frac{1}{2}} = \frac{dy}{dt}$$

$$\sqrt{\frac{K}{m}}\int_0^{t_2} dt = \int_0^x \frac{dy}{\left[\frac{m}{K}\left(v_0^2 - 2gh\right) + \frac{m^2g^2}{K^2} - \left(\frac{mg}{K} + y\right)^2\right]^{\frac{1}{2}}}$$

$$\sqrt{\frac{K}{m}}\,t_2 = \frac{\pi}{2} - \sin^{-1}\frac{\frac{mg}{K}}{\sqrt{\frac{m}{K}\left(v_0^2 - 2gh\right) + \frac{m^2g^2}{K^2}}}$$

During descend the equation of motion of the ball is given by

$$\frac{mdv}{dt} = mg + K(x - y_1);\ y_1 \text{ is the opening in the spring}$$

$$m \cdot \frac{dv}{dy_1} \cdot v = (mg + Kx) - Ky_1$$

$$m\int_0^V v\,dv = \int_0^{y_1}(mg + Kx)\,dy_1 - \int_0^{y_1} Ky_1 dy_1$$

$$mV^2 = 2y_1\left[m^2g^2 + mK\left(v_0^2 - 2gh\right)\right]^{\frac{1}{2}} - Ky_1^2$$

$$\frac{dy_1}{dt} = \sqrt{2y_1\left[g^2 + \frac{K}{m}\left(v_0^2 - 2gh\right)\right]^{\frac{1}{2}} - \frac{K}{m}\cdot y_1^2}$$

$$\int_0^x \frac{dy_1}{\sqrt{2y_1\frac{m}{K}\left[g^2+\frac{K}{m}\left(v_0^2-2gh\right)\right]^{1/2}-y_1^2}}=\sqrt{\frac{K}{m}}\int_0^{t_3}dt$$

$$\left|\sin^{-1}\frac{y_1-\frac{m}{K}\left[g^2+\frac{K}{m}\left(v_0^2-2gh\right)\right]^{1/2}}{\frac{m}{K}\left[g^2+\frac{K}{m}\left(v_0^2-2gh\right)\right]^{1/2}}\right|_0^x=\sqrt{\frac{K}{m}}\cdot t_3$$

$$t_3=\left[\frac{\pi}{2}-\sin^{-1}\frac{\frac{mg}{K}}{\left[\frac{m^2g^2}{K^2}-\frac{m}{K}\left(v_0^2-2gh\right)\right]^{1/2}}\right]\sqrt{\frac{m}{K}}$$

Free descend time $t_4 = t_1$

$$T=2\left[\frac{v_0\pm\sqrt{v_0^2-2gh}}{g}+\sqrt{\frac{m}{K}}\left(\frac{\pi}{2}-\sin^{-1}\frac{\frac{mg}{K}}{\left[\frac{m^2g^2}{K^2}-\frac{m}{K}\left(v_0^2-2gh\right)\right]^{1/2}}\right)\right]$$

7.99. A solid cylinder of mass m is kept in equilibrium on a horizontal surface with the help of two springs of force constants K_1 and K_2 as shown in the Fig. 7.69. Find the frequency of small oscillations.

Solution: Let the cylinder be rotated about an axis passing through O and perpendicular to the plane of paper.

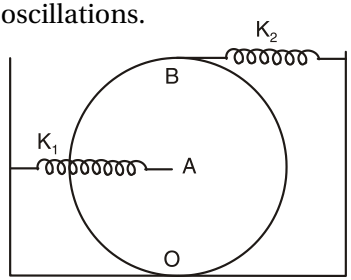


Fig. 7.69

Linear displacement of A and B respectively = $R\theta$ and $2R\theta$

where R = radius of the cylinder

$$v_A = v = \frac{Rd\theta}{dt},\ v_B = 2v = 2\frac{Rd\theta}{dt}$$

Potential energy of the spring of force constant $K_1 = \frac{1}{2}K_1R^2\theta^2$

Potential energy of the spring of force constant $K_2 = \frac{1}{2}K_2\cdot 4R^2\theta^2$

Moment of inertia of the cylinder about an axis passing through O

$$= \frac{3}{2}mR^2$$

K.E. of cylinder $= \frac{1}{2} \cdot \frac{3}{2} mR^2\omega^2 = \frac{3}{4} mv^2$

According to the conservation of energy

$$\frac{3}{4} mv^2 + \frac{1}{2} K_1 R^2\theta^2 + \frac{1}{2} K_2 R^2 \cdot 4\theta^2 = \text{constant}$$

or $$\frac{3}{2} mv \cdot A + K_1 R^2\theta \cdot \omega + 4K_2 R^2\theta \cdot \omega = 0$$

$$\frac{3}{2} mA = -[K_1 + 4K_2]\, R\theta$$

or $$\frac{3}{2} m \cdot R\alpha = -[K_1 + 4K_2]\, R\theta$$

$$\alpha = \frac{-2(K_1 + 4K_2)}{3m} \cdot \theta$$

$$\omega = 2\pi n = \sqrt{\frac{2(K_1 + 4K_2)}{3m}}$$

or $$n = \frac{1}{2\pi} \sqrt{\frac{2(K_1 + 4K_2)}{3m}}$$

7.100. A ball of mass m, a massless rod of length L and two springs of force constant K are arranged in the Figs. 7.70a and 7.70b. Find the frequency of small oscillations in the two cases.

Solution: In Fig. 7.70a we have the energy conservation as

$$\frac{1}{2} Kl^2\theta^2 + mgL(1 - \cos\theta) = \frac{1}{2} mL^2\omega^2 - \frac{1}{2} Kl^2 \cdot \theta^2$$

; θ is the angle made by the rod from the vertical

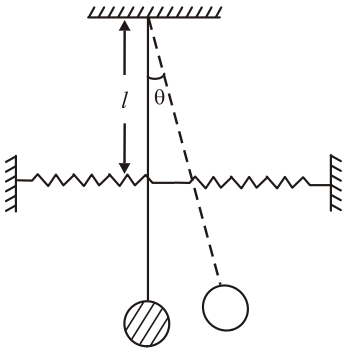


Fig. 7.70a

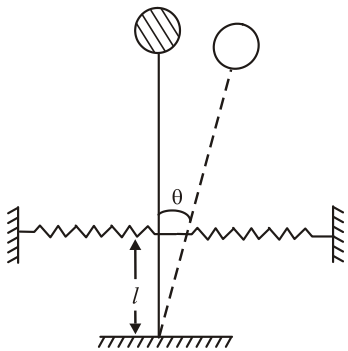


Fig. 7.70b

On differentiation we have

$$Kl^2\theta\cdot\omega + mgL\sin\theta\cdot\omega = mL^2\cdot\omega\,\alpha - Kl^2\cdot\theta\cdot\omega$$

$$\alpha = \frac{mgL+2Kl^2}{mL^2}\theta\,;\ \sin\theta=\theta \text{ for small angles}$$

$$\omega_1 = 2\pi n_1 = \sqrt{\frac{mgL+2Kl^2}{mL^2}}$$

or $$n_1 = \frac{1}{2\pi}\sqrt{\frac{2K}{m}\left(\frac{l}{L}\right)^2+\frac{g}{L}}$$

In Fig. 7.70b we have the energy equation as

$$\frac{1}{2}Kl^2\theta^2 - mgL(1-\cos\theta) = \frac{1}{2}mL^2\omega^2 - \frac{1}{2}Kl^2\theta^2$$

On differentiation we have

$$Kl^2\theta\cdot\omega - mgL\cdot\theta\cdot\omega = mL^2\,\omega\,\alpha - Kl^2\,\theta\cdot\omega;\ \sin\theta=\theta$$

$$\alpha = \frac{2Kl^2-mgL}{mL^2}\cdot\theta$$

$$\omega_2 = 2\pi n_2 = \sqrt{\frac{2K}{m}\cdot\left(\frac{l}{L}\right)^2-\frac{g}{L}}$$

or $$n_2 = \frac{1}{2\pi}\sqrt{\frac{2K}{m}\left(\frac{l}{L}\right)^2-\frac{g}{L}}$$

7.101. Two simple pendulums of equal length L are attached through a spring of force constant K as shown in the Fig. 7.71a. Determine the frequency of small oscillation and solve the equation of motion by assuming that at the initial instant of time one pendulum was deflected through an angle ϕ_0 from the vertical.

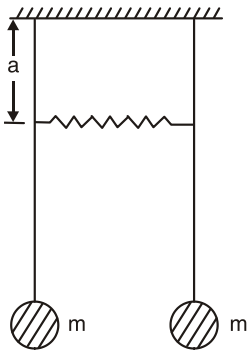


Fig. 7.71a

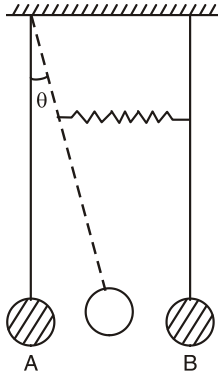


Fig. 7.71b

Solution: Let the pendulum *A* (Fig. 7.71b) be deflected through an angle θ from the vertical. The energy equation is given by

$$\frac{1}{2} Ka^2\theta^2 + mgL(1-\cos\theta) = \frac{1}{2} mL^2\omega^2 - \frac{1}{2} Ka^2\theta^2$$

$$\text{or} \quad Ka^2\theta^2 + mgL(1-\cos\theta) = \frac{1}{2} mL^2\omega^2$$

On differentiation we have

$$Ka^2 \cdot 2\theta \cdot \omega + mgL \cdot \theta \cdot \omega = mL^2 \omega \cdot \alpha; \sin\theta = \theta$$

$$\text{or} \quad \alpha = \frac{mgL + 2Ka^2}{mL^2}\theta$$

$$\omega_1 = \sqrt{\frac{g}{L} + \frac{2Ka^2}{mL^2}} = \sqrt{\frac{g}{L} + \frac{Ka^2}{mL^2} + \frac{Ka^2}{mL^2}}$$

$$= \sqrt{\omega_0^2 + \frac{Ka^2}{mL^2}}$$

where $\omega_0^2 = \frac{g}{L} + \frac{Ka^2}{mL^2}$

For the pendulum *B* we have

$$-\frac{1}{2} Ka^2\theta^2 + mgL(1-\cos\theta) = \frac{1}{2} mL^2\omega^2 - \frac{1}{2} Ka^2\theta^2$$

On differentiation we have

$$-Ka^2\theta + mgL\theta = mL^2\alpha - Ka^2\theta$$

$$\alpha = \left(\frac{mgL + Ka^2 - Ka^2}{mL^2}\right)\theta$$

$$\omega_2 = \sqrt{\frac{g}{L} + \frac{Ka^2}{mL^2} - \frac{Ka^2}{mL^2}}$$

$$= \sqrt{\omega_0^2 - \frac{Ka^2}{mL^2}}$$

$$\phi_1 = \frac{\phi_0}{2}[\cos\omega_1 t + \cos\omega_2 t], \phi_2 = \frac{\phi_0}{2}[\cos\omega_1 t - \cos\omega_2 t]$$

7.102. A rectangular lamina *ABCD* of length *x* and breadth *y* and mass *m* is pivoted at *B*. The end *A* is attached to a spring of force constant *K* as shown in the figure. If the end *A* is pressed a little downward and released it starts oscillating. Find the period of small oscillations.

Solution: On pressing the end *A* let *AB* makes an angle θ with the horizontal.

Torque about $B = Kx\theta \cdot x$

Moment of inertia of the lamina about

$$B = \frac{m}{3}\left(x^2 + y^2\right)$$

$$\frac{m}{3}\left(x^2 + y^2\right)\alpha = -Kx^2\theta$$

$$\omega = \sqrt{\frac{3Kx^2}{m\left(x^2 + y^2\right)}}$$

$$T = 2\pi\sqrt{\frac{m\left(x^2 + y^2\right)}{3Kx^2}}$$

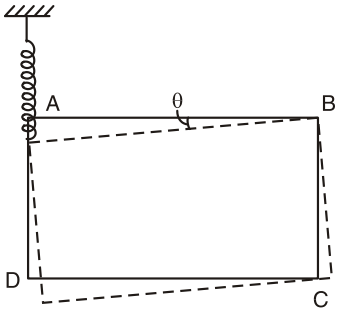


Fig. 7.72

7.103. A rod *AB* of length *L* and mass *m* is pivoted at its mid-point *O*. It is kept horizontal with the help of two identical springs of force constant *K*. The end *A* of the rod is pressed downward and released find the frequency of small oscillations.

Solution: Let θ be the angle which the rod makes with the horizontal.

Force acting on the spring = $K \cdot \frac{L}{2}\theta$

On taking the moment about *O* we have

$$\frac{mL^2}{12} \cdot \alpha = -K \cdot \frac{L}{2} \cdot \theta \cdot \frac{L}{2} \times 2$$

or $\alpha = -\frac{6K}{m} \cdot \theta$

$$\omega = 2\pi n = \sqrt{\frac{6K}{m}}$$

or $n = \frac{1}{2\pi}\sqrt{\frac{6K}{m}}$

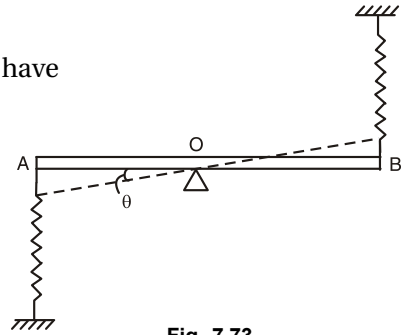


Fig. 7.73

7.104. The equal masses *m* are connected by springs having force constant *K* (see Fig. 7.74a). The masses are free to slide on the frictionless table *AB*. Find the differential equation and its solution.

Solution: Let x_1 and x_2 be the displacements of masses *P* & *Q* respectively.

Force on mass $P = m\frac{d^2x_1}{dt^2} = K(x_2 - x_1) - Kx_1$

Force on mass $Q = m\frac{d^2x_2}{dt^2} = K(x_1 - x_2) - Kx_2$

or $m\frac{d^2x_1}{dt^2} = K(x_2 - 2x_1)$ and $m\frac{d^2x_2}{dt^2} = K(x_1 - 2x_2)$

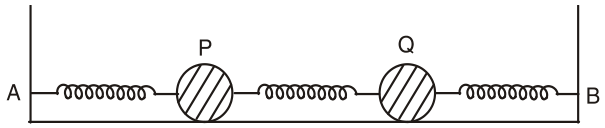


Fig. 7.74a

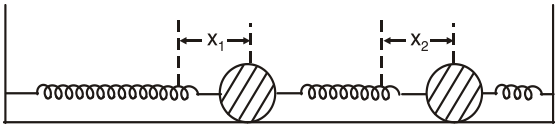


Fig. 7.74b

To solve these equations let $x_1 = A_1 \cos \omega t$ and $x_2 = A_2 \cos \omega t$ then

$-m A_1 \omega^2 \cos \omega t = K(A_2 - 2A_1) \cos \omega t;$

$-mA_2 \omega^2 \cdot \cos \omega t = K(A_1 - 2A_2) \cos \omega t$

$(2K - m\omega^2) A_1 - KA_2 = 0$

$-KA_1 + (2K - m\omega^2) A_2 = 0$

As A_1 & A_2 are amplitudes they will never be zero, hence

$$\begin{vmatrix} 2K - m\omega^2 & -K \\ -K & 2K - m\omega^2 \end{vmatrix} = 0$$

or $(2K - m\omega^2) = +K^2$

or $2K - m\omega^2 = \pm K$

$$\omega = \sqrt{\frac{K}{m}} \text{ and } \omega = \sqrt{\frac{3K}{m}}$$

7.105. A thin bar of mass m and length l is suspended from a point P through one of its end. It is displaced through a small vertical angle and released. Find the period of small oscillations.

Solution: Moment of inertia of the thin bar about $P = \frac{1}{3} ml^2$ (see Fig. 7.75).

Let θ be the angular displacement given to the bar then

$$\frac{1}{3} ml^2 \alpha = -mg \cdot \frac{l}{2} \cdot \sin \theta = -\frac{mg}{2} l \cdot \theta; \sin \theta = \theta$$

$$\therefore \alpha = -\frac{3g}{2l} \cdot \theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{3g}{2l}}$$

$$T = 2\pi \sqrt{\frac{2l}{3g}}$$

Fig. 7.75

7.106. A homogeneous circular disc of radius r and mass m oscillates in its own plane about an axis passing through a point on its edge and perpendicular to its plane. Calculate the period of small oscillations.

Solution: Moment of inertia of the disc about P is given by (see Fig. 7.76)

$$I = \frac{1}{2} mr^2 + mr^2 = \frac{3}{2} mr^2$$

Let the disc be displaced through a small vertical angle θ then

$$\frac{3}{2} mr^2 \cdot \alpha = -mg\, r \sin\theta = -mgr\theta$$

$$\alpha = -\frac{2g}{3r}\theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{2g}{3r}}$$

$$\therefore\ T = 2\pi\sqrt{\frac{3r}{2g}}$$

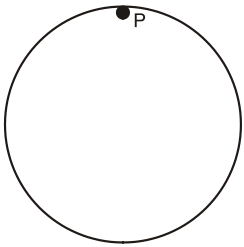


Fig. 7.76

7.107. A thin circular hoop of radius r is hanged over an edge P as shown in the Fig. 7.77. If the hoop is displaced through a small vertical angle θ it oscillates in its own plane. Calculate the frequency of oscillation.

Solution: Moment of inertia of the hoop about an axis passing through P and perpendicular to its plane is equal to

$I = mr^2 + mr^2$; m = mass of the hoop

Let the hoop be displaced through a small vertical angle θ, then

$$2mr^2 \cdot \alpha = -mgr \cdot \sin\theta = -mgr\theta$$

$$\alpha = -\left(\frac{g}{2r}\right)\theta$$

$$\omega = 2\pi n = \sqrt{\frac{g}{2r}}$$

$$n = \frac{1}{2\pi}\sqrt{\frac{g}{2r}}$$

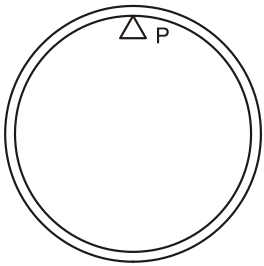


Fig. 7.77

7.108. A thin hoop of radius r and mass m is suspended as shown in the Fig. 7.78. A particle of mass m is also attached at the rim of the hoop. Determine the period of small oscillation when it oscillates (i) in the plane of the hoop (ii) in a direction perpendicular to the plane of the hoop.

Solution:

(i) Moment of inertia of the hoop about an axis passing through O and perpendicular to its plane = mr^2

Moment of inertia of the hoop about $P = mr^2 + mr^2 = 2\,mr^2$

Moment of inertia of the particle attached at the rim about

$$P = m \cdot 4r^2$$

Total moment of inertia = $6mr^2$

Centre of mass of the system from $P = \dfrac{mr + m \cdot 2r}{2m} = \dfrac{3r}{2}$

Let the hoop be displaced in the vertical plane through an angle θ. The equation of motion is given by

$$6\,mr^2 \cdot \alpha = -2\,mg \cdot \frac{3r}{2} \sin\theta$$

$$= -3\,mgr\,\theta$$

$$\alpha = -\frac{g}{2r} \cdot \theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{g}{2r}}$$

$$\therefore\ T = 2\pi\sqrt{\frac{2r}{g}}$$

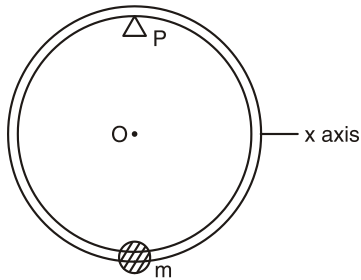


Fig. 7.78

(ii) Moment of inertia of the hoop about O along x axis = $\dfrac{1}{2}mr^2$

Moment of inertia of the hoop about P parallel to x axis

$$= \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2$$

Moment of inertia of the mass about $P = 4\,mr^2$

Net moment of inertia = $\dfrac{11mr^2}{2}$

Let the hoop be displaced through an angle θ in a direction perpendicular to the plane of the hoop. Equation of motion is given by

$$\frac{11mr^2}{2} \cdot \alpha = -2mg \cdot \frac{3r}{2} \cdot \sin\theta$$

$$\alpha = -\frac{6g}{11r} \cdot \theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{6g}{11r}}$$

or $$T = 2\pi\sqrt{\frac{11r}{6g}}$$

7.109. One end of a thin rod of length l and mass m is attached at the rim of a thin disc of mass M and radius r. The other end of the rod is pivoted at O. If the rod is slightly displaced from the vertical and set free it starts oscillating. Find the period of oscillation.

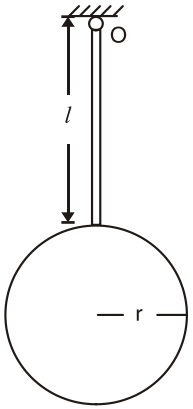


Fig. 7.79

Solution: The total moment of inertia of the disc and rod about the axis of suspension O is given by (see Fig. 7.79).

$$I = \frac{1}{2}Mr^2 + M(l+r)^2 + \frac{1}{3}ml^2$$

$$\text{Centre of mass from } O = \frac{m\cdot\frac{l}{2} + M(l+r)}{(M+m)}$$

Let the rod be displaced through an angle θ from the vertical. The equation of motion is given by

$$\left[\frac{1}{2}Mr^2 + M(l+r)^2 + \frac{1}{3}ml^2\right]\cdot\alpha = -(M+m)g\frac{\left[m\cdot\frac{l}{2} + M(l+r)\right]}{(M+m)}\sin\theta$$

$$\text{or}\quad \left[\frac{1}{2}Mr^2 + M(l+r)^2 + \frac{1}{3}ml^2\right]\cdot\alpha = -\left[m\cdot\frac{l}{2} + M(l+r)\right]g\cdot\theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{m\cdot\frac{l}{2} + M(l+r)}{\frac{1}{2}Mr^2 + M(l+r)^2 + \frac{1}{3}ml^2}}$$

$$\text{or}\quad T = 2\pi\sqrt{\frac{\frac{1}{2}Mr^2 + M(l+r)^2 + \frac{1}{3}ml^2}{m\cdot\frac{l}{2} + M(l+r)}}$$

7.110. A thin circular ring of mass m, inner radii r and outer radii R rests with inner face of its rim on a transverse horizontal knife edge as shown in Fig. 7.80. If the ring is slightly displaced from the vertical it starts oscillating in the vertical plane. Find the period of oscillation.

Solution: Moment of inertia about $P = m\frac{\left(R^2 + r^2\right)}{2} + mr^2$

Let the ring be displaced through an angle θ from the vertical. The equation of motion is given by

$$m\left[\frac{\left(R^2 + r^2\right)}{2} + r^2\right]\cdot\alpha = -mgr\sin\theta = -mgr\,\theta$$

$$\alpha = \frac{-mgr\theta}{m\left[\frac{R^2 + r^2}{2} + r^2\right]}$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{2gr}{R^2 + 3r^2}}$$

$$T = 2\pi\sqrt{\frac{R^2 + 3r^2}{2gr}}$$

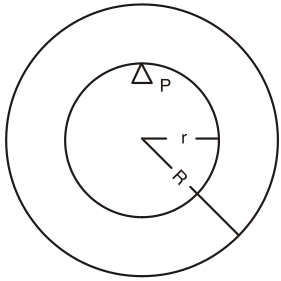


Fig. 7.80

7.111. A thin disc of mass m and radius r is fixed at the rim of another thin disc of mass M and radius R ($R > r$) as shown in the Fig. 7.81. The disc of radius R is mounted at its centre on an axle. The assembly of the two disc is rotated through an angle θ from the vertical and released. Find (i) the speed of centre of disc of radius r at the equilibrium position (ii) the period of oscillation.

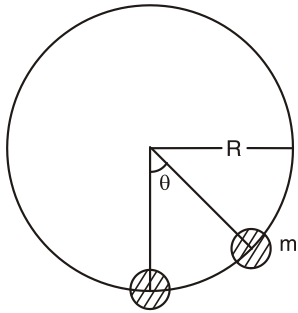


Fig. 7.81

Solution:

(i) In view of conservation of energy we have

$$mgR\,[1 - \cos\theta] = \frac{1}{2}\left[\frac{1}{2}mr^2 + mR^2\right]\omega^2 + \frac{1}{2}\times\frac{1}{2}MR^2\omega^2$$

$$gR\,[1 - \cos\theta] = \frac{r^2\omega^2}{4R^2} + \frac{1}{2}R^2\omega^2 + \frac{1}{4}\frac{M}{m}R^2\omega^2$$

$$4\,gR\,[1 - \cos\theta] = \left[\frac{r^2}{R^2} + 2 + \left(\frac{M}{m}\right)\right]v^2$$

$$v = 2\sqrt{\frac{gR(1 - \cos\theta)}{\left(\frac{r^2}{R^2} + 2 + \frac{M}{m}\right)}}$$

(ii) $4\,mgR^3\,(1 - \cos\theta) = mr^2v^2 + 2\,mR^2v^2 + MR^2v^2$

On differentiation we have

$$4\,mgR^3 \sin\theta \cdot \omega = mr^2 \cdot 2v \cdot a + 4\,mR^2\,v \cdot a + 2MR^2\,v \cdot a$$

$$\therefore a = R\alpha = \frac{2mgR^2\theta}{mr^2 + (M + 2m)R^2}$$

$$\alpha = \frac{2mgR\theta}{mr^2 + (M + 2m)R^2}$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{2mgR}{mr^2 + (M+2m)R^2}}$$

$$T = 2\pi\sqrt{\frac{mr^2 + (M+2m)R^2}{2mgR}}$$

7.112. A pendulum as shown in the Fig. 7.82 consists of two masses m_1 and m_2 fixed on a light thin rod at a respective distance of l_1 and l_2 from the point of suspension O. The pendulum is displaced from the vertical and released. Find the period of oscillation.

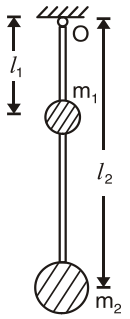


Fig. 7.82

Solution: Total moment of inertia $I = m_1 l_1^2 + m_2 l_2^2$

Centre of mass from $O = \dfrac{m_1 l_1 + m_2 l_2}{m_1 + m_2}$

Let the pendulum be displaced through a small angle θ then equation of motion is given by

$$\left(m_1 l_1^2 + m_2 l_2^2\right)\cdot\alpha = -\left(m_1 + m_2\right) g\left(\frac{m_1 l_1 + m_2 l_2}{m_1 + m_2}\right)\sin\theta$$

or $$\alpha = -\frac{g\left(m_1 l_1 + m_2 l_2\right)\theta}{m_1 l_1^2 + m_2 l_2^2}$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{g\left(m_1 l_1 + m_2 l_2\right)}{m_1 l_1^2 + m_2 l_2^2}}$$

or $$T = 2\pi\sqrt{\frac{m_1 l_1^2 + m_2 l_2^2}{g\left(m_1 l_1 + m_2 l_2\right)}}$$

7.113. A plane lamina in the form of an equivalent triangle of side a = 4.9 cm is suspended vertically keeping its base AB horizontal. The lamina is rotated through an angle θ and released (see Fig. 7.83a). Find the period of oscillation.

Solution:

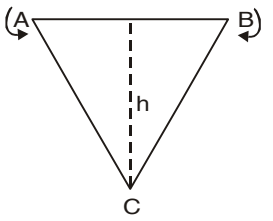


Fig. 7.83a

Moment of inertia of the lamina of mass M about $AB = \dfrac{1}{6}Mh^2 = \dfrac{M}{6}\cdot\left(\dfrac{a\sqrt{3}}{2}\right)^2$

$$= \frac{Ma^2}{8}$$

Centre of mass $= \dfrac{\int_0^h x\,dm}{M} = \int_0^h \dfrac{x\cdot M\cdot DE\cdot dx}{M\cdot\dfrac{a\sqrt{3}a}{4}}$ (see Fig 7.83b)

$$= \int_0^h \frac{4DE \cdot x dx}{a^2 \cdot \sqrt{3}}$$

$$= \frac{4AB}{a^2\sqrt{3}h}\left[\int_0^h \left(xh - x^2\right)\right] dx$$

$$= \frac{4}{a\sqrt{3}h}\left[\frac{h^3}{2} - \frac{h^3}{3}\right]$$

$$= \frac{4h^2}{a\sqrt{3}\cdot 6} = \frac{a\sqrt{3}}{6}$$

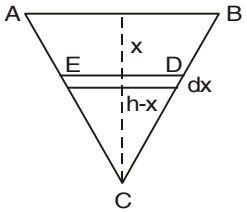


Fig. 7.83b

Let the lamina be displaced through an angle θ. The equation of motion is given by

$$M \cdot \frac{a^2}{8} \cdot \alpha = - Mg \cdot \frac{a\sqrt{3}}{6} \cdot \sin\theta$$

$$\alpha = - \frac{8g\sqrt{3}}{6a} \cdot \theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{8g\sqrt{3}}{6a}}$$

or $$T = \pi\sqrt{\frac{6a}{2g\sqrt{3}}}$$

$$= \pi 3^{1/4}\sqrt{\frac{a}{g}}$$

On substituting the values we have

$$T = 3.14 \times 3^{1/4}\sqrt{\frac{4.9 \times 10^{-2}}{9.8}}$$

$$= \frac{3.14 \times 3^{1/4}}{1.414} = 0.292 \text{ sec}$$

7.114. A 29.4 cm long thin wire is bent in the form of an equilateral triangular frame and is hanged from a pin at *O* (see Fig. 7.84). The frame is displaced through a small angle and released. Find the period of oscillation.

Solution: Let m be the mass of the wire and $3l$ be its length then total moment of inertia of each wire about the centroid O' is given by

$$\frac{m}{3} \cdot \frac{l^2}{12} + \frac{m}{3} \cdot \frac{l^2}{\left(2\sqrt{3}\right)^2} = \frac{ml^2}{18}$$

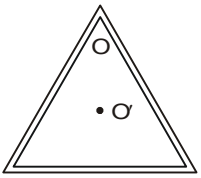


Fig. 7.84

Total moment of inertia of the frame about $O' = 3 \cdot \frac{ml^2}{18} = \frac{ml^2}{6}$

Moment of inertia of the frame about $O = \frac{ml^2}{6} + \frac{ml^2}{3} = \frac{ml^2}{2}$

Let the frame be displaced through a small angle θ from the vertical the equation of motion is given by

$$\frac{ml^2}{2} \cdot \alpha = - mg \cdot \frac{2}{3} \cdot \frac{l\sqrt{3}}{2} \sin\theta$$

$$\frac{l\alpha}{2} = - \frac{g}{\sqrt{3}} \cdot \theta$$

$$\alpha = - \frac{2g}{\sqrt{3}l} \cdot \theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{2g}{\sqrt{3}l}}$$

$$\therefore T = 2\pi \sqrt{\frac{l \cdot \sqrt{3}}{2g}}$$

On substituting the values of l and g we have

$$T = 2 \times 3.14 \sqrt{\frac{9.8 \times 10^{-2} \times 1.732}{2 \times 9.8}}$$

$$= 0.2 \times 3.14 \times \sqrt{0.866}$$

$$= 0.584 \text{ sec}$$

7.115. A thin plate of length l and mass m rests on a half cylindrical glass rod of radius r. The plate is pressed on one side of its length and set free. It starts oscillating. Find the frequency of oscillation.

Solution: When the plate is pressed it forms a new contact which subtends an angle θ. In view of conservation of energy (see Fig. 7.85) we have

$$mgr(1 - \cos\theta) = \frac{1}{2} \cdot \frac{ml^2}{12} \times \omega^2$$

On differentiation we have

$$mgr \sin\theta \cdot \omega = \frac{ml^2}{12} \omega \cdot \alpha$$

$$\alpha = \frac{12gr}{l^2} \theta$$

$$\omega = 2\pi n = 2\sqrt{\frac{3gr}{l^2}}$$

$$n = \frac{1}{\pi} \sqrt{\frac{3gr}{l^2}}$$

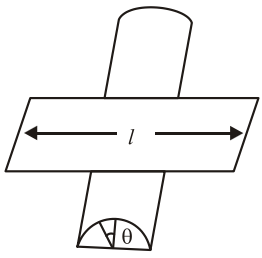


Fig. 7.85

7.116. A homogenous *L* shaped bar of mass 3 kg is supported in a vertical plane through a pin *O* and a spring of force constant $K = 100$ N/m at *A* as shown in Fig. 7.86. The length of the portion *OA* is $l = 1$ m. The bar *OB* is rotated through an angle θ with the vertical and set free. Find the period of rotational oscillation.

Solution: Let *m* be the mass of portion *OA* of the bar. The mass of portion *OB* = 2 m.

$\therefore\ m = 1$ kg

On equation the torque we have

$$\left[\frac{ml^2}{3} + 2m \cdot \frac{4l^2}{3}\right]\alpha = mg \cdot \frac{l}{2} - \left(\frac{mg}{2} + Kl\theta\right) l - 2mgl \sin\theta$$

or $3ml^2 \cdot \alpha = -\left[Kl^2 + 2mgl\right]\theta$; $\sin\theta = \theta$

$$\alpha = -\frac{[Kl + 2mg]}{3ml} \cdot \theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{Kl + 2mg}{3ml}}$$

$$\therefore T = 2\pi\sqrt{\frac{3ml}{Kl + 2mg}}$$

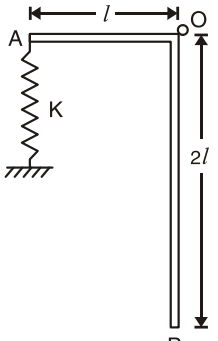


Fig. 7.86

On substituting the values of various terms we have

$$T = 2 \times 3.14\sqrt{\frac{3 \times 1 \times 1}{100 \times 1 + 2 \times 1 \times 9.8}}$$

$$= 6.28\sqrt{\frac{3}{119.6}}$$

$= 1$ sec

7.117. The homogenous bar of the above problem is arranged as shown in the Fig. 7.87. The bar *OB* is rotated through a small angle θ with the vertical and released. Find the period of rotational oscillations.

Solution: On equating the torque we have

$$\left(\frac{ml^2}{3} + \frac{4ml^2}{3}\right)\alpha$$

$$= mg \cdot \frac{l}{2} - \left(\frac{mg}{2} + Kl\theta\right) l + 2mgl \sin\theta$$

$3\,ml^2\,\alpha = -[Kl^2 - 2\,mgl]\,\theta$

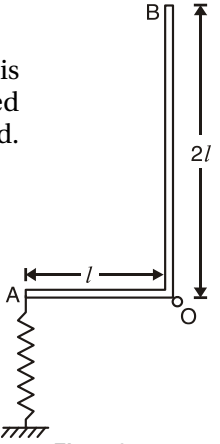


Fig. 7.87

$$\alpha = -\left[\frac{Kl - 2mg}{3ml}\right]\theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{Kl - 2mg}{3ml}}$$

$$T = 2\pi\sqrt{\frac{3ml}{Kl - 2mg}}$$

On putting the values of the various terms we have

$$T = 2 \times 3.14\sqrt{\frac{3 \times 1 \times 1}{100 \times 1 - 2 \times 1 \times 9.8}}$$

$$= 6.28\sqrt{\frac{3}{80.4}}$$

$$= 1.21 \text{ sec}$$

7.118. Two similar thin bars AB and OC of mass $m/2$ and length l are joined to form a T as shown in the Fig. 7.88. On suspending the system of the two bars from O, AB is horizontal. The bar OC is slightly displaced from the vertical and released. Find the period of oscillation.

Solution: Moment of inertia of AB about $C = \frac{m}{2} \cdot \frac{l^2}{12}$

Moment of inertia of AB about $O = \frac{m}{2} \cdot \frac{l^2}{12} + \frac{ml^2}{2} = \frac{13ml^2}{24}$

Moment of inertia of OC about $O = \frac{m}{2} \cdot \frac{l^2}{3} = \frac{ml^2}{6}$

Net moment of inertia about $O = \frac{13ml^2}{24} + \frac{ml^2}{6} = \frac{17ml^2}{24}$

Centre of mass from $O = \frac{\frac{m}{2} \cdot \frac{l}{2} + \frac{m}{2} \cdot l}{m} = \frac{3l}{4}$

Let the bar be displaced through an angle θ then the equation of motion is given by

$$\frac{17ml^2}{24} \cdot \alpha = -mg \cdot \frac{3l}{4} \cdot \sin\theta$$

$$\alpha = -\frac{18g}{17l} \cdot \theta$$

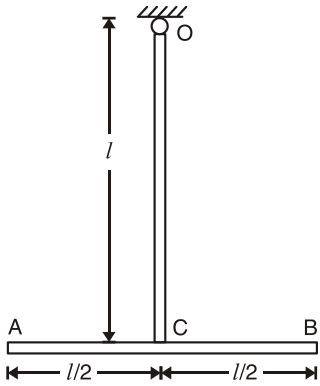


Fig. 7.88

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{18g}{17l}}$$

$$T = 2\pi\sqrt{\frac{17l}{18g}}$$

7.119. A uniform cylinder of mass m and radius r rolls without slip inside cylindrical surface of radius R. Show that it will execute simple harmonic motion. Find its time period.

Solution: Linear velocity of the centre of the cylinder (see Fig. 7.89)

$$v = r\omega = (R-r)\,\Omega$$

$$\therefore \omega = \left(\frac{R-r}{r}\right)\Omega$$

$$\text{Total K.E. of the cylinder} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}m(R-r)^2\Omega^2 + \frac{1}{2}\frac{mr^2}{2}\times\frac{(R-r)^2}{r^2}\Omega^2$$

$$= \frac{3}{4}m(R-r)^2\Omega^2$$

$$\text{Loss in P.E. of the cylinder} = mg(R-r)(1-\cos\theta)$$

$$\therefore \frac{3}{4}m(R-r)^2\Omega^2 = mg(R-r)(1-\cos\theta)$$

On differentiation we have

$$\frac{3}{4}(R-r)\cdot 2\Omega\alpha = g\sin\theta\cdot\frac{d\theta}{dt}$$

$$\frac{3}{2}(R-r)\alpha = g\theta$$

$$\alpha = \frac{2g}{3(R-r)}\cdot\theta$$

$$\frac{2\pi}{T} = \sqrt{\frac{2g}{3(R-r)}}$$

$$\therefore T = 2\pi\sqrt{\frac{3(R-r)}{2g}}$$

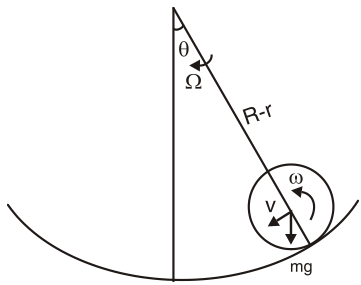


Fig. 7.89

7.120. A homogeneous semi circular ring of radius $r = 4.9$ cm rests on the knife edge P as shown in the Fig. 7.90. When it is slightly displaced from the equilibrium position by pressing one of its end towards the centre it starts oscillating. Assuming that friction is large enough to prevent slipping at P. Calculate the period of small oscillations.

Solution: Let m be the mass of the half ring. Its moment of inertia about $O = mr^2$

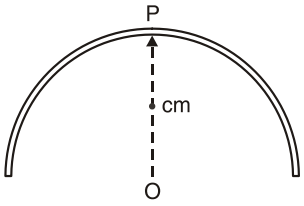


Fig. 7.90

Centre of mass of the half ring is at a distance of $\frac{2r}{\pi}$ above O.

$$I_{c_m} + m\left(\frac{2r}{\pi}\right)^2 = mr^2$$

$$\therefore I_{c_m} = mr^2\left(1 - \frac{4}{\pi^2}\right)$$

Moment of inertia about $P = mr^2\left(1 - \frac{4}{\pi^2}\right) + mr^2\left(1 - \frac{2}{\pi}\right)^2$

$$= 2mr^2\left(1 - \frac{2}{\pi}\right)$$

Distance between P and centre of mass $= r\left(1 - \frac{2}{\pi}\right)$

Let an angular displacement θ be given to the half ring then its equation of motion is given by

$$2mr^2\left(1 - \frac{2}{\pi}\right)\cdot\alpha = -\, mgr\left(1 - \frac{2}{\pi}\right)\sin\theta$$

$$\alpha = -\frac{g}{2r}\cdot\theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{g}{2r}}$$

$$\therefore T = 2\pi\sqrt{\frac{2r}{g}}$$

On substituting the values of r and g we have

$$T = 2\times 3.14\sqrt{\frac{2\times 4.9\times 10^{-2}}{9.8}}$$

$$= 0.628 \text{ sec}$$

7.121. A homogenous half section of pipe of radius r is placed on a rough surface AB. One of the two ends of the pipe is pressed so as to rotate it through a small angle θ and then released. It starts oscillating. Find the period of small oscillations.

Solution: In this case the centre of mass will oscillate. The loss in potential energy is equal to gain in kinetic energy (see Fig. 7.91).

$$mg \cdot \frac{2r}{\pi}(1-\cos\theta) = \frac{1}{2} I_\rho \cdot \omega^2 = \frac{1}{2} \cdot 2mr^2 \left(1-\frac{2}{\pi}\right)\omega^2$$

$$mg \cdot 2r\,(1-\cos\theta) = mr^2\,(\pi-2)\,\omega^2$$

On differentiation we have

$$mg \cdot 2r \cdot \sin\theta \cdot \omega = mr^2\,(\pi-2) \cdot 2\omega \cdot \alpha$$

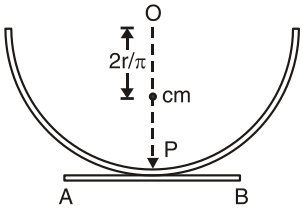


Fig. 7.91

$$\alpha = \frac{g\theta}{r(\pi-2)}$$

$$\frac{2\pi}{T} = \sqrt{\frac{g}{r(\pi-2)}}$$

$$\therefore T = 2\pi\sqrt{\frac{r(\pi-2)}{g}}$$

7.122. The two ends *A* and *B* of a thin homogeneous half ring of radius $r = 15.8$ cm are connected to a weightless thread (see Fig. 7.92a). The thread together with the half ring rests on a cylindrical peg at *O*. The half ring is depressed at one of its ends and released. Find the period of oscillation.

Solution: The *y* component of centroid (see Fig. 7.92b)

$$y_c = \frac{\int y dL}{\int dL}$$

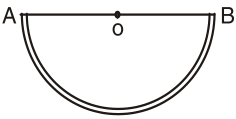


Fig. 7.92a

Now $dx^2 + dy^2 = dL^2$ and $x^2 + y^2 = r^2$

$$x dx + y dy = 0$$

$$\therefore dx = -\frac{y}{x} \cdot dy$$

$$dL^2 = \left(1 + \frac{y^2}{x^2}\right) dy^2$$

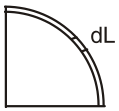


Fig. 7.92b

or $$dL = \frac{r}{x} \cdot dy$$

$$y_c = \frac{\int_0^r y \cdot \frac{r}{x} \cdot dy}{\int_0^{\frac{\pi r}{2}} dL} = \frac{\int_0^r \frac{ry \cdot dy}{\sqrt{r^2 - y^2}}}{\frac{\pi r}{2}}$$

$$= \frac{2}{\pi} \int_0^r \frac{y\,dy}{\sqrt{r^2 - y^2}} = \frac{2r}{\pi}$$

Thus the centroid of the half ring lies at a distance of $\frac{2r}{\pi}$ below *O*.

Let *m* be the mass of the half ring.

Moment of inertia of half ring about an axis passing through O and perpendicular to it = mr^2.

For small angular displacement θ the equation of motion is given by

$$mr^2 \cdot \alpha = -mg \cdot \frac{2r}{\pi} \cdot \sin\theta$$

$$r\alpha = -\frac{2g}{\pi} \cdot \theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{2g}{\pi r}}$$

or $$T = 2\pi\sqrt{\frac{\pi r}{2g}}$$

On substituting the values of r and g we have

$$T = 2 \times 3.14 \sqrt{\frac{3.14 \times 15.8 \times 10^{-2}}{2 \times 9.8}}$$

$= 1$ sec

7.123. A square plate of length l having a centred circular hole of radius r is suspended from P as shown in the Fig. 7.93. The plate is displaced through an angle θ from the vertical and released. Find the period of oscillation.

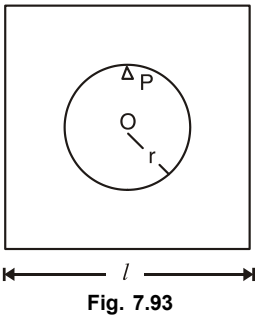


Fig. 7.93

Solution: Let M be the mass of the plate

Moment of inertia of the plate about the centre

$$O = \frac{Ml^2}{6}$$

Moment of inertia of the plate about $P = \frac{Ml^2}{6} + Mr^2$

Moment of inertia of the circular disc of radius r about $O = \frac{M}{l^2}\pi r^2 \cdot \frac{r^2}{2}$

Moment of inertia of the circular disc about $P = \frac{M}{l^2}\frac{\pi r^4}{2} + \frac{M}{l^2}\pi r^2 \cdot r^2$

$$= \frac{3}{2}\frac{M\pi r^4}{l^2}$$

Moment of inertia of the plate with circular hole about

$$P = \frac{Ml^2}{6} + Mr^2 - \frac{3}{2}\frac{M\pi r^4}{l^2}$$

Mass of the plate with hole = $\left[M - \frac{M}{l^2}\pi r^2\right]$

On displacing the plate through a small angle θ the equation of motion is given by

$$\left[\frac{Ml^2}{6}+Mr^2-\frac{3}{2}\frac{M\pi r^4}{l^2}\right]\alpha=-\left[M-\frac{M\pi r^2}{l^2}\right]gr\sin\theta$$

$$\therefore\ \alpha=-\frac{\left(1-\frac{\pi r^2}{l^2}\right)gr\,\theta}{\left(\frac{l^2}{6}+r^2-\frac{3}{2}\frac{\pi r^4}{l^2}\right)}$$

$$\omega=\frac{2\pi}{T}\sqrt{\left(1-\frac{\pi r^2}{l^2}\right)gr\Big/\left(\frac{l^2}{6}+r^2-\frac{3}{2}\frac{\pi r^4}{l^2}\right)}$$

$$\therefore\ T=2\pi\sqrt{\left(\frac{l^2}{6}+r^2-\frac{3\pi r^4}{2l^2}\right)\Big/\left(1-\frac{\pi r^2}{l^2}\right)gr}$$

7.124. A solid disc of mass m and radius R is kept horizontal by connecting it through three wires of length l. The wires are equally spaced round the circumference of a circle of radius r. When the disc is turned through small angle and released it performs simple harmonic motion. Find its time period.

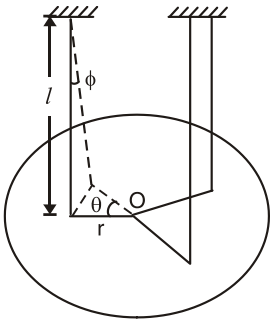


Fig. 7.94

Solution: Let the disc be rotated through a small angle θ the corresponding angular displacement of the wire be ϕ then

$r\theta = l\phi$ or $\phi=\frac{r\theta}{l}$ (see Fig. 7.94)

Tension in each wire $=\frac{mg}{3}$

Component of the tension perpendicular to $r=\frac{mg}{3}\sin\phi$

$$=\frac{mg}{3}\phi=\frac{mg}{3}\cdot\frac{r\theta}{l}$$

Total restoring couple $=3\left(\frac{mg}{3}\cdot\frac{r\theta}{l}\cdot r\right)=mg\frac{r^2\theta}{l}$

Equation of motion of the disc is given by

$I\alpha=-mg\frac{r^2\theta}{l}$; $I=$ moment of inertia of the disc about an axis passing through its centre and perpendicular to its plane.

$$\text{or} \quad \frac{1}{2}mR^2\alpha = -mg\frac{r^2\theta}{l}$$

$$\text{or} \quad \alpha = -\frac{2gr^2}{lR^2}\theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{2gr^2}{lR^2}}$$

$$\text{or} \quad T = 2\pi\sqrt{\frac{lR^2}{2gr^2}}$$

7.125. A rod AB is suspended symmetrically by two equal but non parallel threads each of length l. The distance between the threads at the top and at the bottom is $2d_1$ and $2d_2$ respectively. The rod is rotated through an angle θ in the horizontal plane and released. It starts oscillating. Assuming $d_2 - d_1 = x$ find the period of oscillation.

Solution: Let T be the tension in the thread then

$$2T\cos\phi = mg \text{ (see Fig. 7.95)}$$

$$LB' = \sqrt{d_1^2 + d_2^2 - 2d_1d_2\cos\theta} \cong d_2 - d_1 \text{ assuming } \theta = 0$$

$$LB' = d_2 - d_1 = x$$

$$\cos\phi = \frac{B'R}{B'Q} = \frac{B'R}{l} = \frac{\sqrt{B'Q^2 - RQ^2}}{l} = \frac{\sqrt{l^2 - x^2}}{l}$$

$$\therefore T = \frac{mgl}{2\sqrt{l^2 - x^2}}$$

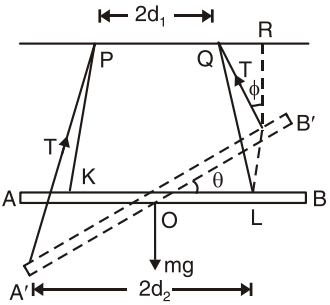


Fig. 7.95

Resolving $T\sin\phi$ acting at A' and B' along $A'K$ and $B'L$ respectively into rectangular components along and at right angles to $A'B'$ we have right angle component $= T\sin\phi \cdot \sin\alpha$; α is the angle between $T\sin\phi$ acting along $B'L$ and $A'B'$.

Moment of restoring couple

$$= T\sin\phi \cdot \sin\alpha \cdot A'B'$$

Restoring torque $= T\sin\phi \cdot \sin\alpha \cdot 2d_2$

$$= 2d_2 \cdot \frac{mgl}{2\sqrt{l^2 - x^2}} \cdot \frac{x}{l}\sin\alpha; \sin\phi = \frac{x}{l}$$

$$= d_2 \cdot \frac{mgx\sin\alpha}{\sqrt{l^2 - x^2}}$$

$$\frac{d_1}{\sin\alpha} = \frac{LB'}{\sin\theta}$$

$$\therefore \frac{d_1}{\sin\alpha} = \frac{x}{\theta}$$

$$\therefore I\frac{d^2\theta}{dt^2} = -\frac{d_1 d_2\, mg\cdot\theta}{\sqrt{l^2 - x^2}}$$

$$\omega = \sqrt{\frac{d_1 d_2\cdot mg}{I\left(l^2 - x^2\right)^{1/2}}} = \frac{2\pi}{T}$$

$$T = 2\pi\sqrt{\frac{I\left(l^2 - x^2\right)^{1/2}}{d_1 d_2\cdot mg}}$$

7.126. A particle of mass m is suspended by a weightless elastic string of length l whose other end is fixed to a rigid support. The modulus of elasticity of string $\lambda = ng$. Show that the particle will be execute simple harmonic motion with time period $t = 2\pi\sqrt{\frac{ml}{ng}}$.

Solution: Let b be the extension in the string when the particle is attached then in equilibrium

$$T_0 = mg = \frac{\lambda b}{l} = \frac{ngb}{l}$$

Let at an instant t, $(b + x)$ be the extension in the string then tension in the string is given by

$$T = ng(b + x)/l$$

The equation of motion of the particle is given by

$$m\frac{d^2x}{dt^2} = mg - T = mg - \frac{ngb}{l} - \frac{ngx}{l}$$

$$= -\frac{ngx}{l};\ mg = \frac{ngb}{l}$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{ng}{ml}\cdot x$$

Thus the acceleration is proportional to displacement hence the motion is simple harmonic in nature.

$$\Omega = \frac{2\pi}{t} = \sqrt{\frac{ng}{ml}}$$

$$\therefore t = 2\pi\sqrt{\frac{ml}{ng}}$$

7.127. Two particles of masses m_1 and m_2 are attached at the lower end of an elastic string of length l whose upper end is fixed to a rigid support. The modulus of elasticity of the string is λ. Out of the two masses m_2 falls off. Calculate the position of the particle m_1 from the upper end of the string at an instant t.

Solution: Let l_1 and l_2 be the extensions in the string with masses m_1 and m_2 respectively.

$$m_1 g = \frac{\lambda l_1}{l}, m_2 g = \frac{\lambda l_2}{l}$$

Equation of motion of the particle is given by

$$m_1 \frac{d^2x}{dt^2} = m_1 g - T = m_1 g - \frac{\lambda x}{l}$$

$$m_1 \frac{d^2x}{dt^2} = m_1 g - m_1 g \cdot \frac{x}{l_1}$$

or $$\frac{d^2x}{dt^2} = -\frac{g}{l_1}(x - l_1)$$

Let the solution of the above differential equation be

$$x - l_1 = A\cos\left(\sqrt{\frac{g}{l_1}} \cdot t\right) + B\sin\left(\sqrt{\frac{g}{l_1}} \cdot t\right)$$

Initially $x = l_1 + l_2$ and at $t = 0$, $\frac{dx}{dt} = 0$

$B = 0$ and $A = l_2$

$$x = l_1 + l_2\cos\left(\sqrt{\frac{g}{l_1}} \cdot t\right)$$

Required position $= l + x = l + l_1 + l_2\cos\left(\sqrt{\frac{g}{l_1}} \cdot t\right)$

7.128. A particle of mass m is attached to a wire which is stretched between two rigid supports O and O' with a tension T (Fig. 7.96). The position of the particle from the two ends are l_1 and l_2 respectively. Find (i) the period of transverse oscillations (ii) the position of the particle at which the time period will be maximum.

Solution:

(i) Restoring force $= T\cos\theta + T\cos\phi$

$$= T\left[\frac{y}{\sqrt{l_1^2 + y^2}} + \frac{y}{\sqrt{l_2^2 + y^2}}\right]$$

For small values of y

$$m \times \text{acceleration} = -T\left[\frac{1}{l_1}+\frac{1}{l_2}\right]y$$

$$\omega = \frac{2\pi}{t} = \sqrt{\frac{T(l_1+l_2)}{ml_1l_2}}$$

or $$t = 2\pi\sqrt{ml_1l_2/T(l_1+l_2)}$$

(ii) Let $l_1 + l_2 = l$ and $l_1 = x$

$$t = 2\pi\sqrt{\frac{m\cdot x(l-x)}{T\cdot l}}$$

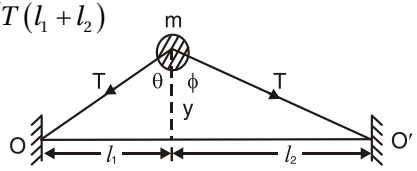


Fig. 7.96

$$\frac{dt}{dx} = 2\pi\sqrt{\frac{m}{Tl}}\left[\frac{l-2x}{2\sqrt{x(l-x)}}\right] = 0$$

$$l - 2x = 0 \text{ or } x = \frac{l}{2}$$

With this value of x, $\frac{d^2t}{dx^2}$ is negative

The time period will be maximum when the particle is at the mid-point of the wire.

7.129. One end A of a rod of mass m is welded at the circumference of a hoop and the other end lies at the centre O of the hoop. The length l of the rod is equal to the radius of the hoop. If the set up is displaced from its equilibrium position it starts oscillating. Neglecting the mass of the hoop find the period of small oscillations.

Solution: Let the assembly be displaced through angle θ from the vertical (see Fig. 7.97). The torque acting on it is $mg\frac{l}{2}\sin\theta$.

Further $$\frac{ml^2}{3}\alpha = -mg\frac{l}{2}\sin\theta$$

$$= -mg\frac{l}{2}\cdot\theta \text{ for small angles}$$

$$\alpha = -\frac{3g}{2l}\cdot\theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{3g}{2l}}$$

or $$T = 2\pi\sqrt{\frac{2l}{3g}}$$

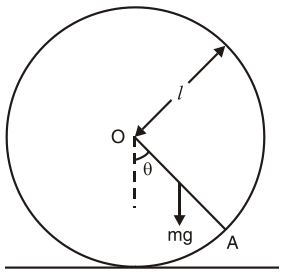


Fig. 7.97

7.130. A homogenous semi circular disc of radius r is suspended at O as a compound pendulum as shown in Fig. 7.98. The pendulum swings with small amplitude of oscillation in its vertical plane. Find the period of oscillation.

Solution: Moment of inertia of the disc about an axis passing through A and $\perp$ to its plane = $\frac{mr^2}{4}$

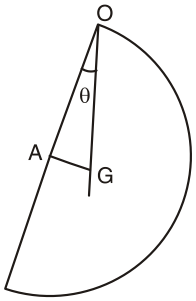


Fig. 7.98

mass of disc = $\frac{m}{2}$

Now $I_G + \frac{m}{2}\cdot\left(\frac{4r}{3\pi}\right)^2 = \frac{mr^2}{4}$

or $I_G = \frac{mr^2}{4} - \frac{m}{2}\cdot\left(\frac{4r}{3\pi}\right)^2$

Moment of inertia about O is given by

$$I_o = I_G + \frac{m}{2}\cdot\left[r^2 + \left(\frac{4r}{3\pi}\right)^2\right]$$

$$= \frac{mr^2}{4} - \frac{m}{2}\cdot\left(\frac{4r}{3\pi}\right)^2 + \frac{mr^2}{2} + \frac{m}{2}\cdot\left(\frac{4r}{3\pi}\right)^2$$

$$= \frac{3mr^2}{4}$$

When the pendulum is rotated through small angle θ from the vertical then

$$\frac{3mr^2}{4}\cdot\alpha = -\frac{m}{2}\cdot g\sqrt{r^2 + \left(\frac{4r}{3\pi}\right)^2}\cdot\sin\theta$$

$$\frac{3mr^2}{4}\alpha = -\frac{mgr}{6\pi}\sqrt{9\pi^2 + 16}\cdot\theta$$

$$\alpha = -\frac{2g}{9\pi r}\sqrt{9\pi^2 + 16}\cdot\theta$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{2g}{9\pi r}\sqrt{9\pi^2 + 16}}$$

$$T = 2\pi\sqrt{\frac{9\pi}{2\sqrt{9\pi^2 + 16}}\cdot\frac{r}{g}}$$

$$= 2\pi\sqrt{\frac{28.26}{20.468}\cdot\frac{r}{g}} = 2\pi\sqrt{1.38\frac{r}{g}}$$

7.131. A uniform semi circular cylinder of radius r lies on a plane surface when it is displaced through small angle θ from its equilibrium position it starts oscillating. Find the period of oscillation.

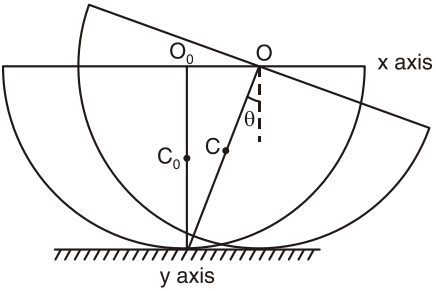


Fig. 7.99

Solution: Let us choose the coordinate axis through point O_0.

The x_c and y_c of the centre of gravity C is given by

$$x_c = r\theta - h\sin\theta,\ y_c = h\cos\theta,\ h = O_0C_0 = OC = \frac{4r}{3\pi}\ \text{(see Fig. 7.99)}$$

During the state of motion from the extreme angular position θ_0 to θ work of all the force is = $mgh(\cos\theta - \cos\theta_0)$.

Change in kinetic energy = $\frac{m}{2}\left(\dot{x}_c^2 + \dot{y}_c^2\right) + m\,i_c^2\,\frac{\dot{\theta}^2}{2}$; i_c = radius of gyration

$$i_c^2 = \left(\frac{r^2}{2} - \frac{16r^2}{9\pi^2}\right)$$

$$\dot{\theta}^2\left[r^2 - 2rh\cos\theta + h^2 + i_c^2\right] = 2gh\left[\cos\theta - \cos\theta_0\right]$$

or $$\dot{\theta}^2\left[(r-h)^2 + i_c^2\right] = 2gh\left(\cos\theta - \cos\theta_0\right)$$

$$2\dot{\theta}\,\ddot{\theta}\left[(r-h)^2 + i_c^2\right] = -2gh\,\theta\cdot\dot{\theta} \text{ on differentiation}$$

$$\ddot{\theta} = \frac{-gh\cdot\theta}{(r-h)^2 + i_c^2}$$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{gh}{(r-h)^2 + i_c^2}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{\frac{4gr}{3\pi}}{\left(r - \frac{4r}{3\pi}\right)^2 + \left(\frac{r^2}{2} - \frac{16r^2}{9\pi^2}\right)}}$$

$$= \sqrt{\frac{8g}{r(9\pi - 16)}}$$

$$T = \pi\sqrt{\frac{r(9\pi - 16)}{2g}}$$

7.132. A solid hemisphere of radius r rests with its convex surface on a horizontal table. If it is displaced slightly show that it will undergo oscillations with period equal to that of a simple pendulum of equivalent length $\frac{4r}{3}$.

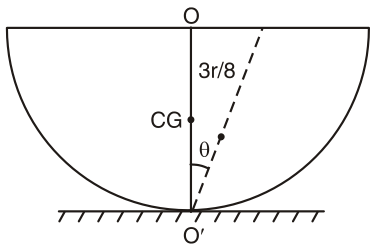


Fig. 7.100

Solution: Let m be the mass of hemisphere. Moment of inertia of the hemisphere about O and perpendicular to the plane of paper $= \frac{1}{5}mr^2$.

Moment of inertia of the hemisphere about $C.G. = \frac{1}{5}mr^2 - m\cdot\left(\frac{3r}{8}\right)^2$

$$= \frac{19mr^2}{320}$$

Moment of inertia of the hemisphere about $O' = \frac{19mr^2}{320} + m\left(\frac{5r}{8}\right)^2$

$$= \frac{144mr^2}{320}$$

Let the sphere be rotated through small angle θ (see Fig. 7.100) then in view of conservation of energy we have

$$\frac{1}{2}\cdot\frac{144}{320}mr^2\omega^2 + \frac{1}{2}m\cdot\left(\frac{5r}{8}\cdot\omega\right)^2 = mg\frac{5r}{8}(1-\cos\theta)$$

$$\frac{269}{640}\cdot r\omega^2 = \frac{5g}{8}(1-\cos\theta)$$

On differentiation we have

$$\frac{269}{640}\cdot r\cdot 2\omega\cdot\alpha = \frac{5g}{8}\cdot\theta\cdot\omega\cdot[\sin\theta = \theta]$$

$$\alpha = \frac{200g}{269r}\cdot\theta$$

$$\Omega = \frac{2\pi}{T} = \sqrt{\frac{200g}{269r}}$$

$$T = 2\pi\sqrt{\frac{269}{200}\cdot\frac{3}{3}\frac{r}{g}}$$

$$= 2\pi\sqrt{\frac{807}{200}\cdot\frac{r}{3g}}$$

$$= 2\pi\sqrt{4.035\cdot\frac{r}{3g}}$$

$\therefore$ Equivalent length of simple pendulum = $\frac{4r}{3}$

7.133. A thin fixed ring of radius 1 m has a positive charge of 1×10^{-5} C uniformly distributed over it. A particle of mass 0.9 g and having a negative charge of 10^{-6} C is placed on the axis at a distance of 1 cm from the centre of the ring. Show that the motion of the negatively charged particle is approximately simple harmonic. Calculate the time period of oscillation. (IIT 1982)

Solution: Consider a ring of radius a on which charge Q is uniformly distributed. The elected field at a distance x from the centre of the ring is given by

$$E = \frac{Qx}{4\pi\varepsilon_0\left(a^2 + x^2\right)^{3/2}}$$

Under the influence of this field the negatively charged particle having charge q will experience a force given by

$$F = \frac{Qqx}{4\pi\varepsilon_0\left(a^2 + x^2\right)^{3/2}}$$

This will cause an acceleration given by

$$A = \frac{Qqx}{4\pi m\varepsilon_0\left(a^2 + x^2\right)^{3/2}}$$

When $a >> x$,

$$A = -\frac{Qqx}{4\pi\varepsilon_0 ma^3}$$

(The negative sign has been introduced because acceleration and displacement are in opposite directions.)

Since the acceleration is proportional to the displacement, it is a case of simple harmonic motion.

Now $\omega = \frac{2\pi}{T} = \sqrt{\frac{Qq}{4\pi\varepsilon_0 ma^3}}$

or $T = 2\pi\sqrt{\frac{4\pi\varepsilon_0 ma^3}{Qq}}$

On substituting the values, we have

$$T = 2 \times 3.14 \sqrt{\frac{9 \times 10^{-4} \times 1}{9 \times 10^{9} \times 10^{-5} \times 10^{-6}}} = 0.63 \text{ s}$$

7.134. An oscillatory circuit contains an inductance 12×10^{-5} H and a variable capacitor whose capacity varies from 3×10^{-8} to 0.75×10^{-8} F. Determine the wavelength and frequency range of this circuit.

Solution: For an *LC* circuit the time period of oscillation is given by

$$T = 2\pi\sqrt{LC_1}$$

$$\therefore\ T_1 = 2 \times 3.14 \sqrt{12 \times 10^{-5} \times 3 \times 10^{-8}} = 2 \times 3.14 \cdot \sqrt{3.6} \times 10^{-6}$$

$$= 11.91 \times 10^{-6} \text{ sec}$$

$$T_2 = 2\pi\sqrt{LC_2} = 2 \times 3.14 \sqrt{12 \times 10^{-5} \times 0.75 \times 10^{-8}}$$

$$= 2 \times 3.14 \cdot \sqrt{0.9} \times 10^{-6} = 5.95 \times 10^{-6} \text{ s}$$

Wavelength $\lambda_1 = 3 \times 10^8 \times 11.91 \times 10^{-6} = 3573$ m

Wavelength $\lambda_2 = 3 \times 10^8 \times 5.95 \times 10^{-6} = 1785$ m

Frequency $\nu_1 = \dfrac{1}{T_1} = 83.96$ kHz

Frequency $\nu_2 = \dfrac{1}{T_2} = 168.06$ kHz

7.135. An oscillator circuit contains an inductance 0.04 H and a capacitor of capacity 16 μF. Determine the maximum current when the maximum voltage across the capacitor is 150 V.

Solution: In view of the conservation of energy,

$$\frac{1}{2} L \cdot I_{\max}^2 = \frac{1}{2} C V_{\max}^2$$

or $$I_{\max} = \sqrt{\frac{C}{L}} \cdot V_{\max}$$

On substituting the values of *C*, *L* and $V_{\max}$ we have

$$I_{\max} = 150 \sqrt{\frac{16 \times 10^{-6}}{0.04}} = 150 \times 10^{-2} \times 2 = 3 A$$

8

SURFACE TENSION AND FLOW OF LIQUIDS

8.1. A liquid film is prepared by pressing a drop of liquid between two glass plates as shown in Fig. 8.1. Calculate the total force required to pull the two plates apart.

Solution: Let d be the thickness of the layers of the liquid between the plates. The radii of curvature of the two concave edges of the layer are both equal to $d/2$. If r be the radius of curvature of the circular area of the film, the excess of pressure p inside the liquid film over the atmospheric pressure is given by

$$p = T\left(\frac{1}{r} - \frac{2}{d}\right)$$

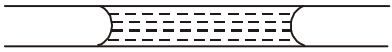


Fig. 8.1

where T is the surface tension of the liquid.

If the thickness d of the layer is very small, r become very large. Therefore,

$$p = T\left(\frac{1}{\infty} - \frac{2}{d}\right) = -\frac{2T}{d}$$

The negative sign indicates that the pressure inside the film is less than the atmospheric pressure. The excess pressure of the atmosphere on the two plates pushes them close together making d still smaller and r still larger. If A be the circular area over which the liquid wets the plates, the total force is given by

$$\frac{2T}{d} \cdot A = \frac{2T \cdot \pi r^2}{d}$$

Therefore $F = 2\pi r^2 T/d$ is the force needed to pull the two plates apart.

8.2. Two vertical plates are kept parallel and close to each other in a liquid. It has been noticed that liquid rises in between them. Calculate the height to which the liquid will rise.

Solution: Let d be the distance between the plates and R the radius of the meniscus (here it is a cylindrical one). If θ be the angle of contact for the liquid and the plate then we have $d = 2R \cos\theta$.

The pressure just below the meniscus is less than the pressure just above it by $T\left(\frac{1}{R}+\frac{1}{R'}\right)$. Where R and R' are the radii of the two curvatures which are at right angles to each other. R is equal to the radius of the cylinder and $R' = \infty$.

Excess pressure just above the meniscus = $\frac{T}{R} = \frac{2T\cos\theta}{d}$

In order to equalise the pressure the liquid column will rise up through a height h given by

$$h\rho g = \frac{2T\cos\theta}{d}; \rho = \text{density of the liquid}$$

or $$h = \frac{2T\cos\theta}{d\rho g}$$

8.3. A glass plate of weighing 8.3 gm *wt* is 15 cm long, 1.56 cm broad and 0.2 cm thick. The plate is immersed in water to half of its volume, keeping it vertical and long side horizontal as shown in Fig. 8.2. What will be its apparent weight ? Surface tension of water = 73 dynes/cm.

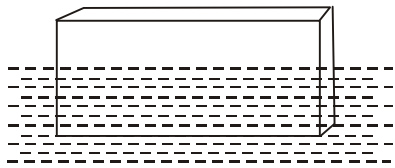


Fig. 8.2

Solution: Three forces act on the glass plate, (i) its weight acting downward, (ii) force due to surface tension acting downward, and (iii) thrust acting upward.

Upward thrust = half the volume of the plate × density of water × g

$$= \frac{15\times1.56\times0.2}{2}\times1\times980$$

= 2.34 gm *wt* (all in CGS system)

Force due to surface tension = $\frac{2\times(15+0.2)\times73}{980}$ = 2.265 gm *wt*

Net upward thrust = 2.34 – 2.265 = 0.075 gm *wt.*

$\therefore$ Apparent weight = 8.300 – 0.075 = 8.225 gm *wt*

8.4. A number of droplets of water each of radius r coalesce to form a single drop of radius R. Calculate the rise in temperature.

Solution: Let there be n droplets each of radius r

The total area of n droplets = $n \times 4\pi r^2$

Loss in area $= n\times4\pi r^2 - 4\pi R^2$

$= 4\pi(nr^2 - R^2)$

Work done $W = 4\pi T (nr^2 - R^2)$

where T is the surface tension of water.

According to first law of thermodynamics,

$W = JH$

where H is heat produced and J is mechanical equivalent of heat.

$$4\pi T (nr^2 - R^2) = J \cdot \frac{4}{3} \pi R^3 \times 1000 \times 1 \times \theta$$

where θ = rise in temperature

$$\therefore \theta = \frac{3T}{1000\, J} \left(\frac{nr^2}{R^3} - \frac{1}{R} \right)$$

On equalizing the volume of a single drop to the total volume of droplets, we have

$$n \cdot \frac{4}{3} \pi r^3 = \frac{4}{3} \pi R^3$$

or $$\frac{nr^2}{R^3} = \frac{1}{r}$$

In the light of the above, we have

$$\theta = \frac{3T}{1000\, J} \left(\frac{1}{r} - \frac{1}{R} \right)$$

8.5. A force if 0.035 N is required to separate a square frame with a side of 8.75 cm from the surface of a liquid. Taking the mass of the frame as 2 gm calculate the surface tension of the liquid.

Solution:

Force due to surface tension = $T \times 8 \times 8.75 \times 10^{-2}$ N ; T = surface tension of the liquid.

In order to separate the frame from the surface of the liquid we must overcome the force of gravity acting of the frame in addition to the force due to surface tension.

$0.035 = 2 \times 10^{-3} \times 9.8 + 8 \times 8.75 \times 10^{-2}\, T$

On solving $T = 0.022$ N/m.

8.6. A soap film is formed on a rectangular frame of dimensions 4 cm × 7 cm by dipping it into a soap solution. The frame is attached to the arm of a balance keeping the short side vertical. An additional weight of 0.38 gm has been placed on the pan in order to balance the pull of the film. Calculate the surface tension of the soap solution.

Solution:

$0.38 \times 10^{-3} \times 9.8 = 2 \times 7 \times 10^{-2} \times T$; T = surface tension

$$\therefore T = \frac{0.38 \times 9.8 \times 10^{-1}}{14}$$
$$= 0.0266 \text{ N/m}$$

8.7. What will the percentage error be in calculating the atmospheric pressure equal to 760 mm of mercury according to the height of a mercury column if the internal diameter of the barometric tube is 5 mm. Surface tension of mercury = 0.5 N/m.

Solution:

$$\frac{2T}{r} = h \rho g$$
$$\therefore h = \frac{2T}{r\rho g}$$
$$= \frac{2 \times 0.5}{2.5 \times 10^{-3} \times 13.6 \times 10^{3} \times 9.8}$$
$$= 3.00 \text{ mm}$$
$$\% \text{ error} = \frac{100 \times 3}{760} = 0.4$$

8.8. Water rises upto a height h in a capillary tube lowered vertically into water to a depth l. The tube is taken out by closing its lower end inside water. Find the length of the water column inside the capillary when (i) $l > h$, (ii) $l < h$.

Solution:

(i) With reference to the water level of the container the upper portion of capillary will have liquid column of height h. The lower portion of the capillary will also have liquid of height h. Hence the total length of water column will be $2h$.

(ii) In the second case the lower portion of the capillary will hold water column of height l and due to this the total length of water column inside the capillary is $l + h$.

8.9. Find the maximum length of a mercury column which can be supported by surface tension in a vertical capillary tube of radius r.

Solution: The maximum length of the mercury column will be obtained when the shape of mercury at the lower end is hemisphere of radius r equal to that of the tube.

Upward force due to surface tension will be balanced by weight of mercury column.

$2 \pi r T - 2 \pi r T \cos(\pi - \theta) = \pi r^2 l \rho g$; T = surface tension and
θ = angle of contact
ρ = density of mercury

$$l = \frac{2T(1+\cos\theta)}{\rho g}$$

8.10. Water rises in a capillary tube of radius r = 1 mm. Calculate the reduction of saturation vapour pressure over a concave surface below that on a flat surface. Density of water $\rho = 10^3$ Kg/m^3. Density of vapour $\sigma = 0.62$ Kg/m^3. Surface tension of water $T = 0.072$ N/m.

Solution:

$$p = \frac{2T}{r}\left(\frac{\sigma}{\rho-\sigma}\right)$$

On substituting the values

$$p = \frac{2\times 0.072\times 0.62}{10^{-3}\times(1000-0.62)}$$

$$= \frac{89.28}{999.38}$$

$$= 0.089 \text{ N/m}^2$$

8.11. Two soap bubbles of radii r_1 and r_2 in vacuum coalesce isothermally to form a new bubble. Find the radius of the bubble formed.

Solution:

$$p_1v_1 + p_2v_2 = pv$$

or $$\left(p_0 + \frac{4T}{r_1}\right)\frac{4}{3}\pi r_1^3 + \left(p_0 + \frac{4T}{r_2}\right)\frac{4}{3}\pi r_2^3 = \left(p_0 + \frac{4T}{R}\right)\frac{4}{3}\pi R^3$$

In vacuum $p_0 = 0$

$$\therefore R^2 = r_1^2 + r_2^2$$

or $$R = \sqrt{r_1^2 + r_2^2}$$

8.12. Two soap bubbles of radii r_1 and r_2 ($r_2 >> r_1$) were blown at the same temperature. The mass of air enclosed in bubble of radius r_1 is m whereas in bubble of radius r_2 as 1.5 m. Taking the atmospheric pressure as P. Calculate the surface tension of the soap solution.

Solution:

$$\left(P + \frac{4T}{r_1}\right)\cdot\frac{4}{3}\pi r_1^3 = \frac{m}{M}\cdot RT \text{ ; } M = \text{mol wt of air}$$

T = surface tension of soap solution

and $$\left(P + \frac{4T}{r_2}\right)\frac{4}{3}\pi r_2^3 = \frac{3m}{2M}\cdot RT$$

$$\left(P+\frac{4T}{r_2}\right)r_2^3 = \frac{3}{2}\left(P+\frac{4T}{r_1}\right)r_1^3$$

$$4T\left(2r_2^2-3r_1^2\right) = \left(3r_1^3-2r_2^3\right)P$$

or $$T = \frac{\left(3r_1^3-2r_2^3\right)}{4\left(2r_2^2-3r_1^2\right)}\cdot P$$

8.13. A material of wire having density $\rho = 1.4$ gm/cc is not wetted by water of surface tension 70 dynes/cm. Find the maximum radius of the wire which can float on the surface of water.

Solution:

$2LT = L\pi r^2\rho g$; L = length of the wire

T = surface tension of water

r = radius of wire

ρ = density of the material of wire

$$r = \left(\frac{2T}{\pi\rho g}\right)^{1/2}$$

On putting the values of the various terms we have

$$r = \left(\frac{2\times 70}{3.14\times 1.4\times 980}\right)^{1/2}$$

$= 0.18$ cm

8.14. A soap bubble is formed from 4×10^{-3} gm of soap solution. The soap bubble filled with hydrogen of density 0.00009 gm/cc floats in air of density 0.00129 gm/cc. Calculate the excess of pressure inside the bubble. Surface tension of soap solution = 30 dynes/cm.

Solution: In equilibrium we have

$$mg+\frac{4}{3}\pi r^3\times 9\times 10^{-5}g = \frac{4}{3}\pi r^3\times 129\times 10^{-5}g$$

$$m = \frac{4\pi}{3}r^3\times 120\times 10^{-5}$$

$$\therefore\ r = \left(\frac{3m}{4\pi\times 120\times 10^{-5}}\right)^{1/3}$$

On putting the values we have

$$r = \left(\frac{3\times 4\times 10^{-3}}{4\times 1.34\times 120\times 10^{-5}}\right)^{1/3}$$

$= 0.9268$ cm

$$\text{Excess of pressure} = \frac{4T}{r}$$
$$= \frac{4 \times 30}{0.9268}$$
$$= 129.5 \text{ dynes/cm}^2$$

8.15. A ring is cut from a platinum tube 8.5 cm internal and 8.7 cm external diameter. It is supported horizontally from a pan of a balance so that it comes in contact with water in a glass vessel. What is the surface tension of water if an extra 3.97 gm weight is required to pull it away from water, $g = 980$ cm/sec^2. (MNR 1992)

Solution: Total force on the ring due to surface tension is given by

$F = T(2\pi r_1 + 2\pi r_2)$; T = surface tension of water

r_1 and r_2 are inner and outer radii of the ring

$$T = \frac{F}{\pi(2r_1 + 2r_2)}$$

On putting the values we have

$$T = \frac{3.97 \times 980}{3.14(8.5 + 8.7)}$$
$$= \frac{3.97 \times 980}{3.14 \times 17.2}$$
$$= 72.13 \text{ dynes/cm.}$$

8.16. Twenty drops of lead were formed by melting a vertically suspended 1 mm thick lead wire at its lower end. By what amount the wire length will be shortened. It is assumed that the diameter of the neck of drop at the time it detaches is equal to the diameter of the wire. Density of lead $\rho = 11300$ Kg/m^3. Surface tension of liquid lead $T = 0.47$ N/m.

Solution: Let the length of the wire be shortened by h then

$$20 \times \frac{2T}{r} = \frac{\pi r^2 h \rho g}{\pi r^2}$$; ρ = density of lead

$$\text{or} \quad h = \frac{40T}{r\rho g}$$

On putting the values we have

$$h = \frac{40 \times 0.47}{0.5 \times 10^{-3} \times 11300 \times 9.8}$$
$$= 34 \text{ cm}$$

8.17. Drops of water fall from a vertical tube with an internal radius of $r = 1$ mm. Find the radius of a drop at the moment when it breaks away,

considering it to be spherical. Assume that the diameter of the neck of a drop at the moment it breaks away is equal to the internal diameter of the tube. Surface tension of water $T = 0.073$ N/m, density of water = 1000 Kg/m^3.

Solution:

$$\frac{2T}{r} \times \pi r^2 = \frac{4}{3}\pi R^3 \rho \cdot g \; ; R = \text{radius of the drop}$$

ρ = density of water

T = surface tension of water

$$R = \left(\frac{3Tr}{2\rho g}\right)^{1/3}$$

On putting the values we have

$$R = \left(\frac{3 \times 0.073 \times 10^{-3}}{2 \times 10^3 \times 9.8}\right)^{1/3}$$

$= 2.235 \times 10^{-3}$m

$= 2.235$ mm

8.18. Calculate the amount of force to be applied to a horizontal aluminium ring with a height $h = 10$ mm, internal diameter $d_1 = 48$ mm and external diameter $d_2 = 50$ mm, away from the surface of water. Density of aluminium $\rho = 2600$ Kg/m^3. Surface tension of water = 0.073 N/m.

Solution:

Net required force F = weight of the ring + force due to surface tension

$$F = \frac{\pi}{4}\left[d_2^2 - d_1^2\right]\rho hg + \pi T\left(d_1 + d_2\right)$$

On substituting the values we have

$$F = \frac{3.14}{4}\left[50^2 - 48^2\right] \times 10^{-6} \times 2600 \times 10^{-2} \times 9.8 + 3.14 \times 0.073\,(50 + 48) \times 10^{-3}$$

$$= \frac{3.14 \times 26 \times 9.8 \times 98 \times 2 \times 10^{-6}}{4} + 3.14 \times 73 \times 98 \times 10^{-6}$$

$= (39.2 + 22.46) \times 10^{-3}$

$= 61.66 \times 10^{-3}$ N

8.19. The movable copper rod *EF* of frame *ABCD* is covered with a soap film (see Fig. 8.3). (i) What should the diameter of the rod *EF* be for it to remain in equilibrium? (ii) What is the length of the bar if a work of 9×10^{-5} J is performed by displacing the bar through a distance of 2 cm? Surface tension of soap solution = 0.045 N/m, Density of copper = 8600 Kg/m^3.

Solution:

(i) Let d be the diameter of the rod EF then in equilibrium

$$\frac{\pi d^2}{4} l \cdot \rho \cdot g = T \cdot 2l; \quad T = \text{surface tension of soap solution}$$

ρ = density of copper

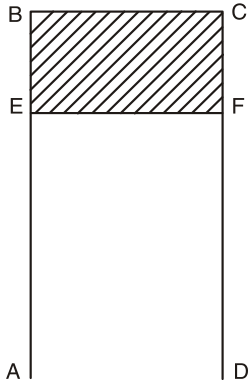


Fig. 8.3

$$d = \sqrt{\frac{8T}{\pi \rho g}}$$

On substituting the values

$$d = \sqrt{\frac{8 \times 0.045}{3.14 \times 8600 \times 9.8}}$$

$$= 1.16 \text{ mm}$$

(ii) $T \cdot 2l \times 2 \times 10^{-2} = 9 \times 10^{-5}$

$$l = \frac{9 \times 10^{-5}}{4 \times 10^{-2} \times 0.045}$$

$$= 5 \text{ cm}$$

8.20. The inner diameter of a barometric tube is 0.75 cm. What changes are required while measuring the atmospheric pressure according to the height of mercury column. Surface tension of mercury = 0.5 N/m.

Solution:

$$\frac{2T}{r} = h \rho g$$

$$\therefore h = \frac{2T}{r \rho g}$$

$$= \frac{4T}{d \rho g}$$

On substituting the values we have

$$h = \frac{4 \times 0.5}{0.75 \times 10^{-2} \times 13.6 \times 10^3 \times 9.8}$$

$$= 2.0 \text{ mm}$$

The height of the mercury column has to be increased by 2 mm.

8.21. Two separate air bubbles (radii 0.002 m and 0.004 m) formed of the same liquid (surface ternsion 0.07 N/m) come to form a double bubble. Find the radius and the sense of curvature of the internal film surface common to both the bubbles. (Roorkee 1989)

Solution: Total pressure inside the smaller and larger bubbles are given by

$$P_1 = P_0 + \frac{4T}{r_1} \;\&\; P_2 = P_0 + \frac{4T}{r_2};$$

P_0 = atmospheric pressure (see Fig. 8.4)

$$P = P_1 - P_2 = \frac{4T}{R} = 4T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\therefore R = \frac{r_1 r_2}{r_2 - r_1}$$

On substituting the values we have

$$R = \frac{0.002 \times 0.004}{0.004 - 0.002} = 0.004 \text{ m}$$

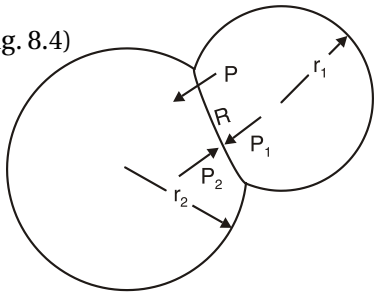


Fig. 8.4

8.22. An air bubble of diameter d_1 = 4μm rises steadily from the bottom of a pond to the surface where its diameter becomes d_2 = 8μm. Assuming the expansion of air to be isothermal find the depth of the pond. Surface tension of water = 73 × 10^{-3} N/m. Atmospheric pressure p_0 = 10^5 N/m². Density of water ρ = 10^3 kg/m³.

Solution: In view of Boyle's Law

$$\left(p_0 + \rho g h + \frac{4T}{d_1}\right)\frac{4\pi}{3}\cdot\frac{d_1^3}{8} = \left(p_0 + \frac{4T}{d_2}\right)\frac{4\pi}{3}\cdot\frac{d_2^3}{8};$$

ρ = density of water

h = depth of the pond

$d_2 = 2d_1$

$$\left(p_0 + \rho g h + \frac{4T}{d_1}\right) = 8\left(p_0 + \frac{4T}{d_2}\right)$$

$$h = \frac{7p_0 - 32T\left(\frac{1}{d_1} - \frac{1}{d_2}\right)}{\rho g}$$

On putting the values

$$h = \frac{7\times10^5 - \frac{32\times73\times10^{-3}}{10^{-6}}\left(\frac{1}{4} - \frac{1}{8}\right)}{9.8\times10^3}$$

$$= \frac{\left[7 - \frac{32\times0.73}{8}\right]}{9.8}\times10^2$$

$$= \left(\frac{7 - 2.92}{9.8}\right)\times10^2$$

$$= 41.63 \text{ m}$$

8.23. A vertical capillary tube of inner diameter 0.5 mm is submerged vertically in water. The length of the tube above the water surface is equal to l = 30 mm. Calculate the curvature radius of the meniscus of water. Surface tension of water = 73×10^{-3} N/m. Density of water = 10^3 Kg/m^3.

Solution: In view of the data given the height upto which water will rise is given by

$$h = \frac{2T}{r\rho g}$$

On putting the values we have

$$h = \frac{2 \times 73 \times 10^{-3}}{0.25 \times 10^{-3} \times 10^3 \times 9.8}$$

$$= 5.96 \text{ cm}$$

This suggests that the length of the capillary tube above the water surface is insufficient. Let r be the radius of the capillary tube and R be the radius of liquid meniscus in the tube then

$$r = R\cos\theta$$

$$\therefore T = \frac{R\cos\theta \cdot h\rho \cdot g}{2\cos\theta} = \frac{Rh\rho g}{2}; h = \text{ height of the liquid column in the tube}$$

Now $Rh = \dfrac{2T}{\rho g}$

In this situation water will spread over the walls of the tube at the top and its meniscus acquire a new radius of curvature R' given by

$$R'l = Rh = \frac{2T}{\rho g}$$

or $$R' = \frac{2T}{l \cdot \rho \cdot g}$$

On substituting the values of the various terms we have

$$R' = \frac{2 \times 73 \times 10^{-3}}{30 \times 10^{-3} \times 10^3 \times 9.8}$$

$$= 0.5 \text{ mm}$$

8.24. A glass capillary sealed at upper end is of length 0.11 m and internal diameter 2×10^{-5} m. The liquid is immersed vertically into a liquid of surface tension 5.06×10^{-2} N/m. To what length the capillary has to be immersed so that the liquid level inside and outside the capillary becomes the same? What will happen to water level inside the capillary if the seal is now broken? Atmospheric pressure = 1.012×10^5 N/m^2. (Rookee 1993)

Solution: Let A be the area of cross-section of tube and l be its length then initial volume of air inside $V_1 = Al$ and initial pressure = p_0.

When the tube is immersed in liquid with its length x inside it, the level of liquid inside and outside is same.

Volume of air inside the tube $V_2 = A\,(l - x)$ (see Fig. 8.5)

Pressure of the gas in the tube $P = p_0 + \dfrac{2T}{r}$

According to Boyle's law

$$p_0 Al = \left(p_0 + \frac{2T}{r}\right) A\,(l - x)$$

or $$x = \frac{l}{\left(1 + \dfrac{r p_0}{2T}\right)}$$

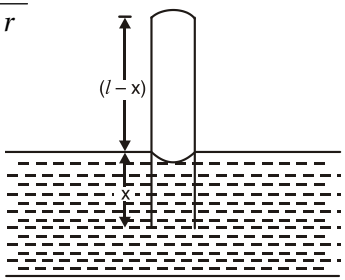


Fig. 8.5

On substituting the values we have

$$x = \frac{0.11}{1 + \dfrac{10^{-5} \times 1.012 \times 10^5}{2 \times 5.06 \times 10^{-2}}} = \frac{0.11}{11} = 0.01\,\text{m}$$

When the seal is broken the pressure inside the tube is equal to atmospheric pressure. Rise of liquid through h will be given by

$$h = \frac{2T}{r\rho g} = \frac{2 \times 5.06 \times 10^{-2}}{10^{-5} \times 10^3 \times 9.8} = 1.03\,\text{m}$$

Length of the tube outside the liquid = 0.11 – 0.01 = 0.1 m. In this way the tube will be of in sufficient length.

The liquid will rise to the top and will form a new meniscus of radius given by

$$r' = \frac{hr}{l}$$

$$= \frac{1.03 \times 10^{-5}}{0.1} = 1.03 \times 10^{-4} m$$

8.25. A large soap bubble of radius r is formed by uniting two soap bubbles of radii r_1 and r_2 without any leakage of air under isothermal conditions. Show that the radii of the three bubbles are connected by the relation

$$P\left(r_1^3 + r_2^2 - r^3\right) = 4\left(r^2 - r_1^2 - r_2^2\right) T$$

where P is the atmospheric pressure and T is the surface tension of the soap solution.

Solution: The pressure inside the three soap bubbles is given by

$$p_1 = P + \frac{4T}{r_1},\ p_2 = P + \frac{4T}{r_2} \text{ and } p = P + \frac{4T}{r}$$

The corresponding volumes are given by $v_1 = \frac{4}{3}\pi r_1^3$, $v_2 = \frac{4}{3}\pi r_2^3$ and $v = \frac{4}{3}\pi r^3$

According to Boyle's law

$p_1 v_1 + p_2 v_2 = pv$

or $\left(P + \frac{4T}{r_1}\right)\frac{4}{3}\pi r_1^3 + \left(P + \frac{4T}{r_2}\right)\frac{4}{3}\pi r_2^3 = \left(P + \frac{4T}{r}\right)\frac{4}{3}\pi r^3$

or $P\left(r_1^3 + r_2^3 - r^3\right) = 4T\left(r^2 - r_1^2 - r_2^2\right)$

8.26. Calculate the amount of work done in spraying a drop of mercury of radius 1 mm into a million identical droplets under isothermal conditions. Surface tension of mercury = 550×10^{-3} N/m.

Solution: Let a drop of radius R be sprayed into n droplets of radius r. Then,

$\frac{4}{3}\pi R^3 = n \cdot \frac{4}{3}\pi r^3$

or $R = n^{1/3} r$

Increase in area $= 4\pi (nr^2 - R^2)$

$$= 4\pi\left(\frac{n \cdot R^2}{n^{2/3}} - R^2\right) = 4\pi R^2\left(n^{1/3} - 1\right)$$

Work done = $T \cdot 4\pi R^2 (n^{1/3} - 1)$

where T is the surface tension.

Work done = $550 \times 10^{-3} \times 4 \times 3.14 \times 10^{-6} (10^{6 \times 1/3} - 1)$

$= 550 \times 10^{-9} \times 4 \times 3.14 \times 99 = 6.839 \times 10^{-4}$ J

8.27. A vessel filled with air under pressure P has a soap bubble of diameter d. The air pressure having been reduced isothermally q-fold, the bubble diameter increased k-fold. Find the surface tension of the soap solution.

Solution: The excess of pressure inside a soap bubble is $\left(P + \frac{4T}{r_1}\right)$ where r_1 is the initial radii of the bubble and T is the surface tension of the soap solution.

The initial volume is $(4/3)\ \pi r_1^3$.

On changing the pressure to P/q the radius increases to kr_1.

Final pressure inside bubble = $\left(\frac{P}{q} + \frac{4T}{kr_1}\right)$ and

Final volume $\frac{4}{3}\pi k^3 r_1^3$

According to Boyle's law, pv = constant

or $\left(P+\frac{4T}{r_1}\right)\frac{4}{3}\pi r_1^3 = \left(\frac{P}{q}+\frac{4T}{kr_1}\right)\frac{4}{3}\pi k^3 r_1^3$

or $\left(P+\frac{4T}{r_1}\right) = \left(\frac{P}{q}+\frac{4T}{kr_1}\right)k^3$

or $P\left(1-\frac{k^3}{q}\right) = \frac{4T}{r_1}(k^2-1)$

or $T = \frac{Pr_1\left(1-k^3/q\right)}{4\left(k^2-1\right)}$

Since $r_1 = d/2$,

$$T = \frac{Pd}{8\left(k^2-1\right)}\left(1-\frac{k^3}{q}\right)$$

8.28. Calculate the force of attraction between two parallel plates separated by a distance $d = 0.2$ mm after a water drop of mass $m = 80$ mg is introduced between them. The wetting is assumed to be complete. Surface tension of water = 0.07 N/m.

Solution: The pressure inside the liquid film in excess of the atmospheric pressure is given by

$$p = \frac{2T}{d}$$

where T is the surface tension.

Force of attraction between the plates = $\frac{2T}{d}A$

where A is the circular area over which the liquid wets the plate.

$$\therefore\ F = \frac{2T}{\rho d^2}Ad\rho = \frac{2Tm}{\rho d^2}$$

On substituting the values of various terms together with density of water $\rho = 10^3$ kg/m^3, we have

$$F = \frac{2\times 0.07\times 80\times 10^{-6}}{10^3\times 4\times 10^{-8}} = 0.28 \text{ N}$$

8.29. A drop of mercury placed between two glass plates acquires the shape of a circular disc of radius R = 3 cm and thickness h = 0.2 mm. Calculate the mass m of a weight which on being placed at the upper

plate reduces the thickness of the drop by a factor $n = 3$. The angle of contact $\theta = 135°$ and the surface tension of mercury $= 490 \times 10^{-3}$ N/m.

Solution: Force F acting between the two plates $= \frac{2T}{h} \pi R^2 \cos\theta$.

On placing the mass m on the upper plate the thickness h' of the drop reduces to h/n and radius R' of the circular area increases to $\sqrt{n}\, R$.

On putting the mass on the plate, the pressure is given by

$$\frac{mg + F}{\pi R'^2} = \frac{2T}{h'} \cos\theta$$

or $$\frac{mg + F}{\pi n R^2} = \frac{2nT}{h} \cos\theta$$

or $$mg + F = \frac{2\pi R^2 T \cos\theta}{h} \cdot n^2$$

or $$mg = \frac{2\pi R^2 T \cos\theta}{h} \cdot n^2 - \frac{2T}{h} \cdot \pi R^2 \cos\theta$$

$$mg = \frac{2\pi R^2 T \cos\theta}{h}\left(n^2 - 1\right) \text{ or } m = \frac{2\pi R^2 T \cos\theta \left(n^2 - 1\right)}{gh}$$

On substituting the values of various terms, we have

$$m = \frac{2 \times 3.14 \times 9 \times 10^{-4} \times 490 \times 10^{-3} \times \frac{1}{\sqrt{2}} \times 8}{9.8 \times 2 \times 10^{-4}}$$

$$= \frac{3.14 \times 9 \times 490 \times 8 \times 10^{-3}}{9.8 \times 1.41}$$

$$= 8.017 \text{ kg}$$

8.30. A vertical capillary is brought in contact with a water surface. What is the amount of heat liberated while the water rises along the capillary? The wetting is assumed to be complete, and the surface tension is T.

Solution: When a capillary of radius r is placed in contact with water, the water rises in it through a distance h given by

$$\frac{2T}{r} = h\rho g$$

Assuming the angle of contact to be zero.

The force acting downwards = weight of the liquid = $\pi r^2 h \rho g$; ρ is the density of water.

Amount of heat liberated = work done = $\pi r^2 h \rho g \frac{h}{2}$

$$= \frac{\pi r^2}{2} \rho g h^2$$

$$= \frac{\pi r^2}{2} \rho \cdot g \cdot \frac{4T^2}{r^2 \rho^2 g^2}$$

$$= \frac{2\pi T^2}{\rho g}$$

8.31. A capillary tube of length l sealed at the top has an inner diameter d. On placing the tube vertically in contact of a liquid of density ρ, the liquid rises through a height h. Find the surface tension of the liquid. The angle of contact is θ and the atmospheric pressure p_0.

Solution: Initially the liquid will rise to height h in the vertical capillary tube. Now the liquid will compress the air contained in it and will cause an excess of pressure $\Delta p = p - p_0$.

According to Boyle's law,

$p \cdot (l - h) \cdot A = p_0 \cdot l \cdot A$. Where A is the area of cross-section of the tube.

or $$p = p_0 \frac{l}{l-h}$$

$$\therefore \Delta p = p - p_0 = p_0 \frac{h}{l-h}$$

Pressure due to surface tension $= \frac{2T}{r} \cos\theta = \left(h\rho g + p_0 \frac{h}{l-h} \right)$, where r is the radius of the capillary tube.

or $$T = \left(h\rho g + p_0 \frac{h}{l-h} \right) \frac{r}{2\cos\theta}$$

$$= \left(h\rho g + p_0 \frac{h}{l-h} \right) \frac{d}{4\cos\theta}$$

8.32. One end of a capillary tube of radius $r = 0.05$ cm is immersed in water to a depth of $h = 3$ cm. Calculate the amount of pressure required to blow an air bubble from the lower end of the tube. Surface tension of water = 0.07 N/m.

Solution: The pressure required should exceed the atmospheric pressure by $\rho gh + (2T/r)$; ρ is the density of water.

Excess pressure $= \rho gh + \frac{2T}{r}$

On putting the values of the various terms, we have

$$= 10^3 \times 9.8 \times 3 \times 10^{-2} + \frac{2 \times 0.07}{0.05 \times 10^{-2}} = 294 + 280 = 574 \text{ N/m}^2$$

8.33. A capillary tube sealed at the top has a length l = 99 mm and an inner diameter d = 28 μm. Calculate the length h up to which the capillary to be submerged vertically in water so that the level inside and outside the capillary of water same. Surface tension of water = 0.07 N/m and atmospheric pressure = 10^5 N/m^2.

Solution: Let the water rise up to a height h. Then we have

$p_0 \cdot l \cdot A = p \cdot (l - h) \cdot A$; p_0 = atmospheric pressure

A = Area of cross-section of the capillary tube

p = Pressure of air enclosed in length $(l - h)$

or $p = \dfrac{p_0 \cdot l}{l - h}$

Owing to the compression of the air by the water of height h, excess pressure of $p - p_0 = \dfrac{p_0 \cdot h}{l - h}$ is created. For the levels inside and outside the capillary to coincide, the excess pressure should be equal to the pressure due to surface tension, i.e.,

$$\frac{2T}{d/2} = p - p_0 = \frac{p_0 \cdot h}{l - h}$$

or $$\frac{4T(l-h)}{d} = p_0 \cdot h$$

or $$(l - h) = \frac{p_0 h}{4T/d}$$

or $$h = \frac{l}{1 + p_0 d/4T}$$

On substituting the values of the various terms, we have

$$h = \frac{99 \times 10^{-3}}{1 + \dfrac{10^5 \times 28 \times 10^{-6}}{7 \times 10^{-2} \times 4}} = 9 \times 10^{-3}\,\text{m} = 0.9\,\text{cm}$$

8.34. The limbs of a manometer consist of uniform capillary tubes of radii 1.44 × 10^{-3} and 7.2 × 10^{-4} m. Find the correct pressure difference if the level of the liquid (density 10^3 kg/m^3, surface tension 7.2 × 10^{-2} N/m) in the narrower tube is 0.2 m above that in the broader tube.

Solution: Let p_1 and p_2 be the pressures in th wide and narrow bore tubes of radii r_1 and r_2 respectively.

The pressures just below the meniscus in the two tube are $p_1 - \dfrac{2T}{r_1}$ and $p_2 - \dfrac{2T}{r_2}$ respectively.

$$\text{Pressure difference} = \left(p_1 - \frac{2T}{r_1}\right) - \left(p_2 - \frac{2T}{r_2}\right) = hdg$$

$$\therefore \text{ Correct pressure difference } \Delta p = p_1 - p_2 = hdg - 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\Delta p = 0.2 \times 10^3 \times 9.8 - 2 \times 7.2 \times 10^{-2}\left(\frac{1}{7.2\times10^{-4}} - \frac{1}{14.4\times10^{-4}}\right)$$

$$= 1860 \text{ N/m}^2$$

8.35. A conical glass capillary tube of length 0.1 m has diameters 10^{-3} and 5×10^{-4} m at the ends. When it is just immersed in a liquid at 0°C with larger diameter in contact with it the liquid rises to 8×10^{-2} m in the tube. If another cylindrical glass capillary tube *B* when immersed in the same liquid at 0°C, the liquid rises to 6×10^{-2} m height. The rise of liquid in the tube *B* is only 5.5×10^{-2} m when the liquid is at 50°C. Find the rate at which the surface tension changes with temperature considering the change to be linear. The density of the liquid is $\frac{1}{14} \times 10^4$ Kg/m^3 and angle of contact is zero. Effect of temperature on density of liquid and glass is negligible. (Roorkee 1994)

Solution: Let *r* be the radius of the meniscus in the conical tube

$$\tan\theta = \frac{r - r_1}{L - h} = \frac{r_2 - r_1}{L} \text{ (see Fig. 8.6)}$$

On substituting the values we have

$$\frac{r - 2.5\times10^{-4}}{0.1\times0.08} = \frac{(5-2.5)\times10^{-4}}{0.1}$$

$$r = 3\times10^{-4}\text{m}$$

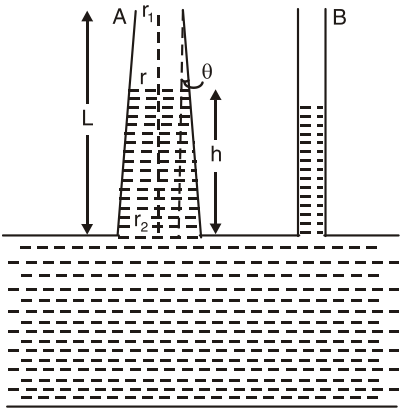


Fig. 8.6

At temperature 0°C

$$h_A \cdot r_A = h_B \cdot r_B = \frac{2T_0}{\rho g} = \text{constt.};$$

T_0 = surface tension at 0°C

$$r_B = \frac{0.08\times3\times10^{-4}}{6\times10^{-2}} = 4\times10^{-4}\text{m}$$

$$T_0 = \frac{h_0 \rho g r}{2} = \frac{1}{2}\left[6\times10^{-2}\times\frac{1}{14}\times10^4\times9.8\times4\times10^{-4}\right] = 8.4\times10^{-2}\text{N/m}$$

$$\frac{T_{50}}{T_0} = \frac{h_{50}}{h_0}$$

$$\therefore T_{50} = \frac{5.5\times10^{-2}}{6\times10^{-2}} \times 8.4\times10^{-2} = 7.7\times10^{-2}$$

$$\frac{\Delta T}{\Delta\theta} = \frac{T_{50} - T_0}{50} = \frac{(7.7-8.4)\times10^{-2}}{50} = -1.4\times10^{-2}\text{N/m/°C}$$

8.36. A conical capillary tube *ABC* with a fine bore at *A* is dipped into water which wets its wall. Calculate the height upto which water will rise.

Solution: In view of Fig. 8.7

$h = \dfrac{2T\cos\theta}{R\rho\cdot g}$; θ is the semi vertical angle of the tube and *R* is the radius of the tube at the meniscus *FG*.

Consider similar triangles *ABD* and *AFE*.

$$\frac{FE}{BD} = \frac{AE}{AD}$$

$$\frac{R}{r} = \frac{l-h}{l}$$

$$\therefore R = (l-h)\,\frac{r}{l}$$

or $$h = \frac{2T\cos\theta\cdot l}{(l-h)r\rho g}$$

or $$hlr - h^2r = \frac{2Tl\cos\theta}{\rho g}$$

or $$h^2 - hl + \frac{2Tl\cos\theta}{\rho g r} = 0$$

$$h = \frac{l}{2} \pm \sqrt{\frac{l^2}{4} - \frac{2lT\cos\theta}{r\rho g}}$$

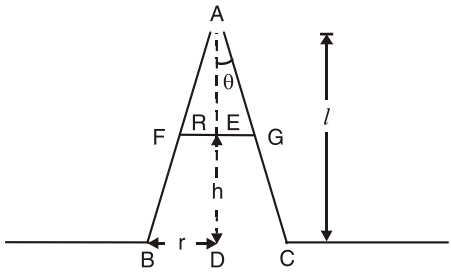


Fig. 8.7

8.37. Two vertical plates *AB* and *CD* submerged partially in water forms a wedge of an angle $\delta\phi$. The edge of this wedge is vertical. Find the height *h* upto which water will rise as a function of the distance *x* measured from the edge.

Solution: Let *r* be the radius of the wedge and *R* be the radius of curvature of the meniscus then

$r = R\cos\theta$ (see Fig. 8.8)

$$\frac{\delta\phi}{2} = \frac{r}{x}$$

$$\therefore R = \frac{r}{\cos\theta} = \frac{\delta\phi\cdot x}{2\cos\theta}$$

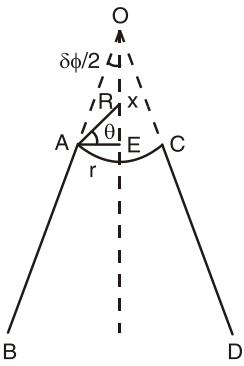


Fig. 8.8

Excess of pressure $p = T\left(\frac{1}{R} + \frac{1}{R'}\right)$

here the two curvatures are at right angles to each other, $R' = \infty$

$$\therefore p = \frac{T}{R}$$

Now $h\rho g = p = \frac{T}{R} = \frac{2T\cos\theta}{\delta\phi x}$

or $$h = \frac{2T\cos\theta}{\delta\phi x \cdot \rho g}$$

8.38. A soap bubble of radius R is connected to a capillary tube of length l and radius r. Calculate the lifetime of the bubble assume the surface tension of soap solution as T and coefficient of viscosity η.

Solution: According to Poiseuilli's formula the quantity of fluid flowing per sec is given by

$V = \frac{\pi P r^4}{8\eta l}$; P = pressure difference across the length l through which the fluid is flowing.

Let x be the radius of the bubble at an instant t and $x - dx$ at an instant $t + dt$.

$$V = \frac{\pi\frac{4T}{x}\cdot r^4}{8\eta l} = \frac{4}{3}\pi\left[x^3 - (x - dx)^3\right]/dt$$

$$= \frac{4\pi}{3}\left[-3x^2\frac{dx}{dt}\right]$$

or $$\frac{4T\pi r^4}{8\eta l x} = -4\pi x^2 \cdot \frac{dx}{dt}$$

or $$\int_0^{\tau} dt = -\frac{8\eta l}{Tr^4}\int_R^0 x^3 dx$$

$$\tau = \frac{8\eta l}{Tr^4}\cdot\frac{R^4}{4}$$

$$= \frac{2\eta l R^4}{Tr^4}$$

8.39. Calculate the change in the free energy of the layer when two similar mercury droplet of diameter $d = 2$ mm merge isothermally. Surface tension of mercury $T = 490 \times 10^{-3}$ N/m.

Solution: Total surface energy of the system of two droplets

$$= 2 \times 4\pi \cdot \frac{d^2}{4} \cdot T = 2\pi d^2 T$$

Now $\frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = 2\times\frac{4\pi}{3}\left(\frac{d}{2}\right)^3$; D is the diameter of the drop formed by coalescing the two drops.

$D = 2^{1/3}d$

Surface energy of the big drop $= \pi D^2 T$

Change in the surface energy $= 2\pi T\left[-\frac{D^2}{2} + d^2\right]$

$$\Delta E = 2\pi T d^2\left[1 - \frac{2^{2/3}}{2}\right]$$

$$= 2\pi T d^2\left[1 - \frac{1.588}{2}\right]$$

On putting the value we have

$\Delta E = 2\times 3.14\times 490\times 10^{-3}\times 4\times 10^{-6}\,[1 - 0.794]$

$= 2\times 3.14\times 490\times 4\times 10^{-9}\times 0.206$

$= 2.536\ \mu J$

8.40. Consider a wave travelling over the surface of water. The velocity of propagation depends both on the force of gravity and surface tension. Calculate the critical wavelength λ_C and minimum velocity V_{min} of the wave at the surface of water beyond which the propagation is mainly due to the force of gravity. Surface tension of water $= 73\times 10^{-3}$ N/m.

Solution: Velocity V of the wave is given by

$$V = \sqrt{\frac{\lambda g}{2\pi} + \frac{2\pi T}{\lambda\rho}}\ ;\ T = \text{Surface tension of water}$$

ρ = density of water

The sum is minimum when

$$\frac{\lambda g}{2\pi} = \frac{2\pi T}{\lambda\rho}$$

Therefore $\lambda_C = 2\pi\sqrt{\frac{T}{g\rho}}$

$$V_{min} = \sqrt{2}\cdot\left(\frac{Tg}{\rho}\right)^{1/4}$$

On substituting the values of various terms we have

$$\lambda_C = 2\times 3.14\left[\frac{73\times 10^{-3}}{9.8\times 10^3}\right]^{1/2}$$

$= 1.71$ cm

$$V_{\min} = 1.414\left(\frac{73\times10^{-3}\times9.8}{1000}\right)^{1/4}$$

$$= 23.1 \text{ cm/sec}$$

8.41. A water drop falls in air with a uniform velocity. Calculate the difference in the radius of curvature at the upper and lower points of the surface of the drop which are separated by a distance *h*.

Solution: Let r_1 and r_2 be the radius of curvature at the two points (see Fig. 8.9).

The pressure inside the surface just below the upper surface is given by

$$p_1 = p_0 + \frac{2T}{r_1}; \; p_0 = \text{atmospheric pressure}$$

Similarly the pressure just above the lower surface is given by

$$p_2 = p_0 + \frac{2T}{r_2}$$

$$p_1 - p_2 = 2T\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \rho g h$$

$$(r_2 - r_1) = \frac{\rho g h \cdot r_1 r_2}{2T}$$

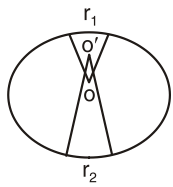


Fig. 8.9

When *h* is small then $r_1 = r_2 \cong \frac{h}{2}$

$$\therefore r_2 - r_1 = \frac{\rho g h^3}{8T}$$

8.42. Find the force of attraction between the two plates of width *l* partially submerged into water keeping them *d* apart (see Fig. 8.10). The wetting is complete and the water lying between the plates is not reaching the upper edge of the plates.

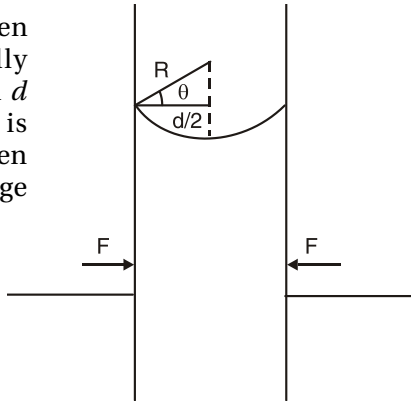


Fig. 8.10

Solution: θ is taken to be zero.

$$\frac{d}{2} = R$$

$$p = T\left[\frac{1}{R} + \frac{1}{R'}\right]; R' = \infty$$

$$\therefore p = \frac{T}{R} = \frac{2T}{d} = h\rho g; \; h = \text{height of water column.}$$

here the pressure just below the meniscus in between the plates is smaller than the pressure due to the atmosphere at the same level outside them and due to this they are pushed towards each other by a force F or in other words they seem to attract each other.

Now $p = \dfrac{2F}{A} = \dfrac{2T}{d}$; $A = lh$

or $F = \dfrac{TA}{d} = \dfrac{T \cdot lh}{d}$

$= \dfrac{Tl}{d} \cdot \dfrac{2T}{d\rho g}$

$= \dfrac{2T^2 l}{\rho g \cdot d^2}$

8.43. Two equal drops of water fall through air with a steady velocity of 10 cm/s. If the drops coalesce, what will be the new terminal velocity ?

Solution: Let R be the radius of the big drop formed and r the radii of the drops which are going to coalesce. The volume of the big drop is equal to the combined volume of the two drops, i.e.,

$\dfrac{4}{3}\pi R^3 = 2 \times \dfrac{4}{3}\pi r^3$

or $R = 2^{1/3} r$

Terminal veolcity $v = \dfrac{2}{9}\dfrac{(\rho - \sigma)}{\eta} g r^2$

where ρ = density of water

σ = density of air

and η = viscosity of air

Now $10 = \dfrac{2}{9}\dfrac{(\rho - \sigma)}{\eta} g \cdot r^2$

and $V = \dfrac{2}{9}\dfrac{(\rho - \sigma)}{\eta} g R^2$

On dividing the two, we have

$\dfrac{V}{10} = \dfrac{R^2}{r^2} = 2^{2/3}$

$\therefore V = 10 \times 2^{2/3} = 10 \times 1.587 = 15.87$ cm/s

8.44. An air bubble of radius 1 cm, rises steadily through a liquid of density 1.5 g/cm^3 at the rate of 0.35 cm/sec. Calculate the coefficient of viscosity of the liquid.

Solution: The terminal velocity is given by the equation

$$v = -\frac{2}{9}\frac{(\rho - \sigma)}{\eta} g \cdot r^2$$

On neglecting the density of air, we have

$$v = -\frac{2}{9}\frac{\sigma}{\eta} g r^2$$

$$\therefore \eta = -\frac{2}{9} \cdot \frac{\sigma}{v} g r^2$$

$$= -\frac{2}{9} \times \frac{1.5 \times 980 \times 1}{-0.35} = 9.33 \times 10^2 \text{ poise (all are in the CGS system)}$$

here v is negative.

8.45. A small sphere is dropped in a Jar containing a liquid. The dragging force has been found to vary with its velocity according to $R = -bv$. Where b is constant. Find its velocity as a function of time.

Solution: The equation of motion of the sphere is given by

$$mg - bv = m\frac{dv}{dt}$$

When $\frac{dv}{dt} = 0$, $v = v_c$ (terminal velocity) $= \frac{mg}{b}$

$$\int dt = \frac{dv}{g - \frac{b}{m}v}$$

$$t = -\frac{m}{b}\log_e\left(g - \frac{bv}{m}\right) + C$$

The value of the constant C is determined using the boundary conditions

At $t = 0$, $v = 0$

$$\therefore C = \frac{m}{b}\log_e g$$

$$t = -\frac{m}{b}\log_e\left(1 - \frac{bv}{mg}\right)$$

$$= -\frac{m}{b}\log_e\left(1 - \frac{v}{v_c}\right)$$

or $$1 - \frac{v}{v_c} = e^{-bt/m}$$

or $$v = v_c\left(1 - e^{-bt/m}\right)$$

8.46. A small sphere of mass m = 3 gm is dropped in a viscous liquid without any initial velocity. The sphere acquires a terminal velocity v_c = 7.5 cm/sec. Calculate the time which the sphere will take to acquire a velocity equal to 0.8 v_c.

Solution:

$$b = \frac{mg}{v_c} = \frac{3 \times 980}{7.5} = 392 \text{ gm/sec}$$

$$\frac{m}{b} = \frac{3}{392} = 7.65 \times 10^{-3} \text{ sec}$$

$$0.8\, v_c = v_c \left(1 - e^{-\frac{t}{7.65 \times 10^{-3}}}\right)$$

$$e^{-\frac{t}{7.65 \times 10^{-3}}} = 0.2$$

$$e^{\frac{t}{7.65 \times 10^{-3}}} = 5$$

$$\frac{t}{7.65 \times 10^{-3}} = 2.303 \log_{10}^{5}$$

$$t = 7.65 \times 10^{-3} \times 2.303 \times 0.6990$$

$$t = 12.3 \times 10^{-3} \text{ sec}$$

8.47. In a region where there is no gravitational field a small sphere moving with an initial velocity v_0 in a viscous medium experiences a draging force $R = -bv$. Determine the velocity of sphere as a function of time.

Solution: Equation of motion of sphere is given by

$$m\frac{dv}{dt} = -bv$$

$$\int \frac{dv}{v} = -\frac{b}{m}\int dt$$

$$\log_e v = -\frac{bt}{m} + C$$

at $t = 0$, $v = v_0$

$$\therefore C = \log v_0$$

or $$v = v_0 e^{-bt/m}$$

8.48. A ball of mass m is placed in a tall vessel containing a liquid and is released without a push. The density of the liquid is $1/\eta$ times that of the ball. When the ball moves a force of resistance of the medium appears that is proportional to the speed; find the velocity of the ball as a function of time.

Solution: The equation of motion of the ball is

$$\frac{4}{3}\pi r^3\rho\frac{dv}{dt}=\frac{4}{3}\pi r^3\rho g-\frac{4}{3}\pi r^3\frac{\rho}{\eta}g-kv$$

or $$m\frac{dv}{dt}=mg-\frac{mg}{\eta}-kv$$

or $$\frac{dv}{dt}=g\left(1-\frac{1}{\eta}\right)-\frac{k}{m}v$$

or $$\frac{dv}{g\left(1-\frac{1}{\eta}\right)-\frac{k}{m}v}=dt$$

On integration we have

$$\log\left[g\left(1-\frac{1}{\eta}\right)-\frac{k}{m}v\right]=-\frac{k}{m}\cdot t+C$$

At $t=0, v=0 \therefore C=\log g\left(1-\frac{1}{\eta}\right)$

With this value of C we have

$$1-\frac{kv}{m}\Big/g\left(1-\frac{1}{\eta}\right)=e^{-kt/m}$$

or $$v=\frac{mg}{k}\left(1-\frac{1}{\eta}\right)\left(1-e^{-kt/m}\right)$$

8.49. A ball rises to the surface at a constant velocity is a liquid whose density is four times greater than that of the material of the ball. How many times is the force of friction acting on the rising ball greater than its weight?

Solution: Let ρ be the density of the material of the ball, the equation of motion is given by

$$\frac{4}{3}\pi r^3\rho g-\frac{4}{3}\pi r^3\cdot 4\rho g+F=0;\quad F=\text{Force of friction}$$
$$r=\text{radius of the ball}$$

$$\therefore F=\frac{4}{3}\pi r^3\cdot 3\rho g$$

= 3 times the weight of the ball

8.50. A cork ball with a radius of 5 mm rises to the surface in a vessel filled with castor oil. Find the dynamic and kinematic viscosities of the castor oil if the ball rises with a constant velocity of 3.5 cm/sec. Density of cork $\rho = 200$ kg/m^3. Density of castor oil $\sigma = 900$ kg/m^3.

Solution: The equation of motion of the ball is given by

$\frac{4}{3}\pi r^3 \rho g - \frac{4}{3}\pi r^3 \sigma g + 6\pi\eta r v = 0$; r = radius of the ball, here v is negative.

η = dynamic viscosity.

$9\,\eta\, v = 2r^2\,(\sigma - \rho)\, g$

$\therefore\ \eta = \dfrac{2r^2\,(\sigma-\rho)\,g}{9v}$

On putting the values we have

$$\eta = \frac{2\times 25\times 10^{-6}\times 700\times 9.8}{9\times 3.5\times 10^{-2}}$$

$$= \frac{0.5\times 7\times 9.8}{9\times 3.5} = 1.09 \text{ N.s/m}^2$$

Kinematic viscosity $v = \dfrac{\eta}{\sigma} = 1.21\times 10^{-3}$ m^2/s

8.51. A sphere of radius $r = 1$ mm is submerged into glycerine contained in a tall and wide vessel from the walls of the vessel without any push. Calculate the distance covered by the sphere in time t. Density of the material of the sphere $\rho = 10\times 10^3$ kg/m^3, density of glycerine $\rho_0 = 1.21\times 10^3$ kg/m^3, viscosity of glycerine $\eta = 0.35$ Pa.s.

Solution: The equation of motion of the sphere is given by

$$\frac{4}{3}\pi r^3 \rho\cdot\frac{dv}{dt} = \frac{4}{3}\pi r^3 \rho g - \frac{4}{3}\pi r^3 \rho_0 g - 6\pi\eta r v$$

or $$\frac{dv}{dt} = g - \frac{\rho_0}{\rho}g - \frac{9}{2}\frac{\eta}{r^2}\cdot\frac{v}{\rho}$$

or $$\int_0^v \frac{dv}{\left[g\left(1-\frac{\rho_0}{\rho}\right) - \frac{9\eta v}{2r^2\rho}\right]} = \int_0^t dt$$

Let us substitute $g\left(1-\dfrac{\rho_0}{\rho}\right) - \dfrac{9\eta v}{2r^2\rho} = x$, then

$$\frac{2r^2\rho}{9\eta}\int_{g\left(1-\frac{\rho_0}{\rho}\right)}^{x}\frac{dx}{x} = -\int_0^t dt$$

or $$\log\left[g\left(1-\frac{\rho_0}{\rho}\right) - \frac{9\eta v}{2r^2\rho}\right] - \log g\left(1-\frac{\rho_0}{\rho}\right) = -\frac{9\eta t}{2r^2\rho}$$

or $$1 - \frac{9\eta v}{2r^2\rho g\left(1-\frac{\rho_0}{\rho}\right)} = e^{-\frac{9\eta t}{2r^2\rho}}$$

or $$v = \frac{ds}{dt} = \frac{2r^2 g(\rho - \rho_0)}{9\eta}\left[1 - e^{-\frac{9\eta t}{2r^2\rho}}\right]$$

On integration

$$s = \frac{2r^2 g(\rho - \rho_0)}{9\eta} \cdot t + \frac{2r^2 g(\rho - \rho_0)}{9\eta} \times \frac{2r^2\rho}{9\eta} e^{-\frac{9\eta t}{2r^2\rho}} + C$$

At $t = 0, s = 0 \therefore C = -\frac{4r^4 g\rho(\rho - \rho_0)}{81\eta^2}$

$$\therefore s = \frac{2r^2 g(\rho - \rho_0)}{9\eta} t - \frac{4r^4 g\rho(\rho - \rho_0)}{81\eta^2}\left[1 - e^{-\frac{9\eta t}{2r^2\rho}}\right]$$

On putting the values of various terms we have

$$s = 0.0547\, t - 0.000347\,[1 - e^{-157.5t}]$$

8.52. A steel ball falls in a broad vessel filled with transformer oil having density $\sigma = 900$ kg/m^3. Find the maximum diameter of the ball. Viscosity of the oil $\eta = 0.8$ N.s/m^2. Density of steel $\rho = 7700$ kg/m^3. Reynold's number $Re = 0.5$.

Solution: The equation of motion of the ball is given by

$$\frac{4}{3}\pi\frac{D^3}{8}(\rho - \sigma)g = 6\pi\eta\frac{D}{2}v = 6\pi\eta\frac{D}{2} \cdot \eta\frac{Re}{D \cdot \sigma};\ Re = \frac{Dv\sigma}{\eta}$$

D = diameter of the ball

$$\frac{D^3}{6}(\rho - \sigma)g = \frac{3\eta^2 Re}{\sigma}$$

$$D = \left(\frac{18 \cdot \eta^2 \cdot Re}{\sigma(\rho - \sigma)g}\right)^{1/3}$$

On putting the values

$$D = \left[\frac{18 \times 0.8^2 \times 0.5}{900 \times (7700 - 900) \times 9.8}\right]^{1/3}$$

$$= 4.58\,\text{mm}$$

8.53. A steady flow of air directed vertically upward and having a speed of $u = 20$ cm/sec set in over a heated section of the Earth's surface. The flow contains a spherical dust particle moving upward at a steady speed of $v = 4$ cm/sec. Determine the radius of the dust particle. Density of the particle $\rho = 5 \times 10^3$ kg/m^3, density of air = 1.29 kg/m^3, viscosity of air $\eta = 1.72 \times 10^{-5}$ Pa.s.

Solution: The equation of motion of the dust particle is given by

$\frac{4}{3}\pi r^3 \rho g = \frac{4}{3}\pi r^3 \sigma g + 6\pi\eta r(v-u)$; r = radius of the dust particle, here v is to be taken –ve.

or $r^2(\rho-\sigma)g = \frac{9}{2}\eta(u-v)$

$$\therefore r = \sqrt{\frac{9\eta(u-v)}{2(\rho-\sigma)g}}$$

On putting the values of various terms we have

$$r = \sqrt{\frac{9}{2}\times\frac{1.72\times10^{-5}\times16\times10^{-2}\times10^{-3}}{9.8[5000-1.29]}}$$

$= 1.59 \times 10^{-2}$ mm

8.54. A pitot tube is fixed to mains of diameter 10 cm, and the pressure difference indicated by the manometer is 5 cm height of water column. Find the volume of water flowing through the mains is 60 s.

Solution: Area of cross-section of the main $= \pi \times 5^2 \times 10^{-4}$ m^2

Loss in kinetic energy per unit mass = gain in pressure energy

or $\frac{1}{2}v^2 = \frac{p}{\rho} = gh$

or $v = \sqrt{2gh}$

Volume of water flowing per second

$$= av = 25\pi \times 10^{-4} \times \sqrt{2\times9.8\times5\times10^{-2}}$$

$$= 25\pi \times 10^{-4} \times \frac{\sqrt{98}}{10}$$

$$= 2.5\pi \times 10^{-4} \times \sqrt{2} \times 7$$

Volume of water flowing in 60 s $= 2.5\pi \times 10^{-4} \times \sqrt{2} \times 7 \times 60$

$= 0.465$ m^3

8.55. A bottle is fitted with a siphon made of capillary tube. Compare the time taken to empty the bottle when it is filled (i) with water, and (ii) with a liquid of specific gravity 0.9. The viscosities of water and the liquid are 0.01 and 0.03 poise respectively.

solution: Volume of liquid flowing in t sec through a capillary tube is given by

$$V = \frac{\pi P a^4 t}{8\eta l}$$

Where l and a are the length and radius of the capillary tube. η is the

coefficient of viscosity. P is the pressure difference across the ends of the tube.

We know that $P \propto \rho$; ρ being the specific gravity of the liquid.

$$V_1 \propto \frac{\pi \rho_1 a^4 t_1}{8 \eta_1 l}$$

and $$V_2 \propto \frac{\pi \rho_2 a^4 t_2}{8 \eta_2 l}$$

here V_1 and V_2 are the volume of water and liquid flowing in t_1 and t_2 s respectively.

$$\therefore \frac{\rho_1 t_1}{\eta_1} = \frac{\rho_2 t_2}{\eta_2}$$

or $$\frac{t_1}{t_2} = \frac{\eta_1}{\eta_2} \cdot \frac{\rho_2}{\rho_1} = \frac{0.01}{0.03} \times 0.9 = 0.3 \text{ s}$$

8.56. A tube with two limbs bent at right angles to each other is held with one end dipped in a stream of water and opposite to the direction of flow. If the velocity of water is 1 m/s, find the height to which water rises in the vertical limb of the tube.

Solution: The flow of water through the tube will stop. This will result in loss of kinetic energy which will be compensated by gain in pressure energy.

Kinetic energy per unit mass = pressure energy per unit mass

or $$\frac{1}{2} v^2 = \frac{p}{\rho} = gh$$

or $$h = \frac{v^2}{2g} = \frac{1}{2 \times 9.8} = 5.1 \text{ cm}$$

8.57. The radii of a pipe are 5 cm and 3 cm at the points where a venturimeter is connected. The difference of pressure at these points corresponds to 10 cm of water column. Find the volume of water following per second.

Solution:

Volume of water flowing per second $V = a_1 a_2 \sqrt{\frac{2(p_1 - p_2)}{(a_1^2 - a_2^2)\rho}}$

where p_1 and p_2 are the pressure at the two points where the areas of cross-section are a_1 and a_2 respectively, and ρ is the density of water.

On substituting the values of various terms, we have

$$V = \pi \times 25 \times \pi \times 9 \times 10^{-8} \sqrt{\frac{2 \times 10 \times 10^{-2} \times 9.8 \times 10^{3}}{\left[(25\pi)^2 - (9\pi)^2\right] \times 10^{-8} \times 10^{3}}}$$

$$= \frac{\pi \times 25 \times 9 \times 10^{-5} \times 2 \times 7}{4 \times \sqrt{34}}$$

$$= 0.004244 \text{ m}^3 = 4244 \text{ c.c.}$$

8.58. Water is flowing through a pipe fitted with a jet at one end at a rate of 10 c.c. per second. The diameter of the orifice is 1 mm. What is the reaction force observed on the hands when one holds it?

Solution: Consider a tube of area of cross-section a. Let us allow water to flow through it with a velocity v m/s.

Volume of water flowing per second = av

Mass of water flowing per second = $a\rho v$, ρ is the density of water

Momentum of water flowing per second = $a\rho v^2$

$$\therefore \text{Force} = a\rho v^2 = \frac{(av)^2 \rho}{a} = \frac{(av)^2 \rho}{\pi r^2}$$

$$= \frac{\left(10 \times 10^{-6}\right)^2 \times 10^3}{3.14 \times 25 \times 10^{-8}} = \frac{10}{3.14 \times 25} = 0.1275 \text{ N}$$

8.59. A fire extinguishing pipe is fitted with a narrow bore tube of diameter 5 cm at one end. Calculate the force exerted by water when it is squirted on a wall with a horizontal velocity of 20 m/s: (i) when water falls vertically downwards after striking the wall, and (ii) when water returns with a horizontal velocity of 10 m/s.

Solution:

(i) In the first case

Momentum of water flowing per second

= force = $a\rho v^2$

= $\pi r^2 \rho v^2 = 3.14 \times (2.5)^2 \times 10^{-4} \times 10^3 \times 20^2$

= 785.0 N

(ii) Required force in the second case

= $a\rho v^2 + a\rho v v'$; v' is the velocity of water at the return

= $3.14 \times (2.5)^2 \times 10^{-4} \times 10^3 \times 20^2 + 3.14 \times (2.5)^2 \times 10^{-4} \times 10^3 \times 20 \times 10$

= 785 + 392.5

= 1177.5 N

8.60. A heart pumps blood at a rate of 5000 c.c. per minute at a pressure of 130 mm of mercury. Calculate the power of the heart.

Solution:

$$\text{Power} = \frac{\text{work}}{\text{time}} = \frac{\text{force} \times \text{displacement}}{\text{time}} = F \times v$$
$$= p \cdot av$$

where p is the pressure and a is the area of cross-section of the vein

Power = p × volume of blood flowing per second

On substituting the values, we have,

$$\text{Power} = 130 \times 10^{-3} \times 13.6 \times 10^3 \times 9.8 \times \frac{5000}{60} \times 10^{-6}$$
$$= \frac{13 \times 13.6 \times 9.8 \times 5}{6} \times 10^{-3}$$
$$= 1.443 \text{ W}$$

8.61. A tank with a small hole at its bottom contains water and kerosene of densities $\rho_1 = 1.0$ and $\rho_2 = 0.8$ g/cm^3 respectively. At what velocity will water flow through the hole if the height of water column $h_1 = 20$ cm and that of kerosene $h_2 = 10$ cm ?

Solution:

$$\text{Total pressure energy per unit mass} = \left(h_1 g + h_2 g \frac{\rho_2}{\rho_1} \right)$$

$$\text{Kinetic energy per unit mass} = \frac{1}{2} v^2$$

$$v^2 = 2g \left(h_1 + h_2 \frac{\rho_2}{\rho_1} \right)$$

or $$v = \sqrt{2g \left(h_1 + h_2 \frac{\rho_2}{\rho_1} \right)}$$

On substituting the values, we have

$$v = \sqrt{2 \times 980 \left(20 + 10 \times 0.8\right)} = \sqrt{2 \times 980 \times 28} \text{ (all in the CGS system)}$$
$$= 234.3 \text{ cm/s}$$

8.62. A L-shaped tube with a small orifice O at the closed end is lowered in a canal in which water is flowing with velocity $v = 5$ m/s (see Fig. 8.11). The orifice is at a height $h = 25$ cm above the water level. To what height H will the water jet spurt ? $g = 10$ m/s^2.

Solution: In this case, total kinetic energy per unit mass = total pressure energy per unit mass

or $\frac{1}{2}v^2 = g(H+h)$

or $H = \frac{v^2}{2g} - h$

On substituting the values, we have

$H = \frac{25}{2\times 10} - 0.25$

$= 1.25 - 0.25 = 1$ m

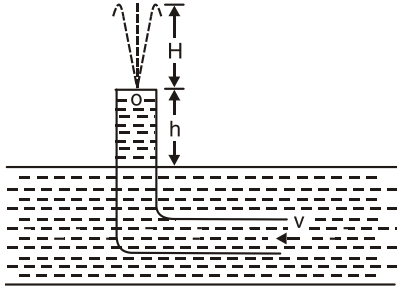


Fig. 8.11

8.63. A horizontally placed cylinder containing a piston has an oriffice O of cross-sectional area $a = 2$ mm^2. The space between the closed end of the cylinder containing the orifice and the piston is filled with water of volume $V = 2000$ c.c. (Fig. 8.12). A constant force is applied on the piston for a duration of $t = 3$ min. Calculate the amount of work done in order to squeeze the entire bulk of water from the cylinder.

Solution: Let v be the velocity of water going out of the orifice. Then,

Volume of water flowing per second = av

Total volume of water flowing in t s = $av \cdot t = V$

Let a force F be applied on the piston of area of cross-section A. Then $F = PA$ where P is the pressure on the liquid.

Now the pressure energy per unit mass = kinetic energy per unit mass

i.e., $\frac{P}{\rho} = \frac{1}{2}v^2$

or $\frac{F}{A} = \frac{1}{2}\rho \cdot v^2 = \frac{1}{2}\rho \frac{V^2}{a^2t^2}$

or $F = \frac{1}{2}\rho \frac{V^2}{a^2t^2} \cdot A$

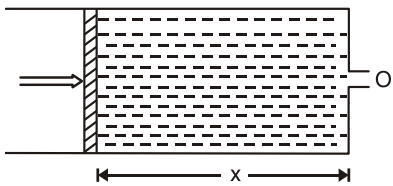


Fig. 8.12

Work done $= F \cdot x = \frac{1}{2}\rho \frac{V^2}{a^2t^2} A \cdot x = \frac{1}{2}\rho \frac{V^3}{a^2t^2}$

On substituting the values of various terms, we have

$$\text{Work done} = \frac{1}{2} \times \frac{10^3 \times \left(2000 \times 10^{-6}\right)^3}{\left(2 \times 10^{-6}\right)^2 \times 180 \times 180}$$

$$= \frac{10^3 \times 8 \times 10^{-9}}{2 \times 4 \times 10^{-12} \times 18 \times 18 \times 10^2}$$

$$= \frac{10^4}{18 \times 18} = 30.86 \text{ J}$$

8.64. A pitot tube (Fig. 8.13) is fixed in a pipeline of cross-sectional area A = 5 cm^2 through which nitrogen gas of density d = 12.5 g/cm^3 is flowing. The difference of pressure corresponds to a length l = 10 cm of water column. Find the volume of gas flowing per second. Density of water d_0 = 1 g/cm^3.

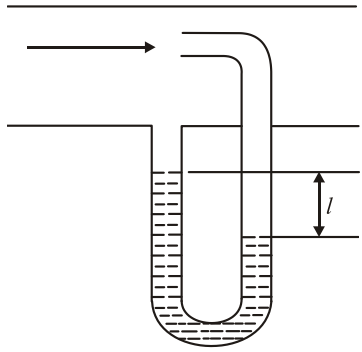


Fig. 8.13

Solution: Let v be the velocity of gas. Then

Volume of gas flowing per second

$= A \cdot v$

The pressure energy per unit mass = kinetic energy per unit mass

or $\frac{P}{d} = \frac{1}{2}v^2$

Now the pressure is measured in terms of a water column of length l, i.e.,

$$P = ld_0g$$

$$\therefore \sqrt{\frac{2ld_0g}{d}} = v$$

Volume of gas flowing per second $= Av = A\sqrt{\frac{2ld_0g}{d}}$

On substituting the values of various terms, we have

Volume of gas flowing per second $= 5 \times 10^{-4}\sqrt{\frac{2\times10\times10^{-2}\times1\times9.8}{1.25}}$

$$= \frac{5\times10^{-4}\times2\times7}{5\sqrt{5}} = \frac{14}{\sqrt{5}}\times10^{-4}$$

$$= 6.26 \times 10^{-4} \text{ m}^3$$

8.65. A horizontal cylindrical tube of length l filled with liquid has an orifice O at one end. The tube rotates with an angular velocity ω about an axis PQ perpendicular to its length and passing through the other end (Fig. 8.14). Find the velocity of the liquid in terms of the distance H.

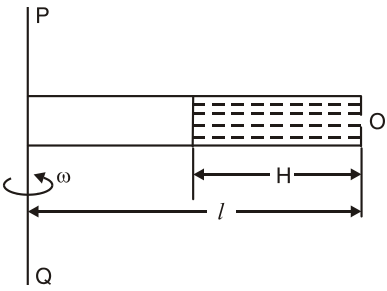


Fig. 8.14

Solution: Let us assume that the entire length of the tube is filled with

a liquid of density ρ. The total mass of the liquid = $A\rho l$ when A is the area of cross-section of the tube.

Mass per unit length = $A\rho$

Consider an element of liquid of length dx situated at a distance x from the axis PQ.

Centrifugal force due to the rotation of the system = $A\rho\, dx\, x\, \omega^2$

Pressure energy per unit mass = $\dfrac{dp}{\rho} = \omega^2\, x\, dx$

Total pressure energy = $\int \dfrac{dp}{\rho} = \dfrac{p}{\rho} = \omega^2 \int_{l-H}^{l} x dx = \dfrac{1}{2}\omega^2 \left[l^2 - (l-H)^2 \right]$

$$= \frac{1}{2}\omega^2\, H(2l - H)$$

This should be equal to kinetic energy per unit mass

i.e., $\dfrac{1}{2}\omega^2 H^2 \left(\dfrac{2l}{H} - 1 \right) = \dfrac{1}{2} v^2$

or $\omega H \sqrt{\left(\dfrac{2l}{H} - 1 \right)} = v$

8.66. In Fig. 8.15 water of density d = 1.0 g/cm^3 flows along a horizontal tube out of a wide tank. Find the velocity of water flow if H_1 = 5 cm, H_2 = 10 cm and H_3 = 20 cm. All distances l are equal.

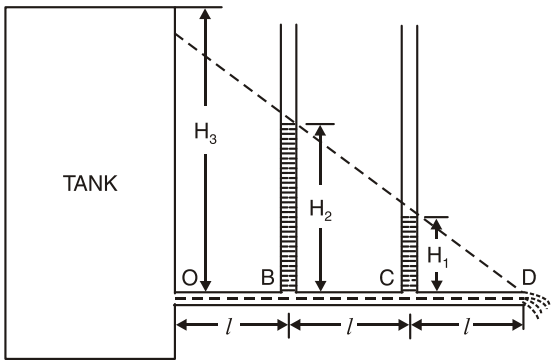


Fig. 8.15

Solution: At O the total energy per unit mass = $H_3 g$.

At B we have

$$H_3 g = H_2 g + \frac{1}{2} v_B^2$$

At C we have

$$\frac{1}{2} v_B^2 = H_1 g + \frac{1}{2} v_C^2$$

or $v_C = \sqrt{2(H_3 - H_2 - H_1)g}$

On substituting the values of H_1, H_2 and H_3, we have

$v_C = \sqrt{2(20-10-5)\times 980}$

$= \sqrt{98\times 100} = 70\sqrt{2} = 70\times 1.41 = 98.7$ cm/s

8.67. Water flows with a velocity v along a rubber tube in the form of a circular ring of radius R (Fig. 8.16a). The diameter of the tube is d. With what force is the rubber tube stretched ?

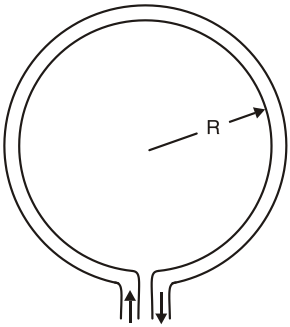


Fig. 8.16a

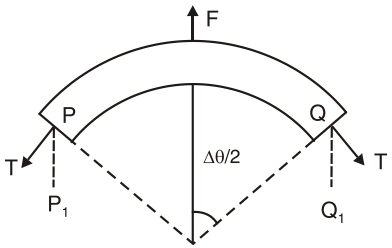


Fig. 8.16b

Solution: Let the required force be T (Fig. 8.16b).

The resolved part of T along PP_1 and

$QQ_1 = T\sin\dfrac{\Delta\theta}{2}$

Net downward force $= 2T\sin\dfrac{\Delta\theta}{2}$

$= 2T\dfrac{\Delta\theta}{2} = T\cdot\Delta\theta$

This force is balanced by the force $F = \dfrac{mv^2}{R}$ where m is the mass of water confined in the part of the tube PQ.

$m = \pi\dfrac{d^2}{4}\times R\Delta\theta\cdot\rho$; where ρ is the density of water.

$\therefore\ T\cdot\Delta\theta = \dfrac{\pi d^2}{4}\cdot R\Delta\theta\cdot\rho\cdot\dfrac{v^2}{R}$

or $T = \dfrac{\pi d^2 R\rho}{4}\cdot\dfrac{v^2}{R} = \dfrac{\pi d^2\rho v^2}{4}$

8.68. In ancient period a typically designed vessel filled with water and having a small orifice O has been used to record the time. With this setup the time was evaluated in view of the height of the level of water in the

vessel. What should be the shape of the vessel which could provide a uniform time scale.

Solution: Let A be the area of water level in the upper vessel and v be the velocity with which the upper level lowers itself. Let a be the area of orifice O.

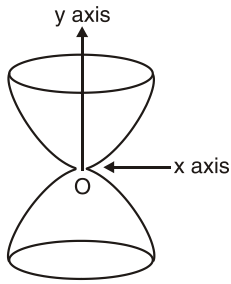


Fig. 8.17

Velocity of water through the orifice is given by

$V = \sqrt{2gy}$ (see Fig. 8.17)

Now $Av = aV$ where $A = \pi x^2$

$$\frac{\pi x^2}{\sqrt{2gy}} = \frac{a}{v} = \text{constant}$$

$$\therefore y = \frac{\pi^2 v^2 x^4}{2ga^2}$$

or $$y = kx^4$$

The shape should be designed by joining apex of two parabolas.

8.69. A cylindrical vessel of height $h = 19.6$ cm and base radius $R = 10$ cm has an orifice of radius $r = 2$ mm at its bottom. The vessel is completely filled with water. Calculate the time required to squeeze all water from the vessel.

Solution:

$$-\pi R^2 \cdot \frac{dh}{dt} = \pi r^2 v;\ v = \sqrt{2gh}$$

$$-R^2 \frac{dh}{dt} = r^2 \sqrt{2gh}$$

$$\int_h^0 -h^{-1/2} dh = \frac{r^2}{R^2}\sqrt{2g} \cdot \int_0^t dt$$

$$2\sqrt{h} = \frac{r^2}{R^2}\sqrt{2g} \cdot t$$

$$\therefore t = \frac{R^2}{r^2} \cdot \sqrt{\frac{2h}{g}}$$

On substituting the values of the various terms we have

$$t = \frac{100 \times 10^{-4}}{4 \times 10^{-6}} \sqrt{\frac{2 \times 19.6 \times 10^{-2}}{9.8}}$$

$$= 500 \text{ sec}$$

8.70. Two circular holes A and B each of radius $r = 5$ mm separated by $\Delta h = 40$ cm are created on the opposite sides of a rectangular vessel filled with water (see Fig. 8.18). Find the resultant reaction force of water going out of the vessel.

Solution:

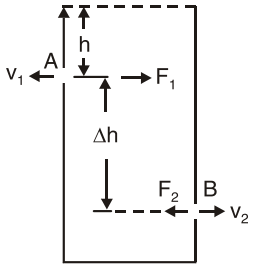


Fig. 8.18

Velocity of water V_1 at $A = \sqrt{2gh}$

Velocity of water V_2 at $B = \sqrt{2g(h+\Delta h)}$

Force $\vec{F_1} = \pi r^2 \rho V_1^2 = \pi r^2 \rho \cdot 2gh$

Force $\overleftarrow{F_2} = \pi r^2 \rho V_2^2 = \pi r^2 \rho \cdot 2g[h+\Delta h]$

Net force $F = \pi r^2 \rho \cdot 2g \cdot [h+\Delta h - h]$

$= 2\pi r^2 \rho g \cdot \Delta h$

On substituting the values of the various terms we have

$F = 2 \times 3.14 \times 25 \times 10^{-6} \times 10^3 \times 9.8 \times 40 \times 10^{-2}$

$= 0.615$ N

8.71. A large open top container of negligible mass and uniform cross-sectional area A has a small hole of cross-sectional area $A/100$ in its side wall near the bottom. The container is kept on a smooth horizontal floor and contains a liquid of density ρ and mass m_o. Assuming that the liquid starts flowing out horizontally through the hole at $t = 0$. Calculate (i) the acceleration of the container and (ii) the velocity when 75% of the liquid has drained out. (IIT 1997)

Solution:

(i) $F = Ma = v \cdot \dfrac{dm}{dt}$; $a =$ acceleration

$$\therefore a = \frac{v}{M} \cdot \frac{dM}{dt}$$

$$= \frac{v}{M}\left(\rho \cdot \frac{A}{100} \cdot v\right)$$

$$a = \frac{\rho A v^2}{100M}$$

$$v = \sqrt{2gh}$$

$$\therefore a = \frac{2\rho A \cdot gh}{100M}$$

(ii) When 75% of the liquid is drained then $h = \dfrac{H}{4}$ and $M = \dfrac{1}{4}$ $(AH\rho)$

$$\therefore a = \frac{2g\rho A \cdot \frac{H}{4}}{100\frac{AH\rho}{4}} = \frac{g}{50}$$

As the system was at rest

$$Mv = 0$$

or $$v\frac{dM}{dt} + M\frac{dv}{dt} = 0$$

$$\frac{dv}{v} = -\frac{dM}{M}$$

or $$dv = \sqrt{2gh}\cdot\frac{dh}{h}; v = \sqrt{2gh} \text{ and } \frac{dM}{M} = -\frac{dh}{h}$$

$$\therefore v = \sqrt{2g}\cdot\int_{H}^{H/4} h^{-1/2}dh$$

$$= 2\sqrt{2g}\left[\sqrt{H} - \sqrt{H}/2\right]$$

$$= \sqrt{2gH}$$

8.72. A wide open tank has an opening at which a narrowing tube is fitted without any leakage. The radius of the attached tube decreases from r_1 to r_2. The tank is filled with water and the level of water is at a height h above the tube (see Fig. 8.19a). Neglecting the viscosity of water find the horizontal force tending to detach the tube out of the tank.

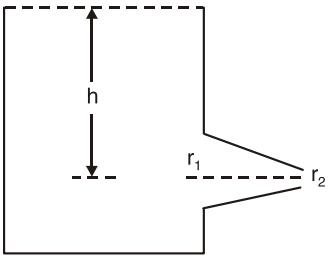


Fig. 8.19a

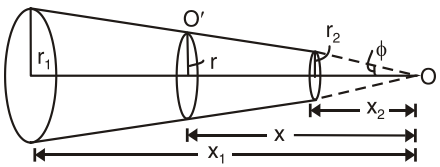


Fig. 8.19b

Solution: In Fig 8.19b consider a section of radius r at a distance x from O.

$$\frac{r_1}{x_1} = \frac{r}{x} = \frac{r_2}{x_2} = \tan\phi$$

$$\therefore r = x\tan\phi$$

In view of Bernoulli's theorem

$$p_0 + \rho gh = \frac{1}{2}\rho v^2 + p = \frac{1}{2}\rho v_2^2 + p_0;$$

v = velocity at O'

p = pressure at O'

p_0 = atmospheric pressure

$$v_2 = \sqrt{2gh}$$

Further $\pi r^2 v = \pi r_2^2 v_2$

$$\therefore v = \frac{r_2^2}{r^2}\sqrt{2gh}$$

$$p - p_0 = \Delta p = \text{change of pressure} = \rho g h - \frac{1}{2}\rho \cdot \frac{r_2^4}{r^4} \cdot 2gh$$

$$= \rho g h \left[1 - \frac{r_2^4}{r^4}\right]$$

Consider an element of area $2\pi r dr$ at O'

$$\text{Force } dF = \rho g h \left[1 - \frac{r_2^4}{r^4}\right] 2\pi r dr$$

$$= \rho g h \left[1 - \frac{r_2^4}{x^4 \tan^4\phi}\right] 2\pi x \tan\phi \cdot \tan\phi \cdot dx$$

$$F = \int dF = 2\pi\rho g h \cdot \tan^2\phi \int_{x_2}^{x_1} x\,dx - \frac{2\pi\rho g h r_2^4}{\tan^2\phi}\int_{x_2}^{x_1}\frac{dx}{x^3}$$

$$= \pi\rho g h \cdot \tan^2\phi\left[x_1^2 - x_2^2\right] - \frac{\pi\rho g h r_2^4}{\tan^2\phi}\left[\frac{1}{x_2^2} - \frac{1}{x_1^2}\right]$$

$$= \pi\rho g h\left[r_1^2 - r_2^2\right] - \frac{\pi\rho g h r_2^4}{r_1^2 r_2^2}\left[r_1^2 - r_2^2\right]$$

$$= \pi\rho g h \frac{\left[r_1^2 - r_2^2\right]^2}{r_1^2}$$

8.73. A tank of height h having a small circular hole of radius r at A is filled with water. The stream of water at A after striking the blade of the wheel of radius R continues to move with the velocity of the blade. Assuming the angular velocity of the wheel as ω calculate the force acting on the blades of the wheel.

Solution: Velocity of water at A

$= \sqrt{2gh}$ (see Fig. 8.20)

Velocity of the tip of the blade

$= \omega R$

Velocity of water relative to the wheel $v = \left(\sqrt{2gh} - \omega R\right)$

F = rate of change of momentum

$= \pi r^2 \rho v^2 = \pi r^2 \rho \left(\sqrt{2gh} - \omega R\right)^2$;

ρ = density of water.

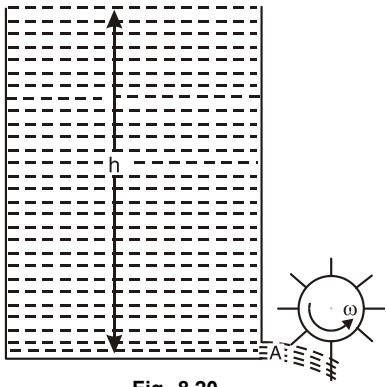


Fig. 8.20

8.74. A liquid of density ρ and viscosity η flows through a tube of length l and radius R. The velocity of the liquid at a distance r from the axis of the tube is given by $v = v_0\left(1-\frac{r^2}{R^2}\right)$. Calculate (i) the volume of liquid flowing per sec (ii) the pressure difference across the ends of the tube.

Solution:

(i) Consider coaxial cylindrical shell of liquid of radius r and $r + dr$.

Area of the shell $= 2\pi r dr$

Volume of liquid flowing per sec $= V = \int_0^R 2\pi r\,dr \cdot v_0\left(1-\frac{r^2}{R^2}\right)$

$$= 2\pi v_0\left[\frac{R^2}{2}-\frac{R^2}{4}\right] = \frac{\pi v_0 R^2}{2}$$

(ii) We know that

$\eta \cdot 2\pi r l \cdot \frac{dv}{dr} = -P \cdot \pi r^2$ (–ve sigh shows that the two forces are in opposite direction)

$\int dv = \int -\frac{P r\,dr}{2\eta l}$ P = pressure difference across the two ends of the tube.

$v = \frac{-Pr^2}{4\eta l} + C$

where C is constant whose value is determined with the help of the condition $r = R$, $v = 0$

$$\therefore C = \frac{PR^2}{4\eta l}$$

$$\therefore v = \frac{P}{4\eta l}\left[R^2 - r^2\right]$$

$$V = \int_0^R 2\pi r\,dr \cdot \frac{P}{4\eta l}\left(R^2 - r^2\right)$$

$$= \frac{2\pi P}{4\eta l}\left[\frac{R^4}{2}-\frac{R^4}{4}\right] = \frac{\pi P R^4}{8\eta l}$$

$$\frac{\pi v_0 R^2}{2} = \frac{\pi P R^4}{8\eta l}$$

$$\therefore P = \frac{4\eta l \cdot v_0}{R^2}$$

8.75. The radius of a pipeline decreases exponentially via relation $r = r_0 e^{-\beta x}$ where β is constant and x is the distance measured from the inlet of the pipeline. Calculate the ratio of Reynold's number for two radii of the pipeline observed at separation Δx.

Solution: The critical velocity is defined as

$$v_C = \frac{K\eta}{\rho r}; \quad K = \text{Reynold's number}$$

η = coefficient of viscosity

r = radius of the tube

ρ = density of liquid

Now $$\frac{K_1\eta}{\rho r_0 e^{-\beta x_1}} = \frac{K_2\eta}{\rho r_0 e^{-\beta x_2}}$$

$$\frac{K_1}{K_2} = e^{-\beta x_1} \cdot e^{\beta x_2} = e^{\beta(x_2 - x_1)} = e^{\beta \cdot \Delta x}$$

8.76. An iron ball of density $\rho_1 = 7.8 \times 10^3$ kg/m^3 is sinking in glycerine of viscosity $\eta = 1.39$ *Pa*. Estimate the maximum diameter of the ball at which the flow around it still remain laminar. Reynold's number $K = 0.5$ when the characteristic length is taken to be the diameter of the sphere. Density of glycerine $\rho_2 = 1.26 \times 10^3$ kg/m^3.

Solution:

Terminal velocity $v_t = \frac{2r^2}{9\eta}(\rho_1 - \rho_2)$; r = radius of the ball

Critical velocity $v_c = \frac{K\eta}{\rho_2 D}$; D = diameter of the ball

For motion to be laminar $v_t = v_c$

$$\therefore \frac{K\eta}{\rho_2 D} = \frac{2r^2}{9\eta}(\rho_1 - \rho_2) g$$

$$\therefore D = 2r = \left[\frac{18 K \eta^2}{\rho_2 (\rho_1 - \rho_2) g}\right]^{1/3}$$

On substituting the values we have

$$D = \left(\frac{18 \times 0.5 \times 1.39^2}{1.26 \times 10^3 \times (7.8 - 1.26) \times 10^3 \times 9.8}\right)^{1/3}$$

$$= 0.5992 \times 10^{-2} \text{ m}$$

8.77. A tank has a small hole at A at a height h_1 from the bottom (see Fig. 8.21). Initially the hole is closed and a liquid of density ρ is filled into the tank upto a height h. The tank is covered from the top. The space

between the top of the tank and the level of the liquid is filled with air at pressure P. The hole is now opened. Determine the speed of liquid at A.

Solution: At A we have total energy as

$$P_0 + \frac{1}{2}\rho v^2 + \rho g h_1; \quad v = \text{velocity of liquid at } A$$

P_0 = atmospheric pressure

At B we have total energy as $P + \rho g h$

$$\therefore P_0 + \frac{1}{2}\rho v^2 + \rho g h_1 = P + \rho g h$$

$$\therefore v = \sqrt{\frac{2}{\rho}\left[P - P_0 + \rho g\left(h - h_1\right)\right]}$$

$$v = \sqrt{\frac{2\left(P - P_0\right)}{\rho} + 2g\left(h - h_1\right)}$$

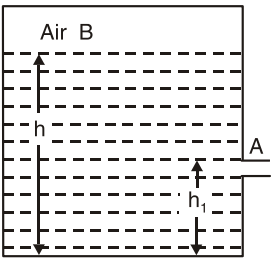


Fig. 8.21

8.78. A spherical ball of radius 1×10^{-4} m and of density 10^4 kg/m^3 falls freely under gravity through a distance h before entering a tank of water. If after entering the water the velocity of the ball does not change, find h. The coefficient of viscosity of water is 9.8×10^{-6} N.sec/m^2.

(Roorkee 1990)

Solution: The velocity of the ball after falling through a distance h is given by

$$V = \sqrt{2gh}$$

According to Stoke's law

$$V = \frac{2}{9}\frac{r^2\left(\rho - \sigma\right)g}{\eta}$$

On putting the values we have

$$V = \frac{2}{9} \times \frac{10^{-8} \times \left(10^4 - 10^3\right) \times 9.8}{9.8 \times 10^{-6}}$$

$$= 20 \text{ m/sec}$$

$$h = \frac{V^2}{2g}$$

$$= \frac{20^2}{2 \times 9.8} = 20.41 \text{ m}$$

8.79. Calculate the rate of flow of glycerine of density 1.25×10^3 kg/m^3 through the conical section of a pipe if the radii at its ends are 0.1 m and 0.04 m and the pressure drop across its length is 10 N/m^2.

(Roorkee 1991)

Solution: Let A_1 and A_2 be the area of cross-section at the two ends and v_1 and v_2 be the velocities of flow of glycerine then

$A_1 v_1 = A_2 v_2$

According to Bernoulli's theorem

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Now $$v_1 = A_2 \sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}$$

Rate of flow $= A_1 v_1 = A_1 A_2 \sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}$

On putting the values we have

$$\text{Rate of flow} = 10^{-2}\pi \cdot 0.16 \times 10^{-2}\pi \sqrt{\frac{2 \times 10}{1.25 \times 10^3 \left[10^{-4} \cdot \pi^2 - 10^{-4}\pi^2 (0.16)^2\right]}}$$

$$= 6.28 \times 10^{-4} \text{ m}^3/\text{sec}$$

8.80. A liquid is kept in a cylindrical vessel which is rotated about its axis. The liquid rises at the sides. If the radius of the vessel is $r = 0.05$ m and the speed of rotation is $\omega = 4\pi$ rad/sec, find the difference in height of the liquid at the centre of the vessel and its sides. (Roorkee 1987)

Solution: Consider a layer of liquid of radius x and thickness dx.

Mass of liquid in the layer $= 2\pi x dx \cdot y \cdot \rho$ (see Fig. 8.22)

Let dp be the difference of pressure across the layer then

$$\int dp = \int \rho \omega^2 x dx$$

$$p = \rho\omega^2 \frac{x^2}{2} + C$$

where C is constant whose value is obtained by the boundary condition At $x = 0$, $p = p_0$

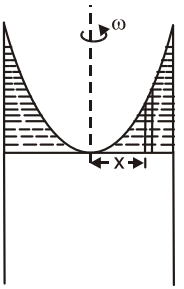


Fig. 8.22

$$\therefore C = p_0$$

$$\therefore p = p_0 + \frac{1}{2}\rho\omega^2 x^2$$

$$p - p_0 = \rho g y = \frac{1}{2}\rho\omega^2 x^2$$

or $$y = \frac{\omega^2}{2g} \cdot x^2$$

This is the equation of parabola.

On substituting the values we have

$$y = \frac{16 \times 3.14^2 \times 0.05^2}{2 \times 9.8}$$

$$= 0.02 \text{ m} = 2 \text{ cm}$$

8.81. A tank filled with water upto a height H has a hole O of area A at its bottom (see Fig. 8.23). The water of the tank is drained out through this hole. Calculate the depth h below O at which the area of the stream reduces to half of its value at O.

Solution: Let v be the velocity of water at O then its velocity at a depth h is given by

$$Av = 0.5 A.V$$

$$V = \frac{v}{0.5} = 2v$$

According to Bernoulli's theorem

$$p_0 + \rho gH = p_0 + \frac{1}{2}\rho V^2 - \rho gh;$$

p_0 = atmospheric pressure

$$V^2 = 2g(H + h)$$

$$4v^2 = v^2 + 2gh; v = \sqrt{2gH}$$

$$3v^2 = 2gh$$

or $$h = \frac{3v^2}{2g} = \frac{3}{2g} \cdot 2gH = 3H$$

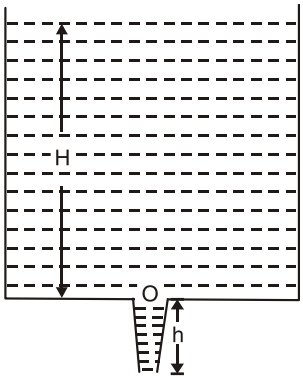


Fig. 8.23

8.82. A vertical slit of length l and width b is cut in the side wall of a wide vertical rectangular vessel of height h. The slit is closed and the vessel is completely filled with water. Calculate the force of reaction of the water coming out of the vessel immediately after the slit is opened.

Solution:

Velocity of water at $A = \sqrt{2g(h-l)}$ (see Fig. 8.24)

Velocity of water at a depth dl below A

$$= \sqrt{2g\left[(h + dl) - l\right]}$$

Consider an element of length dl and width b within the slit.

$$\text{Force } dF = \rho bdl\left[2g(h + dl - l)\right]$$

$$\text{Net force } F = \int dF = \rho b \cdot 2gh \int_0^l dl - 2g\rho b \int_0^l l\,dl$$

$$= \rho bg\left[2hl - l^2\right] = \rho bgl[2h - l]$$

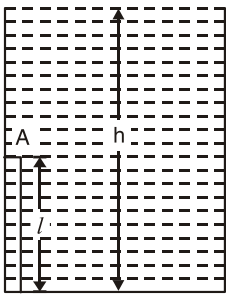


Fig. 8.24

8.83. Two similar holes each of area Am^2 are drilled one above the other on one side of a vessel. The two holes are Hm apart. The vessel is completely filled with water. In order to maintain a constant level of water in the vessel per sec Qm^3 of water is supplied to the vessel from external sources. Find the coordinates of the point of intersection of the two streams.

Solution: Let the two holes be at a height h and $h+H$ below the water level. Velocity of water going out of the two holes is given by

$$v_1 = \sqrt{2gh}\ ,\ v_2 = \sqrt{2g(H+h)} \quad \text{(see Fig. 8.25)}$$

$$x = v_1 t_1 = v_2 t_2$$

$$y = h + \frac{1}{2} g t_1^2 = h + H + \frac{1}{2} g t_2^2$$

$$\frac{1}{2} g \left(t_1^2 - t_2^2\right) = H$$

or $$\frac{1}{2} g \left(\frac{x^2}{v_1^2} - \frac{x^2}{v_2^2} \right) = H$$

or $$\frac{1}{2} g x^2 \frac{\left[v_2^2 - v_1^2\right]}{v_1^2 v_2^2} = H$$

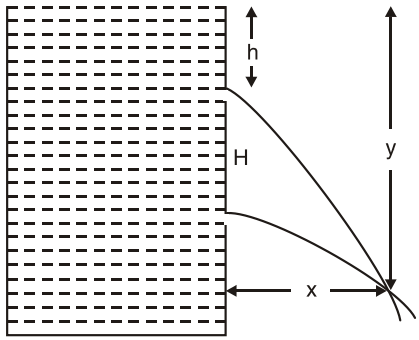


Fig. 8.25

or $$g \frac{x^2}{2} \frac{\left[2g(H+h) - 2gh\right]}{2gh \ \ 2g(H+h)} = H$$

$$\frac{x^2 H}{4h(H+h)} = H$$

or $$x = 2\sqrt{h(H+h)}$$

Now $$2y = h + \frac{1}{2} g t_1^2 + h + H + \frac{1}{2} g t_2^2$$

$$= H + 2h + \frac{g}{2} \left[\frac{x^2}{v_1^2} + \frac{x^2}{v_2^2} \right]$$

$$= H + 2h + g \frac{x^2}{2} \left[\frac{v_1^2 + v_2^2}{v_1^2 v_2^2} \right]$$

$$= H + 2h + g \frac{x^2}{2} \frac{\left[2g(H+h) + 2gh\right]}{2gh \ \ 2g(H+h)}$$

$$y = H + 2h$$

Now $$\frac{Q^2}{A^2} = \left(v_1 + v_2\right)^2 = 2gh + 2g(H+h) + 2 \ \ 2g\sqrt{h(H+h)}$$

$$\therefore \frac{Q^2}{2gA^2} = H + 2h + 2\sqrt{h(H+h)}$$

$$= y + x$$

Now $H^2 \quad 2g \quad \frac{A^2}{Q^2} = \frac{H^2 \quad 2g}{(v_1 + v_2)^2}$

$$= \frac{2gH^2}{2g(H+h) + 2gh + 2.2g\sqrt{h(H+h)}}$$

$$= \frac{H^2}{H + 2h + 2\sqrt{h(H+h)}}$$

$$= \frac{H^2\left[H + 2h - 2\sqrt{h(H+h)}\right]}{(H+2h)^2 - 4h(H+h)}$$

$$= (H + 2h) \quad 2\sqrt{h(H+h)}$$

$$= y \quad x$$

$$\therefore x = \frac{1}{2}\left[\frac{Q^2}{2gA^2} - H^2 \quad 2g \quad \frac{A^2}{Q^2}\right]$$

and $$y = \frac{1}{2}\left[\frac{Q^2}{2gA^2} + H^2 \quad 2g \quad \frac{A^2}{Q^2}\right]$$

8.84. A horizontal pipeline carries water in a stream line flow. At a point along the pipe where the cross-sectional area is 10 cm^2, the water velocity is 1 m/s and the pressure is 2000 Pa. What is the pressure of water at another point where the cross-sectional area is 5 cm^2. (IIT 1994)

Solution:

$$A_1 v_1 = A_2 v_2$$

$$\therefore v_2 = \frac{10}{5} = 2 \text{ m/sec.}$$

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

$$2000 + \frac{1}{2} \times 10^3 \times 1 = p_2 + \frac{1}{2} \times 10^3 \times 2^2$$

$$p_2 = 500 \text{ Pa}$$

8.85. The space between two long coaxial cylinders of radii r_1 and r_2 respectively is filled with water of viscosity η as shown in Fig. 8.26. The inner cylinder of radius r_1 is displaced along its axis with a constant velocity v_0. Assuming the flow to be laminar find the velocity of water as a function of r, the distance from the axis of the two cylinders.

Solution: Let l be the length of the cylinder, consider a cylindrical shell of liquid of radius r and $r + dr$. The viscous force is given by

$$F = 2\pi r l \eta \frac{dv}{dr}$$

$$F \int_{r_1}^{r_2} \frac{dr}{r} = 2\pi l \eta \int_{v_0}^{0} dv$$

$$F \log_e \frac{r_2}{r_1} = -2\pi l \eta v_0$$

or $$F \log_e \frac{r_1}{r_2} = 2\pi l \eta v_0$$

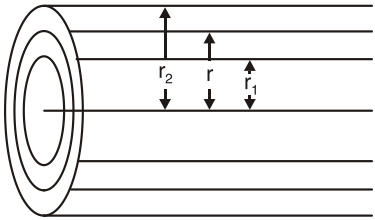


Fig. 8.26

Now $$\frac{2\pi l \eta v_0}{\log_e \frac{r_1}{r_2}} = 2\pi r l \eta \frac{dv}{dr}$$

$$v_0 \int_{r_2}^{r} \frac{dr}{r} = \log_e \frac{r_1}{r_2} \int_0^V dv$$

$$v_0 \log_e \frac{r}{r_2} = \log_e \frac{r_1}{r_2} \; V$$

$$\therefore V = \frac{v_0 \log_e \frac{r}{r_2}}{\log_e \frac{r_1}{r_2}}$$

8.86. A cylindrical tank 1 m in radius rests on a plateform 5 m high. Initially the tank is filled with water upto a height of 5 m. A plug whose area is 10^{-4} m^2 is removed from an orifice on the side of the tank at the bottom. Calculate (a) initial speed with which the water flows from the orifice (b) initial speed with which the water strikes the ground and (c) time taken to empty the tank to half its original value. $g = 10$ m/sec^2.

(Roorkee 1995)

Solution:

(a) $V_H = \sqrt{2gh} = \sqrt{2 \times 10 \times 5} = 10$ m/sec (see Fig. 8.27)

(b) $V_V = \sqrt{2gh} = 10$ m/sec

The initial speed with which water strikes the ground

$V = \sqrt{V_H^2 + V_V^2} = 10\sqrt{2} = 14.14$ m/sec

(c) Let A be the area of the tank then

$$A \frac{dy}{dt} = A_0 \sqrt{2gy} \; ; A_0 = \text{area of the orifice}$$

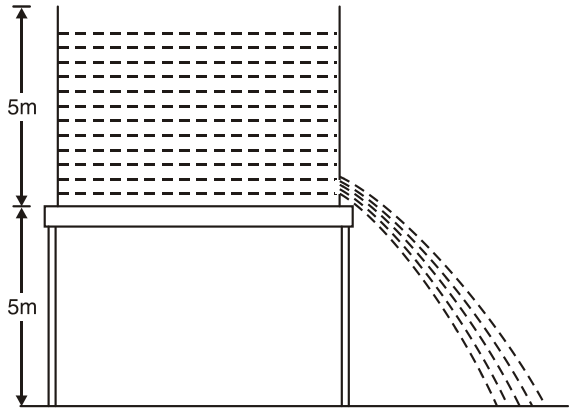


Fig. 8.27

$$-\int_{H}^{H/2} -\frac{dy}{\sqrt{y}} = \frac{A_0}{A}\sqrt{2g}\int_0^t dt \; ; H = \text{height of the tank}$$

$$-\left| 2\sqrt{y}\right|_{H}^{H/2} = \frac{A_0}{A}\sqrt{2g}\;\; t$$

$$\frac{A}{A_0}\frac{\sqrt{2}}{\sqrt{g}}\left[\sqrt{H} - \sqrt{H/2}\right] = t$$

On putting the values we have

$$t = \frac{\pi \times 1}{10^{-4}}\frac{\sqrt{2}}{\sqrt{10}}\left[\sqrt{5} - \sqrt{2.5}\right] = 9.2 \times 10^3 \text{ sec.}$$

8.87. A container of large uniform cross-section area A resting on a horizontal surface, holds two immiscible, non-viscous and incompressible liquids of density d and $2d$ each of height $H/2$ as shown in Fig. 8.28. The lower density liquid is open to the atmosphere having pressure p_0. (a) A homogenous solid cylinder of length L ($L < H/2$) cross-sectional area $A/5$ is immersed such that it floats with its axis vertical at the liquid liquid interface with length $L/4$ in the denser liquid. Determine (i) The density of solid and (ii) the total pressure at the bottom of the container. (b) The cylinder is removed and original arrangement is restored. A tiny hole of area s ($s < A$) is punched on the vertical side of the container at a height h ($h < H/2$). Determine (i) the initial speed of efflux of the liquid at the hole (ii) the horizontal distance x travelled by the liquid initially and (iii) the height h_m at which the hole should be punched so that the liquid travels the maximum distance x_m initially. Also calculate x_m. (IIT 1995)

Solution:

(i) $V\rho g = V_1 d_1 g + V_2 d_2 g$

$$L \quad \frac{A}{5} \quad \rho$$

$$= \frac{3L}{4} \quad \frac{A}{5} \quad d + \frac{L}{4} \quad \frac{A}{5} \quad 2d$$

$$\rho = \frac{5d}{4}$$

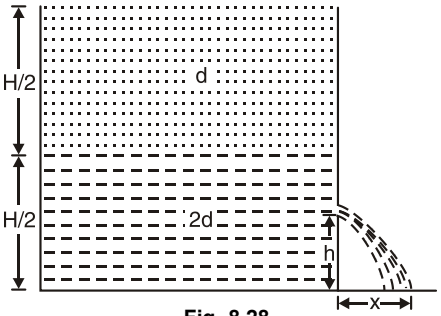


Fig. 8.28

(ii) Total pressure

$$= p_0 + \frac{\text{weight of liquid + weight of solid}}{A}$$

or $$p = p_0 + \frac{H}{2} d \; g + \frac{H}{2} \; 2d \; g + \frac{5d}{4} \quad \frac{AL}{5} \times \frac{g}{A}$$

$$= p_0 + \frac{3}{2} H d g + \frac{L}{4} d g = p_0 + \frac{1}{4} (6H + L) d g$$

b(i) According to Bernoulli s theorem we have

$$p_0 + \frac{H}{2} d g + \left(\frac{H}{2} - h \right) 2d \; g = p_0 + \frac{1}{2} \; (2d) v^2$$

$$g\,(3H \quad 4h) = 2v^2$$

or $$v = \sqrt{\frac{g}{2} (3H - 4h)}$$

(ii) Time taken to reach the ground is given by

$$t = \sqrt{\frac{2h}{g}}$$

$$\therefore x = vt = \sqrt{\frac{g}{2} (3H - 4h)} \sqrt{\frac{2h}{g}}$$

$$x = \sqrt{h (3H - 4h)}$$

(iii) $$\frac{d}{dh} \left(x^2 \right) = 0 = h \times -4 + 3H - 4h$$

$$\therefore h = \frac{3}{8} H$$

$$x_{\max} = \sqrt{\frac{3H}{8} \left(3H - \frac{3}{2} H \right)}$$

$$= \frac{3H}{4}$$

8.88. A tube of inner radius r bent at right angles is connected to a tank filled with water as shown in the Fig. 8.29. The horizontal portion of the tube has length l. Water flows through it at a rate of Q lit/sec. Find the moment of the reaction forces of water acting on the walls of the tube relative to the point O.

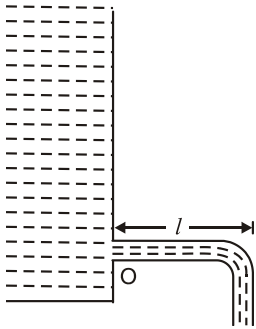


Fig. 8.29

Solution:

$$Q = Av = \pi r^2 v;\ v = \text{velocity of water}$$

$$\therefore\ v = \frac{Q}{\pi r^2}$$

$$\text{Force } F = \rho\ \pi r^2\ v^2 = \rho\ \pi r^2\ \frac{Q^2}{\pi^2 r^4}$$

$$= \frac{\rho}{\pi r^2}\ Q^2$$

$$\text{Moment of the force} = \frac{\rho\ Q^2 l}{\pi r^2}$$

8.89. A cylindrical cavity as shown in the Fig. 8.30 is filled with oil of viscosity η. Within the cavity a thin disc of radius R is rotating with a constant angular velocity ω. The separation between the disc and the horizontal plane of the cavity is y. Calculate the power developed by the viscous forces acting on the disc.

Solution: Consider an element of radius r and thickness dr.

Viscous force acting on it is given by

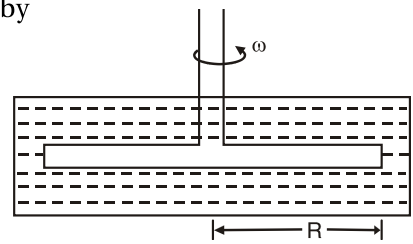


Fig. 8.30

$$dF = 2\eta\ \ 2\pi r dr\ \frac{\omega r}{y}$$

$$d\tau = \text{torque} = r\ \ dF = 4\pi\eta\ \frac{\omega}{y}\ r^3 dr$$

$$\text{Total torque } \tau = \frac{4\pi\eta\omega}{y}\int_0^R r^3 dr$$

$$= \frac{\pi\eta\omega R^4}{y}$$

$$\text{Power developed} = \tau\ \ \omega = \frac{\pi\eta\omega^2 R^4}{y}$$

8.90. A liquid of viscosity η is filled in the space between the two coaxial cylinders of radii r_1 and r_2 respectively. The inner cylinder of radius r_1 is kept stationary and the outer one is rotated with a constant angular velocity ω_0 by external means. Calculate (i) the torque acting per unit length of the cylinder of radius r_2 (ii) the angular velocity of the liquid as a function of distance from the axis of the cylinder.

Solution:

(i) Consider a shell of radius r and $r + dr$. The viscous force is given by

$$F = 2\pi r l \eta \frac{d}{dr}(r\omega) = 2\pi r^2 l \eta \frac{d\omega}{dr}\ ;\ l = \text{length}$$

Torque acting on the shell $\tau = 2\pi r^2 l \eta \ \frac{d\omega}{dr}\ r$

or $$\tau \int_{r_2}^{r_1} \frac{dr}{r^3} = 2\pi l \eta \int_{\omega_0}^{0} d\omega$$

$$\tau \left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right] = 4\pi l \eta \omega_0$$

$$\tau = 4\pi l \eta \omega_0 \frac{r_1^2 r_2^2}{r_2^2 - r_1^2}$$

(ii) $$4\pi l \eta \omega_0 \frac{r_1^2 r_2^2}{r_2^2 - r_1^2} = 2\pi l \eta r^3 \frac{d\omega}{dr}$$

$$2\omega_0 \frac{r_1^2 r_2^2}{r_2^2 - r_1^2} \int_{r}^{r_1} \frac{dr}{r^3} = \int_{\omega}^{0} d\omega$$

or $$\omega = \frac{\omega_0 r_1^2 r_2^2}{r_2^2 - r_1^2}\left[\frac{1}{r_1^2} - \frac{1}{r^2}\right]$$

8.91. Amongst the two coaxial cylinders A and B of radii r_1 & r_2 respectively cylinder A of length l is suspended through a fibre whose twisting couple per unit angle of twist is C (see Fig. 8.31). On rotating the cylinder B with constant angular velocity ω_0 from external efforts the mirror m attached to the suspension fibre rotates through an angle θ. Calculate the viscosity of air.

Solution: Consider a cylindrical shell of air of radius r and $r + dr$

The viscous force acting on it is given by

$$F = 2\pi r l \eta \frac{dv}{dr} = 2\pi r l \eta r \frac{d\omega}{dr}$$

$$\text{Torque } \tau = 2\pi r^2 l \eta \frac{d\omega}{dr}\ r$$

$$\tau \int_{r_1}^{r_2} \frac{dr}{r^3} = 2\pi l \eta \int_{0}^{\omega_0} d\omega$$

$$\frac{\tau}{2}\left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right] = 2\pi l \eta \omega_0$$

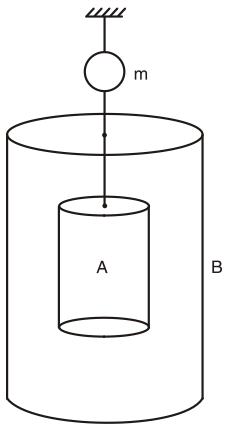


Fig. 8.31

$$\tau = 4\pi l \eta \omega_0 \frac{r_1^2 \ r_2^2}{r_2^2 - r_1^2}$$

$$\tau = C\theta = 4\pi l \eta \omega_0 \frac{r_1^2 \ r_2^2}{r_2^2 - r_1^2}$$

$$\therefore \eta = \frac{\left(r_2^2 - r_1^2\right) C \theta}{4\pi l \omega_0 \ r_1^2 \ r_2^2}$$

8.92. Two similar circular disc of radius R are placed horizontally one above the other keeping them d apart. The space between them is filled with liquid of viscosity η. The upper disc is kept stationary while the lower one is rotated with an angular velocity ω by a string wound around it. Calculate the tension in the string.

Solution: Consider two concentric rings of radius x and $x + dx$.

$$\text{Viscous force} = \eta \ 2\pi x dx \ \frac{v}{d}; \ v = \text{velocity} = x\omega$$

$$= \frac{2\pi\omega}{d} \eta x^2 dx$$

$$\text{Torque} = \frac{2\pi\omega\eta \ x^2 dx \ x}{d} = \frac{2\pi\omega\eta}{d} x^3 dx$$

$$\text{Net torque } \tau = \frac{2\pi\omega\eta}{d} \int_0^R x^3 dx$$

$$= \frac{\pi\omega\eta R^4}{2d}$$

$$\tau = T \ R = \frac{\pi\omega\eta R^4}{2d}; \ T = \text{tension in the string}$$

$$\therefore T = \frac{\pi\omega\eta R^3}{2d}$$

8.93. A soap bubble of radius R is blown at the end of a capillary tube of length l and internal radius r. Calculate the time required to reduce its radius to half of its initial value.

Solution: Consider a shell of radius x and $x + dx$

$$\text{Volume confined in the shell} = \frac{4}{3}\pi\left[(x + dx)^3 - x^3\right]$$

$$= 4 \pi x^2 dx$$

Volume of air flowing per sec

$$-4\pi x^2 \ \frac{dx}{dt} = \frac{\pi \ \frac{4T}{x} \ r^4}{8\eta l}; \ T = \text{surface tension}$$

$$\text{or} \quad \int_{R}^{R/2} -x^3 dx = \frac{Tr^4}{8\eta l} \int_0^t t$$

$$R^4 - \frac{R^4}{16} = \frac{Tr^4}{2\eta l}\ t$$

$$\text{or} \quad t = \frac{15\,\eta l R^4}{8Tr^4}$$

8.94. A vessel of cross-sectional area A has a capillary tube of length l and inner radius r. The tube is fitted at the bottom of the vessel by keeping it horizontal. The vessel is filled with water upto a certain height above the capillary tube. Calculate the time required to empty one half of the vessel.

Solution: Let h be the height upto which water is filled into the vessel. Let the level goes down by x in time t.

$$\text{Volume of water flowing/sec} = A\ \frac{dx}{dt} = \frac{\pi(h-x)\rho g r^4}{8\eta l}$$

$$\text{or} \quad \int_0^{h/2} \frac{dx}{h-x} = \frac{\pi g \rho r^4}{8\eta A l} \int_0^t dt$$

$$-\left|\log(h-x)\right|_0^{h/2} = \frac{\pi\rho g r^4\ t}{8\eta A l}$$

$$\log_e^2 = \frac{\pi\rho g r^4\ t}{8\eta A l}$$

$$\text{or} \quad t = \frac{8\eta A l\ \log_e^2}{\pi\rho g r^4}$$

8.95. Two bulbs A and B of same volume v are connected by a capillary tube of length l and radius r. The entire system is filled with a gas. The pressures in the two bulbs are p_1 and p_2 respectively such that $p_1 > p_2$. Calculate the time in which the pressure in the bulb A reduces by 10% of its initial value as a result of flow of the gas through the capillary tube to the bulb B.

Solution: Let P and p be the instantaneous pressure in the two bulbs then the instantaneous value of the volume of gas flowing per sec is given by

$$V = \frac{\pi r^4 \left(P^2 - p^2\right)}{16\eta l P}$$

In view of the gas law we have $Pv = \frac{m}{M} RT$; m = mass of gas in bulb A.

After a lapse of time dt let dP be the fall in pressure in bulb A then

$$P\ \ dP = \frac{m - V\rho dt}{M}\ RT\ ;\ \rho = \text{density of the gas}$$

On dividing the two we have

$$\frac{P - dP}{P} = \frac{m - V\rho dt}{m}$$

or $$\frac{dP}{P} = \frac{V\,dt\,\rho}{m} = \frac{V\,dt\,\rho}{\rho v} = \frac{V\,dt}{v}$$

or $$dt = \frac{v}{P}\frac{dP}{V} = \frac{v}{\pi r^4}\frac{16\eta l\,dP}{\left(P^2 - p^2\right)}$$

Further $p = p_1 + p_2 \quad P$

$$\therefore dt = \frac{v}{\pi r^4}\frac{16\eta l}{(p_1 + p_2)\left[2P - (p_1 + p_2)\right]}\frac{dP}{}$$

On integration we have

$$\int_0^t dt = \frac{16\eta l v}{\pi r^4 (p_1 + p_2)} \int_{p_1}^{0.9p_1} \frac{dP}{\left[2P - (p_1 + p_2)\right]}$$

$$t = \frac{8\eta l v}{\pi r^4 (p_1 + p_2)} \log_e \left(\frac{0.8\,p_1 - p_2}{p_1 - p_2}\right)$$

8.96. A cylindrical vessel with a liquid moves linearly with an acceleration a as shown in the Fig. 8.32. Find the shape of the minuscus of the liquid.

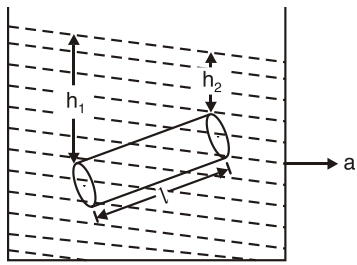


Fig. 8.32

Solution: Consider a cylinder of area A and length l.

$$F = ma$$

$$p_1 A \quad p_2 A = (Al\rho)\,a$$

$$(h_1 \quad h_2)\,\rho\,g\,A = Al\rho \quad a$$

or $$\frac{h_1 - h_2}{l} = \frac{a}{g} = \text{constant}$$

The shape of the minuscus will therefore be a straight line.

8.97. 24 cc of water flows per sec through a capillary tube of radius r and length l connected to a constant pressure head. Calculate the volume of water flowing per sec through another capillary tube of radius $r/2$ and length $l/2$ connected to the same pressure head.

Solution:

$$Q = 24 \text{ cc} = \frac{\pi P r^4}{8\eta l}$$

$$Q' = \frac{\pi P}{8\eta} \frac{r^4}{16 \times l/2} = \frac{1}{8}\left(\frac{\pi P r^4}{8\eta l}\right)$$

$$= \frac{24}{8} = 3 \text{ cc}$$

8.98. Two capillary tube of same length and radius r and $r/2$ joined in series are connected to a constant pressure head as shown in Fig. 8.33. Calculate the ratio of quantity of liquid flowing per sec through the two tubes.

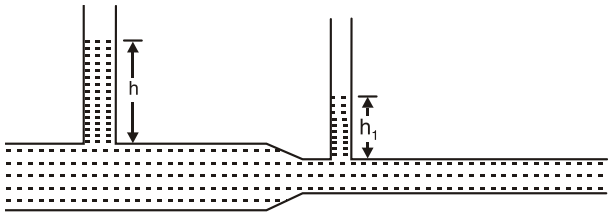


Fig. 8.33

Solution: Volume of liquid flowing through the two tubes will have the same value hence

$$Q = \frac{\pi(h - h_1)\rho g}{8\eta l} \frac{r^4}{} = \frac{\pi h_1 \rho g}{8\eta l} \frac{r^4}{16}$$

$$\text{or } h \quad h_1 = \frac{h_1}{16}$$

$$\therefore h = \frac{17 h_1}{16}$$

$$\eta = \frac{\pi h \rho g}{8 l Q_1} \frac{r^4}{} = \pi \frac{16}{17} h \frac{\rho g}{8l} \frac{r^4}{16} \frac{1}{Q_2}$$

$$\therefore Q_1 = 17 Q_2$$

$$\text{or } \frac{Q_1}{Q_2} = 17$$

8.99. Three capillary tubes of same radius and of lengths l_1, l_2 and l_3 joined in series are connected to a constant pressure head as shown in Fig. 8.34. Calculate the equivalent length of the capillary of same radius.

Solution: Quantity of liquid flowing per sec through each tube is same hence

$$Q = \frac{\pi(h_1 - h_2)\rho g r^4}{8\eta l_1} = \frac{\pi(h_2 - h_3)\rho g r^4}{8\eta l_2} = \frac{\pi h_3 \rho g r^4}{8\eta l_3}$$

$$\frac{h_2 - h_3}{h_3} = \frac{l_2}{l_3}$$

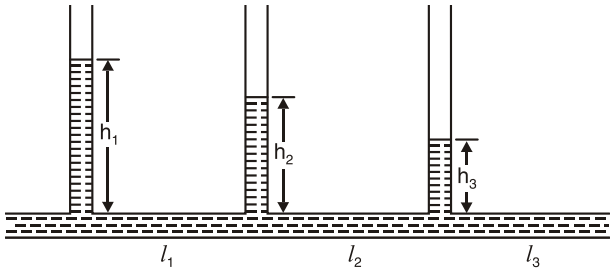


Fig. 8.34

$$\therefore \frac{h_2}{h_3} = \frac{l_2 + l_3}{l_3}$$

$$\frac{h_1 - h_2}{h_3} = \frac{l_1}{l_3}$$

$$\frac{h_1}{h_3} = \frac{l_1 + l_2 + l_3}{l_3}$$

$$Q = \frac{\pi \rho g r^4}{8\eta l_1}\left[h_1 - \frac{l_2 + l_3}{l_3}\ h_3\right]$$

$$= \frac{\pi \rho g r^4}{8\eta l_1}\left[h_1 - \frac{l_2 + l_3}{l_3}\ \frac{l_3}{(l_1 + l_2 + l_3)}\ h_1\right]$$

$$= \frac{\pi \rho g r^4\ h_1}{8\eta (l_1 + l_2 + l_3)}$$

$\therefore$ Equivalent length $= l_1 + l_2 + l_3$

8.100. In the above problem if the three capillary tubes are connected in parallel then calculate the equivalent length of the capillary which can replace the three capillary tubes.

Solution:

$$Q_1 = \frac{\pi h \rho g r^4}{8\eta l_1},\ Q_2 = \frac{\pi h \rho g r^4}{8\eta l_2},\ \text{and}\ Q_3 = \frac{\pi h \rho g r^4}{8\eta l_3}$$

$$Q = Q_1 + Q_2 + Q_3 = \frac{\pi h \rho g r^4}{8\eta}\left[\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3}\right]$$

$$\therefore \text{Equivalent length} = l = \frac{l_1 l_2 l_3}{l_1 l_2 + l_2 l_3 + l_1 l_3}$$

8.101. A horizontal capillary tube is inserted into the side of a vessel placed on a table at a height of $h_1 = 5$ cm from the bottom. The length of the capillary tube $l = 1$ cm and its radius $r = 1$ mm. The vessel is filled with a

liquid of density 900 kg/m^3 upto a height of $h_2 = 50$ cm above the capillary tube. Find the distance from the end of the capillary tube to the place where the liquid drops on the table. Viscosity of liquid = 0.5 N.s/m^2.

Solution:

Volume of liquid flowing per sec

$$= \pi r^2 v = \frac{\pi r^4 \rho g h_2}{8\eta l} \; ; \; v = \text{velocity of liquid}$$

$$\therefore v = \frac{r^2 \rho g h_2}{8\eta l}$$

Further $h_1 = \frac{1}{2} g t^2$

$$\therefore t = \sqrt{\frac{2h_1}{g}}$$

$$\therefore \text{Required distance } x = vt = \frac{r^2 \rho g h_2}{8\eta l} \times \sqrt{\frac{2h_1}{g}}$$

$$= \frac{r^2 \rho g h_2}{8\eta l} \sqrt{2h_1 g}$$

On putting the various terms we have

$$x = \frac{10^{-6} \times 900 \times 0.5 \sqrt{2 \times 5 \times 10^{-2} \times 9.8}}{8 \times 10^{-2} \times 0.5}$$

$$= \frac{9}{8} \sqrt{0.98} \text{ cm}$$

$$= 1.1 \text{ cm}$$

8.102. A stream of water of radius $r = 2$ cm moving with a velocity $v = 20$ m/s strikes a plate which is moving in the same direction with velocity $v' = 5$m/s (see Fig. 8.35). The water moves parallel to the plate. Calculate the force exerted on the plate.

Solution: Mass of water flowing per sec $m = A \quad v \quad \rho$ where v is the speed of water relative to the plate = 20 5 = 15.

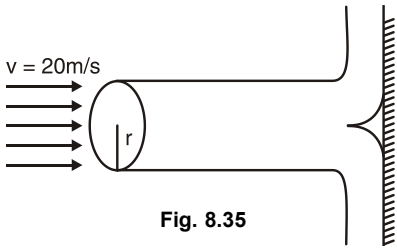


Fig. 8.35

$m = \pi (2 \times 10^{\ 2})^2 \times 15 \times 10^3$

$= 3.14 \times 4 \times 15 \times 10^{\ 1}$

$= 18.84$ kg

Force of the plate on the water $= F = 18.84$ (5 20)

$= 18.84 \times 15$

$= 282.6$ N

$\therefore$ Force of water on the plate = 282.6 N

8.103. A stream of water having radius $r = 2$ cm moving with a velocity $v = 15$ m/sec strikes a fixed blade as shown in figure 8.36. Assuming the speed of water relative to the blade is constant. Calculate the force of the blade on the stream.

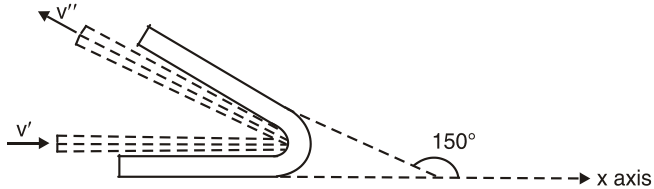


Fig. 8.36

Solution: Mass of water flowing per sec $m = Av\rho$

$m = \pi \times (2 \times 10^{-2})^2 \times 15 \times 10^3$

$= 3.14 \times 6$

$= 18.8$ kg

Force along x axis $= F_x = 18.84 [\ 15 \cos 30 \quad 15]$

$$= 18.84 \left[-15 \frac{\sqrt{3}}{2} - 15 \right]$$

$= \quad 527.3$ N

Force along y axis $= 18.84 [15 \sin 30 \quad 0]$

$= 18.84 \times 7.5$

$= 141.3$ N

9

SOUND

9.1. Longitudinal sound waves produced at the surface of a sea return after being reflected at the bottom in 2.5 s. Calculate the depth of that sea. The coefficient of compression of water is 4.6×10^{-10} m^2/N and the density of sea water 1,030 kg/m^3.

Solution: Let the depth of sea be h. Then

$$\frac{h}{1.25} = v\text{; where } v \text{ is the velocity of sound.}$$

$$\therefore h = 1.25\, v = 1.25 \sqrt{\frac{1}{4.6 \times 10^{-10} \times 1030}}$$

$$= 1816 \text{ m}$$

9.2. Calculate the time taken by sound travel a distance x between point P and Q. The air temperature between these points varies linearly between θ_1 and θ_2. The velocity of sound in air is given by $v = a\sqrt{\theta}$ where a is a positive constant.

Solution: Let v_1 and v_2 be the velocities at temperatures θ_1 and θ_2 respectively then $v_1 = a\sqrt{\theta_1}$ and $v_2 = a\sqrt{\theta_2}$

$$\text{Average velocity } v = \frac{v_1 + v_2}{2} = \frac{a}{2}\left[\sqrt{\theta_1} + \sqrt{\theta_2}\right]$$

$$\therefore v = \frac{x}{t} = \frac{a}{2}\left[\sqrt{\theta_1} + \sqrt{\theta_2}\right]$$

$$\text{or} \qquad t = \frac{2x}{a\left[\sqrt{\theta_1} + \sqrt{\theta_2}\right]}$$

9.3. A man is sitting close to a wall and beating a drum. When the rate is 90 per min, it is not possible to differentiate between the sound and its echo. Find the distance between the wall and the place of beating the drum. Velocity of sound is 330 m/s.

Solution: Rate of beating drum $= \frac{90}{60} = \frac{3}{2}$ per second

Time taken between two consecutive beats of drum = 2/3 s

Now in this time the wave has travelled a distance $2x$ where x is the distance between the wall and the place of beating the drum.

$$\text{Now } \frac{2x}{2/3} = 330$$
$$\text{or } 3x = 330$$
$$\text{or } x = 110 \text{ m}$$

9.4. An engine is approaching a cliff at a constant speed. When it is at a distance of 0.9 km from the cliff, it sounds a whistle. The echo of the sound is heard by the driver after 5 s. Determine the speed of the engine, assuming the velocity of sound in air to be 330 m/s.

Solution: Let O and B be the initial and final positions of the engine (Fig. 9.1).

Now $OB = 5v$ where v is the velocity of the engine.

During the period of 5 s, distance travelled by the sound waves

$$= OA + AB = (1800 - 5v) \text{ m}$$

$$\text{Now, Velocity} = \frac{\text{distance}}{\text{time}}$$
$$\text{or } 330 = \frac{1800 - 5v}{5}$$
$$\text{or } 1650 = 1800 - 5v$$
$$\text{or } v = 30 \text{ m/s}$$

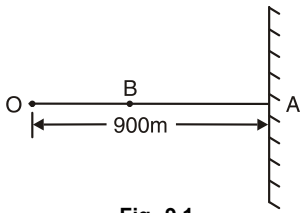


Fig. 9.1

9.5. Find the refractive index of a sound wave at the air glass interface and the limit angle of internal reflection. Young's modulus for glass $Y = 6.9 \times 10^{10}$ N/m^2. Density of glass $\rho = 2.6$ gm/c.c. Air temperature is 27°C. Velocity of sound at 0°C = 330 m/sec.

Solution:

$$n = \frac{\text{velocity of sound waves in air}}{\text{velocity of sound waves in glass}}$$

$$= \frac{V_0\sqrt{\frac{300}{273}}}{\sqrt{\frac{6.9\times10^{10}}{2.6\times10^{3}}}}; \; V_0 = \text{velocity of sound at 0°C.}$$

$$= \frac{300\sqrt{300}\cdot\sqrt{26}}{\sqrt{273}\cdot\sqrt{690}\times10^{3}}$$

$$= \frac{330\times17.32\times5.099}{16.52\times26.267\times10^{3}} = 0.067$$

$$\frac{\sin C}{\sin 90} = {}^{a}\mu_{g} = 0.067$$

$$C = 3^\circ 48'$$

9.6. Calculate the ratio of the speed of sound in neon to that in water vapours at any temperature. [Mol. *wt* of neon = 2.02×10^{-2} kg/mol and for water vapours = 1.8×10^{-2} kg/mol.] (Roorkee 1992)

Solution: The velocity of sound in neon and water vapours is given by

$$V_{Ne} = \sqrt{\gamma_{Ne} \cdot \frac{RT}{M_{Ne}}}$$

$$V_{W} = \sqrt{\gamma_{W} \cdot \frac{RT}{M_{W}}}$$

$$\frac{V_{Ne}}{V_{W}} = \sqrt{\frac{\gamma_{Ne} \times M_{W}}{M_{Ne} \times \gamma_{W}}}$$

$$\gamma_{Ne} = \frac{5}{3} \text{ (mono atomic gas)}$$

$$\gamma_{W} = \frac{4}{3} \text{ (poly atomic gas)}$$

$$\frac{V_{Ne}}{V_{W}} = \sqrt{\frac{\frac{5}{3} \times 1.8 \times 10^{-12}}{\frac{4}{3} \times 2.02 \times 10^{-2}}} = \sqrt{\frac{5 \times 1.8}{4 \times 2.02}} = 1.055$$

9.7. The speed of sound in dry air at N.T.P is 332 m/s. Assuming air as composed of 4 parts of N_2 and one part of O_2. Calculate the velocity of sound in O_2 under similar conditions when the densities of O_2 and N_2 at N.T.P. are in the ratio of 16:14 respectively.

Solution:

$$d_{\text{air}} = \frac{V_{N_2} \cdot d_{N_2} + V_{0_2} \cdot d_{0_2}}{V_{N_2} + V_{0_2}}$$

$$V_{N_2} = 4V, V_{0_2} = V$$

$$d_{\text{air}} = \frac{4d_{N_2} + d_{0_2}}{5}$$

$$= \frac{d_{0_2}}{5}\left[1 + 4\frac{d_{N_2}}{d_{0_2}}\right]$$

$$= \frac{d_{0_2}}{5}\left[1 + 4 \times \frac{14}{16}\right]$$

$$d_{\text{air}} = \frac{9}{10} d_{0_2}$$

$$332 = \sqrt{\frac{3P}{d_{\text{air}}}} = \sqrt{\frac{3P \cdot 10}{9 \cdot d_{0_2}}} = V_{0_2} \sqrt{\frac{10}{9}}$$

$$V_{0_2} = 332\sqrt{0.9} = 314.96 \text{ m/s}$$

9.8. The planet Jupiter has an atmosphere mainly of methane at a temperature –130°C. Calculate the velocity of sound on this planet assuming γ for the mixture to be 1.3. Gas constant $R = 8.3$ J/mol/°C.

Solution:

$$V = \sqrt{\frac{\gamma RT}{M}}$$

$$T = 273 - 130 = 143°\text{K}$$

$$M = 12 + 4 = 16 \text{ gm}$$

$$V = \sqrt{\frac{1.3 \times 8.3 \times 143}{16 \times 10^{-3}}} = 310.5 \text{ m/s}$$

9.9. (i) Find the speed of sound in a mixture of 1 mol of helium and 2 mol of oxygen at 27°C. (ii) If the temperature is raised by 1 K from 300 K find the percentage change in the speed of sound in the gaseous mixture ($R = 8.31$ J/mol/K). (IIT 1995)

Solution:

(i) Molecular weight of mixture $M_{\text{mix}} = \dfrac{n_1 M_1 + n_2 M_2}{n_1 + n_2}$

$$= \frac{1 \times 4 + 2 \times 32}{1 + 2} = \frac{68}{3} \times 10^{-3} \text{ kg/mol}$$

$$C_{V_{\text{mix}}} = \frac{n_1 C_{V_1} + n_2 C_{V_2}}{n_1 + n_2} = \frac{1 \times \frac{3}{2} R + 2 \times \frac{5}{2} R}{3} = \frac{13R}{6}$$

$$C_{P_{\text{mix}}} = \frac{13R}{6} + R = \frac{19R}{6}$$

$$\gamma_{\text{mix}} = \frac{19R}{6} \Big/ \frac{13R}{6} = \frac{19}{13}$$

$$V_{\text{mix}} = \sqrt{\frac{\gamma_{\text{mix}} R.T}{M_{\text{mix}}}}$$

$$= \sqrt{\frac{19}{3} \times \frac{8.31 \times 300 \times 3}{68 \times 10^{-3}}}$$

$$= 400.9 \text{ m/s}$$

(ii) $V = \sqrt{\dfrac{\gamma RT}{M}}$

$$\frac{dV}{V}\times 100 = \frac{1}{2}\cdot\frac{dT}{T}\times 100$$

$$= \frac{1}{2}\cdot\frac{1}{300}\times 100 = 0.167\%$$

9.10. An observer is at a distance of one meter from a point of light source whose power output is 1 KW. Calculate the magnitude of electric and magnetic fields assuming that the source is monochromatic, it radiates uniformly in all directions and that at the point of observation it behaves like a travelling plane wave $\dfrac{\mu_0}{4\pi} = 10^{-7}$ H/m and $C = 3\times 10^8$ m/s.

Solution:

$$I = \frac{P}{4\pi r^2} = \frac{10^3}{4\pi\times 1^2} = \frac{10^3}{4\pi}\ \text{W/m}^2$$

In case of electromagnetic waves

$$I = \frac{1}{\mu_0}EB \text{ and } \frac{E}{B} = C$$

$$I = \frac{E}{\mu_0}\times\frac{E}{C}$$

$$\therefore E = \sqrt{I\,\mu_0 C}$$

On putting the values we have

$$E = \sqrt{\frac{10^3}{4\pi}\cdot 4\pi\times 10^{-7}\times 3\times 10^8} = 100\sqrt{3} = 173\,\text{V/m}$$

and $B = \dfrac{E}{C} = \dfrac{100\sqrt{3}}{3\times 10^8} = \dfrac{1}{\sqrt{3}}\times 10^{-6} = 5.77\times 10^{-7}\,\text{W/m}^2$

9.11. A transverse wave is travelling along a string from left to right. Fig. 9.2 represents the shape of the string (snap shot) at a given instant. At this instant (a) which points have an upward velocity? (b) which points will have downward velocity? (c) which points have zero velocity? (d) which points have maximum magnitude of velocity? (IIT 1970)

Solution:

$v_{\text{Particle}} = -v \times$ slope of the curve

(a) For upward velocity v_{Particle} will be positive. This is possible when the slope is negative. For point D, E & F the slope is negative.

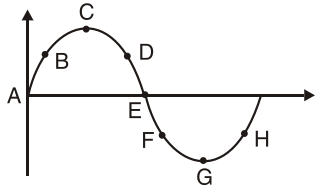


Fig. 9.2

(b) For downward velocity v_{Particle} will be negative. This is possible when the slope is positive. For points *A*, *B* & *H* the slope is positive.

(c) For zero velocity slope must be zero. For points *C* and *G* the slope is zero.

(d) For maximum velocity slope must be maximum. For points *A* & *E* the slope is maximum.

9.12. The amplitude of a wave disturbance propagating in the positive *x* direction is given by

$$y = \frac{1}{1+x^2} \text{ at } t = 0 \text{ and } y = \frac{1}{\left[1+(x-1)^2\right]} \text{ at } t = 2 \text{ sec}$$

where *x* and *y* are in *m*. The shape of the wave disturbance does not change during the propagation. What is the velocity of the wave. (IIT 1990)

Solution: Fig. 9.3 shows a wave pulse whose shape does not change during propagation. The pulse is represented by

$$y = \frac{a}{b+(x \mp vt)^2}$$

where *a* & *b* are constants.

Comparing the above equation with the problem equation we get

$$(x \mp vt)^2 = (x-1)^2 \text{ at } t = 2 \text{ sec}$$

$$vt = 1$$

$$v = \frac{1}{2} = 0.5 \text{ m/sec}$$

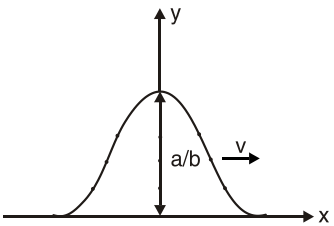


Fig. 9.3

9.13. A wave pulse at $t = 0$ is described by the equation

$$y = \frac{4}{x^2+2}$$

Assuming that the pulse is moving towards the positive side of *x* axis with a velocity of 1 m/sec. write the equation of the pulse as a function of time.

Solution: For the motion of the pulse along positive x axis replace *x* by $x - vt$. On putting $v = 1$ m/sec the required function is $y = \frac{4}{(x-t)^2+2}$

9.14. The mean kinetic energy of translational motion of the molecules of one kilo mole of nitrogen is 3.4×10^6 J. Find the velocity of sound in nitrogen under these conditions.

Solution:

$$\frac{1}{2}MC^2 = 3.4 \times 10^6$$

$$C = \sqrt{\frac{2 \times 3.4 \times 10^6}{28}} = \sqrt{\frac{3P}{d}}$$

$$V_s = \sqrt{\frac{\gamma P}{d}} = \sqrt{\frac{1.4 \times 6.8 \times 10^6}{3 \times 28}} \ 336.6 \text{ m/sec}$$

9.15. A whistle emits a note of frequency 1000 Hz in hydrogen gas, but in an unknown gas it emits a sound of frequency 225 Hz. Calculate the molecular weight of the unknown gas.

Solution:

$$\frac{n_1}{n_2} = \sqrt{\frac{M_2}{M_1}}$$

$$n_2 = n_1 \sqrt{\frac{M_1}{M_2}}$$

$$225 = 1000 \sqrt{\frac{2}{M_2}}$$

$$\sqrt{M_2} = \frac{1000}{225} \sqrt{2}$$

$$M_2 = \left(\frac{1000}{225}\right)^2 \cdot 2$$

$$= 39.5$$

9.16. Let us assume that the temperature of the air varies with height linearly from the value $T_1 = 300$ K at $y_1 = 0$ to $T_2 = 250$ K at $y_2 = 10$ km. Find the time t needed for a sound wave produced at the height y_2 to reach to the earth's surface. Velocity of sound at 0°C = 330 m/sec.

Solution:

$$\sqrt{\frac{3R \cdot T}{M}} = \text{velocity of sound}$$

$$\sqrt{\frac{3R \times 273}{M}} = 330$$

$$\therefore \sqrt{\frac{3R}{M}} = \frac{330}{\sqrt{273}} = 20$$

The variation of temperature with height is given by

$$T = T_1 - \frac{T_1 - T_2}{y_2} \cdot y$$

On differentiation we have

$$1 = -\frac{(T_1 - T_2)}{y_2} \cdot \frac{dy}{dT}$$

$$v = \sqrt{\frac{3RT}{M}} = \frac{dy}{dT} \cdot \frac{dT}{dt}$$

$$20\sqrt{T} = -\frac{y_2}{T_1 - T_2} \cdot \frac{dT}{dt}$$

$$-\frac{20(T_1 - T_2)}{y_2} \int_t^0 dt = \int_{T_1}^{T_2} \frac{dT}{\sqrt{T}}$$

$$\frac{20(T_2 - T_1)}{y_2} \cdot t = \frac{\sqrt{T_2} - \sqrt{T_1}}{\frac{1}{2}}$$

or
$$t = \frac{y_2}{10\left(\sqrt{T_2} + \sqrt{T_1}\right)}$$

$$= \frac{10 \times 1000}{10[17.52 + 15.81]}$$

$$= \frac{1000}{33.13} = 30 \text{ sec}$$

9.17. If in air a plane wave of frequency 250 Hz and amplitude 10^{-3} mm is produced, calculate the radiated energy/volume and rate of transmission of energy across unit area of the wave front. Velocity of sound = 330 m/sec and density of air $\rho = 1.29$ kg/m^3.

Solution:

The energy density E = total energy per unit volume = $2\pi^2 a^2 n^2 \rho$

On putting the values

$E = 2 \times 3.14^2 \times (10^{-6})^2 \times 250^2 \times 1.29$

$= 1.59 \times 10^{-6}$ J/m^3

The rate of transmission of energy across unit area of the wave front

$= 2\pi^2 a^2 n^2 \rho v = Ev$

$= 1.59 \times 10^{-6} \times 330$

$= 5.247 \times 10^{-4}$ Jm^{-2} sec^{-1}

9.18. Find the pressure due to plane progressive sound waves in air at a point where the particle velocity is 1 mm/sec in the forward direction. Velocity of sound = 330 m/sec. Atmospheric pressure p_0 = 75 cm of mercury, $r_{air} = 1.4$.

Solution:

$$p = -E\frac{dy}{dx}$$

$$= \frac{E}{V}\cdot\frac{dy}{dt}\text{; } V = \text{velocity of sound and } \frac{dy}{dt} = \text{particle velocity}$$

$$= \frac{\gamma p_0}{V}\cdot\frac{dy}{dt}$$

On putting the values we have

$$p = \frac{1.4\times 0.75\times 13.6\times 10^3\times 9.8\times 10^{-3}}{330}$$

$$= 0.424 \text{ N/m}^2$$

9.19. A stationary wave in a rod is represented by the equation

$y = 0.01 \sin 13x \sin 200t$

Calculate the maximum particle velocity and tensile stress at the point $x = 4$ cm. Young's modulus of the material of the rod = 10^{11}N/m^2.

Solution:

$$\text{Velocity} = \frac{dy}{dt} = 0.01\times 200 \sin 13x \cos 200t$$

Maximum velocity $= 0.01 \times 200 \sin 13x$

Now $13x = 13 \times 0.04 = 0.52$ rad $\cong 30°$

$$\text{Sin } 30 = \frac{1}{2}$$

$$\text{Maximum velocity} = 0.01\times 200\times\frac{1}{2} = 1 \text{ m/sec.}$$

$$\text{Tensile stress} = Y\frac{dy}{dx} = 10^{11}\times 0.01\times 13\cos(13x)\sin(200t)$$

Maximum tensile stress $= 10^{11} \times 0.01 \times 13 \cos 13x$

$$= 13\times 10^9\times\frac{\sqrt{3}}{2}$$

$$= 11.26\times 10^9 \text{ N/m}^2$$

9.20. The equation of motion of a progressive wave is represented as $y = 50 \cos (1200\, t - 6.28\, x)$, where y is in micrometres t in seconds and x in metres. Find (i) the ratio of the amplitude and the wavelength, (ii) the ratio of velocity amplitude to the wave propagation velocity, and (iii) the oscillation amplitude of relative deformation of the medium and its relation to the velocity amplitude of particles.

Solution:

(i) On comparing the given wave equation with the standard wave equation

$y = a \cos(\omega t - kx)$, we have

$$a = 50 \times 10^{-6}, \frac{2\pi}{\lambda} = k = 6.28$$

$$\therefore \lambda = 1 \text{ m}$$

$$\therefore \frac{a}{\lambda} = \frac{50 \times 10^{-6}}{1} = 50 \times 10^{-6}$$

(ii) $v_{max} = \omega a = 1200 \times 50 \times 10^{-6} = 6 \times 10^{-2}$ m/s

$$\frac{v_{max}}{v} = \frac{\omega a}{\omega / k} = k \cdot a = 50 \times 10^{-6} \times 6.28 = 314 \times 10^{-6}$$

(iii) On differentiating the given equation, we have

$$\left(\frac{\delta y}{\delta t}\right)_{max} = a\omega$$

We know that

$$\left(\frac{\delta y}{\delta x}\right)_{max} = \left(\frac{\delta y}{\delta t}\right)_{max} \cdot \left(\frac{\delta t}{\delta x}\right) = \frac{(\delta y / \delta t)_{max}}{v} = \frac{a\omega}{\omega / k} = ak = 314 \times 10^{-6}$$

9.21. In the case of loud sound a normal ear can bear a maximum pressure variation of nearly 25 N/m^2. Calculate the maximum displacement for a sound wave of frequency 1500 c/s in air. Atmospheric pressure = 10^5 N/m^2. Density of air = 1.293 kg/m^3. Velocity of sound in air = 330 m/s.

Solution:

The amplitude P is given by the relation

$$P = k\rho v^2 y_m$$

$$\therefore y_m = \frac{P}{k\rho v^2} = \frac{P}{\frac{2\pi n}{v} \cdot \rho v^2}; k = \frac{2\pi}{\lambda} = \frac{2\pi n}{v}$$

$$= \frac{P}{2\pi n \rho v}$$

On substituting the values of the various terms, we have

$$y_m = \frac{25}{2 \times 3.14 \times 1500 \times 1.293 \times 330} = 0.62 \times 10^{-5} \text{ m}$$

9.22. A source of sound of frequency 500 c/s and an intensity of 1.5 μW/m^2 is propagating waves in air. Calculate the amplitude of air vibrations. Velocity of sound = 330 m/s, density of air 1.293 kg/m^3.

Solution:

Intensity $I = 2\pi^2 \rho a^2 n^2 v$

$$\therefore a = \sqrt{\frac{I}{2\pi^2 \rho n^2 v}}$$

On putting the various values, we have

$$a = \sqrt{\frac{1.5 \times 10^{-6}}{2 \times 3.14^2 \times 1.293 \times 500^2 \times 330}}$$
$$= 2.670 \times 10^{-8} \text{ m}$$

9.23. Standing waves are formed after reflection from a less denser medium of a wave of wavelength 16 cm. Calculate the position of nodes and antinodes.

Solution: Antinode will be formed at $l = 0, \lambda/2, \lambda, 3\lambda/2$ etc.

Node will be formed at $l = \lambda/4, 3\lambda/4, 5\lambda/4$ etc.

Taking $\lambda = 16$ cm

Antinodes will be obtained at $l = 0, 8, 16, 24$ cm etc.

Nodes will be obtained at $l = 4, 12, 20$ cm etc.

9.24. A train of sound waves is propagated along a wide pipe and it is reflected from an open end. If the amplitude of the wave is 0.002 cm, the frequency 1000 Hz and wavelength 0.4 m find the amplitude of vibration at a point 10 cm from the open end.

Solution: The equation for the incident and reflected waves are represented as

$$y_1 = a \sin \frac{2\pi}{\lambda} (vt - x) \text{ and } y_2 = a \sin \frac{2\pi}{\lambda} (vt + x)$$

The superposition of the two waves yields

$$y = 2a \cos \frac{2\pi x}{\lambda} \cdot \sin \frac{2\pi v t}{\lambda}$$

$$\text{Amplitude} = 2a \cos \frac{2\pi x}{\lambda}$$

On putting $\lambda = 0.4$ and $x = 0.1$ m

Amplitude = 0

9.25. A stationary wave given by equation $\frac{d^2y}{dt^2} = 11.56 \times 10^4 \frac{d^2y}{dt^2}$ is established in $L = 1.7$ m long pipe filled with a gas and closed at both the ends. Calculate the loop length of the wave formed and permissible frequencies.

Solution: Let l be the length of a loop and n be the number of loops then

$nl = 1.7$

or $l = \dfrac{1.7}{n} m$

Velocity $v = \sqrt{11.56 \times 10^4} = 340$ m/sec

$N = \text{frequency} = \dfrac{340}{2 \times 1.7} = 100$

The possible frequencies are 100, 200, 300 etc.

9.26. A wave $y_i = A_i \sin(\omega t - k_1 x)$ is travelling from a dense to a rare medium. The interface of the two mediums lies at $x = 0$. At the boundary the wave is partly reflected and partly transmitted. Assuming the incidence to be a normal one find the amplitude of the reflected and transmitted waves.

Solution: Let v_1 be the velocity of the wave in the dense medium. The displacement of the incident wave is given by

$$y_i = A_i \sin \omega \left(t - \frac{x}{v_1} \right)$$

The displacement of the reflected and transmitted wave is given by

$$y_r = A_r \sin \omega \left(t + \frac{x}{v_1} \right)$$

$$y_t = A_t \sin \omega \left(t - \frac{x}{v_2} \right); \; v_2 = \text{velocity in rare medium.}$$

At $x = 0$, $y_i + y_r = y_t$ and $A_i + A_r = A_t$

Now $\dfrac{dy_i}{dx} + \dfrac{dy_r}{dx} = \dfrac{dy_t}{dx}$

$$-\frac{A_i \omega}{v_1} \cos \omega t + \frac{A_r \omega}{v_1} \cos \omega t = -\frac{A_t \omega}{v_2} \cos \omega t$$

or $A_i - A_r = \left(\dfrac{v_1}{v_2} \right) A_t$

On solving

$$A_r = \frac{v_2 - v_1}{v_1 + v_2} A_i \text{ and } A_t = \frac{2v_2}{v_1 + v_2} A_i$$

9.27. Fundamental vibrations are excited in a string of mass m fixed at both the ends using a source of angular velocity ω and having a maximum displacement amplitude a_{max}. Find (i) the maximum kinetic energy of the string, and (ii) the mean kinetic energy of the string averaged over one oscillation period.

Solution:

(i) Let l be the length of the string. Mass per unit length = m/l.

Let us consider the stationary waves to be represented by the equation

$$y = A \sin\left(\frac{2\pi}{\lambda} \cdot x\right) \cos \omega t$$

Kinetic energy of an element of length dx

$$= \frac{1}{2}\frac{m}{l} \cdot dx \left(\frac{dy}{dt}\right)^2 = \frac{m}{2l}\left\{-A\omega \sin\left(\frac{2\pi}{\lambda} x\right) \sin \omega t\right\}^2 dx$$

$$= \frac{m}{2l} A^2\omega^2 \sin^2\left(\frac{2\pi x}{\lambda}\right) \sin^2 \omega t \cdot dx$$

$$\text{Total kinetic energy} = \frac{1}{2}\frac{m}{l} A^2\omega^2 \cdot \sin^2 \omega t \int_0^{\lambda/2} \sin^2\left(\frac{2\pi x}{\lambda}\right) dx$$

$$= \frac{1}{2}\frac{m}{l} A^2\omega^2 \cdot \sin^2 \omega t \int_0^{\lambda/2} \left[\frac{1 - \cos\left(\frac{4\pi x}{\lambda}\right)}{2}\right] dx$$

$$= \frac{1}{8}\frac{m}{l} \cdot A^2\omega^2 \lambda \sin^2 \omega t$$

Now $A = a_{max}$, $\lambda = 2l$ and $\sin^2 \omega t = 1$. On substituting these values,

$$\text{total kinetic energy} = \frac{1}{8}\frac{m}{l} \cdot a_{max}^2 \omega^2 \cdot 2l$$

$$= \frac{1}{4} m \cdot a_{max}^2 \omega^2$$

and will be at the antinode.

(ii) Mean kinetic energy of string $= \frac{1}{2} \times \frac{1}{4} m\omega^2 a_{max}^2 = \frac{1}{8} m\omega^2 a_{max}^2$

9.28. The vibrations of a string of length 60 cm fixed at both ends are represented by the equation

$$y = 4 \sin\left(\frac{\pi x}{15}\right) \cos (96\,\pi t)$$

where x and y are in cm and t in seconds.

(i) What is the maximum displacement of a point at $x = 5$ cm?

(ii) Where are the nodes located along the strings?

(iii) What is the velocity of the particle at $x = 7.5$ cm at $t = 0.25$ s?

(iv) Write down the equations of the component waves whose superposition gives the above equation. (IIT 1985)

Solution:

(i) $y_{max} = 4 \sin\left(\frac{5\pi}{15}\right) = 4 \sin 60 = \frac{4\sqrt{3}}{2} = 3.464$ cm

(ii) At nodes $\sin\left(\frac{\pi x}{15}\right)$ must be zero. This is possible when

$\frac{\pi x}{15} = 0, \pi, 2\pi, 3\pi.$

$\therefore x = \frac{15}{\pi} \times 0, \frac{15}{\pi} \times \pi, \frac{15}{\pi} \times 2\pi,$

$x = 0, 15, 30, 45$ and 60 cm

(iii) $y = 4 \sin\left(\frac{\pi x}{15}\right) \cos(96\pi t)$

Velocity $= \frac{dy}{dt} = -4 \times 96\pi \sin\left(\frac{\pi x}{15}\right) \sin(96\pi t)$

$= -4 \times 96\pi \sin\left(\frac{\pi x}{15} \times 7.5\right) \sin\left(96\pi \times \frac{1}{4}\right)$

$= -4 \times 96\pi \times 1 \times \sin(24\pi) = 0$

(iv) The equation $y = 4 \sin\left(\frac{\pi x}{15}\right) \cos(96\pi t)$ can be represented as

$y = 2a \sin\alpha \cos\beta$

where $a = 2$ cm, $\alpha = \frac{\pi x}{15}$, $\beta = 96\pi t$

Now $2a \sin\alpha \cos\beta = a[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

or $y = y_1 + y_2$

$\therefore y_1 = 2 \sin\left(\frac{\pi x}{15} + 96\pi t\right)$ represents one wave, and

$\therefore y_2 = 2 \sin\left(\frac{\pi x}{15} - 96\pi t\right)$ the other wave.

9.29. A standing wave $y = a \sin kx \cos\omega t$ is maintained in a homogeneous rod with cross-sectional area A and density ρ. Find the total mechanical energy confined between the sections corresponding to the adjacent displacement nodes.

Solution:

$y = a \sin\left(\frac{2\pi}{\lambda} x\right) \cos\omega t$

Let m be the mass per unit length of the rod.

Kinetic energy in element of length $dx = \frac{1}{2} m dx \left(\frac{dy}{dt}\right)^2$

$$= \frac{1}{2} A \cdot \rho \left[-a\omega \sin\left(\frac{2\pi}{\lambda} x\right) \sin \omega t \right]^2 dx$$

$$= \frac{1}{2} A\rho \cdot a^2\omega^2 \sin^2\left(\frac{2\pi x}{\lambda}\right) \sin^2 \omega t \cdot dx$$

$$\text{Total kinetic energy} = \frac{1}{2} A\rho a^2\omega^2 \sin^2 \omega t \int_0^{\lambda/2} \frac{\left[1 - \cos\left(4\pi x/\lambda\right)\right]}{2} dx$$

$$= \frac{1}{2} A\rho a^2\omega^2 \cdot \frac{\lambda}{4} \cdot \sin^2 \omega t$$

$$= \frac{\pi}{4k} A\rho a^2\omega^2 \sin^2 \omega t; k = \frac{2\pi}{\lambda}$$

At the antinode, $\sin^2 \omega t = 1$

$$\therefore \text{ Kinetic energy} = \frac{\pi}{4k} A\rho a^2\omega^2$$

$$\text{Potential energy in the element } dx = \frac{1}{2} T \left(\frac{dy}{dx}\right)^2 dx$$

where T is the tension $= mv^2$

$$\therefore PE = \frac{1}{2} mv^2 \left\{ \frac{d}{dx} \left[a \sin\left(\frac{2\pi n}{v} \cdot x\right) \cos \omega t \right] \right\}^2 dx$$

$$= \frac{1}{2} mv^2 a^2 \cdot \frac{4\pi^2 n^2}{v^2} \cos^2\left(\frac{2\pi n x}{v}\right) \cos^2 \omega t \cdot dx$$

$$= \frac{1}{2} A\rho a^2\omega^2 \cdot \cos^2\left(\frac{2\pi n x}{v}\right) \cos^2 \omega t \cdot dx$$

$$\text{Total potential energy} = \frac{1}{2} A\rho a^2\omega^2 \cos^2 \omega t \int_0^{\lambda/2} \frac{\left[1 + \cos\left(4\pi n x/v\right)\right]}{2} dx$$

$$= \frac{1}{8} A\rho a^2\omega^2 \lambda \cos^2 \omega t$$

$$= \frac{\pi}{4k} \cdot A\rho a^2\omega^2 \cos^2 \omega t; k = \frac{2\pi}{\lambda}$$

At nodes, $\cos^2 \omega t = 1$.

$$\text{Potential energy} = \frac{\pi}{4k} \cdot A\rho a^2\omega^2$$

At nodes,

Kinetic energy = 0,

$$PE = \frac{\pi}{4k} \cdot A\rho a^2\omega^2$$

Total mechanical energy = $\frac{\pi}{4k} A\rho a^2\omega^2$

At antinodes

Kinetic energy = $\frac{\pi}{4k} A\rho a^2\omega^2$

$PE = 0$

Total mechanical energy = $\frac{\pi}{4k} \cdot A\rho a^2\omega^2$

Therefore,

Total mechanical energy = $\frac{\pi}{4k} \cdot A\rho a^2\omega^2$

9.30. A rod of length $l = 100$ cm is fixed at its midpoint. Find the harmonic oscillations of the rod lying in the frequency range of 10 to 30 kHz. Young's modulus of the material of the rod 13×10^{10} N/m^2 and density of the rod 8.9×10^3 kg/m^3.

Solution: In this case longitudinal stationary waves are formed with node at middle point and antinode at its ends.

The fundamental frequency will be obtained when $l = \lambda/2$ or $\lambda = 2l$.

$v = \nu_1 \lambda$

$\therefore \nu_1 = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{E}{d}}$

The next frequency will be obtained when $l = 3\lambda/2$ or $\lambda = \frac{2l}{3}$

$\nu_3 = \frac{3v}{2l} = \frac{3}{2l}\sqrt{\frac{E}{d}}$

The vibrations may be represented by the following general equation.

$\nu_n = \frac{(2n+1)}{2l}\sqrt{\frac{E}{d}}$

where $n = 0, 1, 2, 3, \ldots$

Now $\sqrt{\frac{E}{d}} = 3.821 \times 10^3$

$\therefore \nu_n = \frac{3.82}{2l}(2n+1)$ kHz

There are five oscillations with frequencies 13.37, 17.19, 20.01, 24.83 and 28.65 kHz.

9.31. Find the ratio of the fundamental frequencies of two similar strings one of which is stretched by $x_1 = 1.0\%$ and the other by $x_2 = 2.0\%$. In each case the tension produced is proportional to the elongation.

Solution: The fundamental frequency $n = \frac{1}{2l}\sqrt{\frac{T}{m}}$ where l is the length of the string, T is the tension applied and m is the mass per unit length of the string.

On stretching the string by x_1 its length becomes $l\ (1 + x_1)$, tension becomes $T = kl\,x_1$ and mass per unit length becomes $\frac{m}{(1+x_1)}$ where m is the mass of unit length of the unstretched string.

Now $$n_1 = \frac{1}{2l(1+x_1)} \cdot \sqrt{\frac{kl\,x_1(1+x_1)}{m}}$$

Similarly on stretching the string by x_2, we have

$$n_2 = \frac{1}{2l(1+x_2)} \cdot \sqrt{\frac{kl\,x_2(1+x_2)}{m}}$$

$$\therefore \frac{n_2}{n_1} = \sqrt{\frac{x_2(1+x_1)}{x_1(1+x_2)}}$$

On substituting the values of x_1 and x_2, we have

$$\frac{n_2}{n_1} = \sqrt{\frac{(2/100)\times(101/100)}{(1/100)\times(102/100)}} = \sqrt{\frac{202}{102}} = \sqrt{\frac{101}{51}} = 1.407$$

9.32. A tuning fork is in unison with a sonometer wire of length 100 cm. When the suspended weight causing tension in the wire is dipped in water then the same tuning fork is in unison with 93.4 cm length of the wire. Find the relative density of the material of the weight suspended.

Solution: Let n be the frequency of a sonometer wire of length l_1 when a tension Mg is applied. Then,

$$n = \frac{1}{2l_1}\sqrt{\frac{Mg}{m}}$$

On dipping the suspended mass in water, we have the same frequency, but with length l_2 and tension $M'g$, where $M'g$ is the weight of the mass suspended in water, i.e.,

$$n = \frac{1}{2l_2}\sqrt{\frac{M'g}{m}}$$

On dividing the two, we have

$$1 = \sqrt{\frac{M}{M'}} \cdot \frac{l_2}{l_1}$$

or $$\frac{M'}{M} = \frac{l_2^2}{l_1^2}$$

or $$\frac{M - M'}{M} = \frac{l_1^2 - l_2^2}{l_1^2}$$

or $$\frac{M}{M - M'} = \frac{l_1^2}{l_1^2 - l_2^2}$$

or $$\frac{Mg}{(M - M')g} = \frac{l_1^2}{l_1^2 - l_2^2}$$

Relative density $d = \dfrac{\text{weight in air}}{\text{loss in weight in water}} = \dfrac{l_1^2}{l_1^2 - l_2^2}$

$$= \frac{100^2}{100^2 - 93.4^2} = \frac{100^2}{193.4 \times 6.6} = 7.834$$

9.33. A metallic ball is suspended in order to provide tension in a sonometer wire. A tunng fork resonates with a 15 cm length of the wire. When the ball is dipped in liquid and then in water, the tuning fork resonates with 12 and 10 cm lengths of the wire respectively. Find the relative density of the liquid.

Solution: Let n be the frequency of sonometer wire of length l_1 when a tension Mg is applied, i.e.,

$$n = \frac{1}{2l_1}\sqrt{\frac{Mg}{m}}$$

On dipping the mass in the liquid and in water, we get the same frequency but with different lengths, and different tensions, i.e.,

$$n = \frac{1}{2l_2} \cdot \sqrt{\frac{M_1 g}{m}}$$

and $$n = \frac{1}{2l_3} \cdot \sqrt{\frac{M_2 g}{m}}$$

where l_2 and l_3 are the lengths of wire in liquid and water and $M_1 g$ and $M_2 g$ are the weights of the ball in the liquid and water respectively.

On solving the above equations, we get

$$\frac{M - M_1}{M} = \frac{l_1^2 - l_2^2}{l_1^2}$$

and $$\frac{M - M_2}{M} = \frac{l_1^2 - l_3^2}{l_1^2}$$

or $$\frac{M-M_1}{M-M_2}=\frac{l_1^2-l_2^2}{l_1^2-l_3^2}$$

Relative density of liquid $= \dfrac{\text{loss in weight in liquid}}{\text{loss in weight in water}}$

$$= \frac{(M-M_1)g}{(M-M_2)g}=\frac{l_1^2-l_2^2}{l_1^2-l_3^2}$$

$$= \frac{15^2-12^2}{15^2-10^2}=\frac{27\times 3}{25\times 5}$$

$$= 0.648$$

9.34. One end of a steel wire (density 8000 kg/m^3 and $Y = 2\times10^{11}$ N/m^2) is fixed on a rigid base. The other end is wound over a cylindrical peg of radius 0.3 cm. There is no tension in the wire when the distance between the support and peg is 60 cm. Calculate the fundamental frequency of the wire when the peg is turned through half the cycle.

Solution: Young's modulus of the material of the wire, $Y = \dfrac{TL}{Al}$

where, T is the tension, L the original length, A the area of cross-section, and l the increase in length.

Velocity of transverse wave $= \sqrt{\dfrac{T}{Ad}}=\sqrt{\dfrac{Yl}{Ld}}$

Fundamental frequency $n = \dfrac{1}{2L}\sqrt{\dfrac{Yl}{Ld}}$

On substituting the values of the various terms, we have

$$n = \frac{1}{2\times 0.6}\sqrt{\frac{2\times10^{11}\times 3.14\times 0.3\times10^{-2}}{0.6\times 8000}}$$

$$= 522.08 \text{ Hz}$$

9.35. A uniform rope of length 12 m and mass 6 kg hangs vertically from a rigid support. A block of mass 2 kg is attached to the free end of the rope. A transverse pulse of wavelength 0.06 m is produced at the lower end of the rope. What is the wavelength of the pulse when it reaches the top of the rope ? (IIT 1984)

Solution: Tension acting at lower end of the rope $T_1 = 2\times 9.8$ N

Tension acting at upper end of the rope $T_2 = 8\times 9.8$ N

Speed of wave at lower end of rope $v_1 = \sqrt{\dfrac{T_1}{m}}=\sqrt{\dfrac{2\times 9.8}{m}}$

Speed of wave at upper end of rope $v_2 = \sqrt{\frac{T_2}{m}} = \sqrt{\frac{8 \times 9.8}{m}}$

Now
$$\begin{aligned} v_2 &= 2v_1 \\ n \cdot \lambda_2 &= 2 \cdot n \cdot \lambda_1 \\ \therefore \lambda_2 &= 2\lambda_1 \\ &= 2 \times 0.06 \\ &= 0.12 \text{ m} \end{aligned}$$

9.36. An aluminium wire of cross-sectional area 1×10^{-6} m^2 is joined to a steel wire of the same cross-sectional area. This compound wire is stretched on a sonometer, pulled by a weight of 10 kg. The total length of the compound wire between the two bridges is 1.5 m of which the aluminium wire is 0.6 m and the rest is the steel wire. Transverse vibrations are set up in the wire by using an external force of variable frequency. Find the lowest excitation for which standing waves are formed such that the joint in the wire is a node. What will be the total number of nodes observed at this frequency excluding the two at the ends of the wire? The density of aluminium is 2.6×10^3 kg/m^3 and that of steel 1.04×10^4 kg/m^3.

(Roorkee 1983)

Solution: Let n_1 and n_2 be the total number of loops in aluminium and steel wire respectively. The frequency n is given by

$$n = \frac{n_1}{2l_1}\sqrt{\frac{T}{m_1}} = \frac{n_2}{2l_2}\sqrt{\frac{T}{m_2}}$$

where l_1 and l_2 represents the lengths of the aluminium and steel wire respectively

or $$\frac{n_1}{2l_1}\sqrt{\frac{T}{Ad_1}} = \frac{n_2}{2l_2}\sqrt{\frac{T}{Ad_2}}$$

where d_1 and d_2 represents the densities of aluminium and steel wire respectively,

or $$\frac{n_1}{n_2} = \frac{l_1}{l_2}\sqrt{\frac{d_1}{d_2}}$$

On substituting the values of l_1, l_2, d_1 and d_2, we have

$$\frac{n_1}{n_2} = \frac{0.6}{0.9}\sqrt{\frac{2.6 \times 10^3}{1.04 \times 10^4}} = \frac{1}{3}$$

Thus the aluminium wire will have one loop and the steel wire three loops. On the total length of 1.5 m there are five nodes. Excluding the nodes at the two ends, the number of nodes is 3.

The lowest frequency, $n = \dfrac{1}{2\times 0.6}\sqrt{\dfrac{10\times 9.8}{10^{-6}\times 2.6\times 10^{3}}}$

$$= \frac{100}{1.2}\sqrt{\frac{4.9}{1.3}} = 161.8 \text{ Hz}$$

9.37. A wire with mass per unit length 5 g/m is stretched between two rigid supports with a tension 4.5 N. It resonates at the frequency 420 Hz. If the next resonance occurs at 490 Hz find the length of the wire.

Solution: In case of the vibration of a string, both even and odd harmonics are obtained. That is, frequencies are n, $2n$, $3n$, $4n$, etc. The difference between the two frequencies gives rise to the fundamental frequency.

In the present case, $n = 490 - 420 = 70$ Hz

$$n = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

$$\therefore L = \frac{1}{2n}\sqrt{\frac{T}{m}} = \frac{1}{2\times 70}\sqrt{\frac{4.5}{5\times 10^{-3}}} = \frac{3}{14}\text{ m}$$

9.38. Two sonometer wires have fundamental frequencies of 500 vibrations per second when kept under the same tension. What fractional increase in the tension of one wire would cause an occurrence of 5 beats per second when both wires vibrate together? (Roorkee 1985)

Solution: The fundamental frequency of the sonometer wire is given by

$$n = \frac{1}{2l}\sqrt{\frac{T}{m}}$$

Now $500 = \dfrac{1}{2l}\sqrt{\dfrac{T}{m}}$

and $505 = \dfrac{1}{2l}\sqrt{\dfrac{T'}{m}}$

where T' is the new tension. Now on dividing the two, we have

$$\frac{505}{500} = \sqrt{\frac{T'}{T}}$$

or $$\frac{505^2}{500^2} = \frac{T'}{T}$$

or $$\frac{T'-T}{T} = \frac{\Delta T}{T} = \frac{505^2 - 500^2}{500^2} = \frac{1005\times 5}{500^2} = 0.0201$$

9.39. The displacement of the medium in a sound wave is given by the equation

$Y = A \cos (ax + bt)$

where A, a and b are positive constants. The wave is reflected by an obstacle situated at $x = 0$. The intensity of the reflected wave is 0.64 times that of the incident wave (a) What is the wavelength and frequency of the incident wave ? (b) Write the equation for the reflected wave. (c) In the resultant wave formed after reflection, find the maximum and minimum values of the particle speed in the medium. (d) Express the resultant wave as a superposition of a standing wave and a travelling wave. What are the positions of the anti nodes of the standing wave ? What is the direction of propagation of the travelling wave? (IIT 1991)

Solution:

(a) Let the equation of the travelling wave

$Y = A \sin (\omega t + kx + \phi)$

On comparing it with the equation given above

$\omega = b,\ k = a$ and $\phi = \dfrac{\pi}{2}$

$f = \dfrac{\omega}{2\pi} = \dfrac{b}{2\pi},\ \lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{a}$

(b) $I_r = 0.64\, I_0$

$\therefore \dfrac{I_r}{I_0} = 0.64$

$\therefore \dfrac{A_r^2}{A^2} = 0.64$

$\therefore A_r = 0.8\,\text{A}$

The equation of reflected wave is given by

$Y_2 = 0.8A \cos (bt - ax + \pi) = -0.8A \cos (bt - ax)$

(c) Velocity of the particle is given by

$v = \omega \sqrt{A^2 - Y^2}$

The particle velocity is maximum when amplitude is maximum i.e. at antinode and Y is min and is equal to 0

$(V_{AN})_{\max} = \omega A_{\max} = b(A + 0.8A) = 1.8bA$

Particle velocity is min when $Y = A$

$V_{\min} = O$

$A = A_{\min} = A_1 - A_2$ and so $(V_N)_{\min} = 0.2bA$

(d) $y = 0.8A\,[\cos(bt + ax) - \cos(bt - ax) + 0.2A\cos(bt + ax)$

$= -1.6\,A \sin ax \sin bt + 0.2A \cos(bt + ax)$

$y = y_{\text{Standing}} + y_{\text{Progressive}}$

$y_{\text{Standing}} = -1.6A \sin ax \sin bt$

$y_{\text{Pregressive}} = 0.2A \cos(bt + ax)$

Now $\sin ax = \pm 1$

$$ax = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

or $$x = \frac{\pi}{2a}, \frac{3\pi}{2a}, \frac{5\pi}{2a}$$

In the equation of progressive wave x is $+ve$ the wave is travelling along negative x axis i.e., in the direction of incident wave.

9.40. The following equations represent transverse waves: $Z_1 = A \cos(kx - \omega t)$, $Z_2 = A \cos(kx + \omega t)$, $Z_3 = A \cos(ky - \omega t)$. Identify the combinations of the waves which will produce, (a) standing waves, (b) a wave travelling in the direction making an angle of 45° with positive X and Y axes. In each cases, find the positions at which the resultant intensity is always zero.

(IIT 1987)

Solution:

(a) The first two waves are travelling along x axis in opposite directions and their superposition give rise to standing waves.

$Z = Z_1 + Z_2 = A \cos(kx - \omega t) + A \cos(kx + \omega t) = 2A \cos kx \cos \omega t$

The resultant intensity $\propto \cos^2 kx$. For zero intensity

$\cos^2 kx = 0$

$$\therefore\ kx = (2n + 1)\frac{\pi}{2} \text{ where } n = 0, 1, 2, \ldots$$

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots$$

(b) The third wave is travelling along Y axis, combination of 1 and 3 will give

$$Z = Z_1 + Z_3 = A \cos(kx - \omega t) + A \cos(ky - \omega t)$$

$$= 2a \cos\left(\frac{kx - ky}{2}\right) \cos\left(\frac{kx + ky}{2} - \omega t\right)$$

This is the equation of a progressive wave of amplitude $2A \cos\left(\frac{kx - ky}{2}\right)$ travelling along the direction having direction cosines l, m and n.

$\frac{kx+ky}{2} = K(lx+my+nz)$; K = wave-vector of the resultant wave.

$Kl = \frac{k}{2}$, $Km = \frac{k}{2}$ and $Kn = 0$

$\because l^2 + m^2 + n^2 = 1$

We have $\frac{k^2}{4K^2} + \frac{k^2}{4K^2} = 1$

$K = \frac{k}{\sqrt{2}}$

$l\cos\alpha = \frac{k}{2k/\sqrt{2}} = \frac{1}{\sqrt{2}} \therefore \alpha = 45°$

$m\cos\beta = \frac{k}{2k/\sqrt{2}} = \frac{1}{\sqrt{2}} \therefore \beta = 45°$

$n = \cos\gamma = 0 \therefore \gamma = \frac{\pi}{2}$

Thus this combination will give a progressive wave in the direction making 45° with the positive X and Y axis

For zero intensity $\cos^2\frac{kx-ky}{2} = 0$

$\frac{kx-ky}{2} = \pm(2n+1)\frac{\pi}{2}$

or $y = x \pm (2n+1)\frac{\pi}{k}$

9.41. A metallic rod of length 1 m is rigidly clamped at its mid-point. Longitudinal stationary waves are set up in the rod in such a way that there are two nodes on either side of the mid-point. The amplitude of an antinode is 2×10^{-6} m. Write the equation of motion at a point 2 cm from the mid-point and those of constituent waves in the rod ($Y = 2 \times 10^{11}$ N/m^2 and $\rho = 8 \times 10^3$ kg/m^3). (IIT 1994)

Solution:

$4 \times \frac{\lambda}{2} + 2 \times \frac{\lambda}{4} = L$ (see Fig. 9.4)

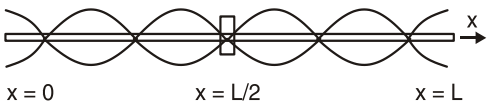


Fig. 9.4

$\lambda = \frac{2}{5}L = 0.4$ m

Velocity $V = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{2\times10^{11}}{8\times10^{3}}} = 5000\ \text{m/s}$

frequency $f = \frac{V}{\lambda} = \frac{5000}{0.4} = 12500\ \text{Hz}$

Let the incident and reflected wave along the rod be $Y_1 = A \sin(\omega t - kx)$ and $Y_2 = A \sin(\omega t + kx + \phi)$

Resultant wave will be

$Y = Y_1 + Y_2 = A[\sin(\omega t - kx) + \sin(\omega t + kx + \phi)]$

$= 2A \cos\left(kx + \frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right)$

At $x = 0$ we have antinode and $\cos\left(k\times 0 + \frac{\phi}{2}\right) = 1$ therefore $\phi = 0$

$A_{max} = 2A = 2 \times 10^{-6}\ m$ (given)

$\therefore Y = 2 \times 10^{-6} \cos kx \sin \omega t$

$Y = 2 \times 10^{-6} \cos 5\pi x \sin 25000\pi t$

For a point 2 cm from the mid-point of the rod we have $x = 0.5 \pm 0.02$

$\therefore y = 2 \times 10^{-6} \cos 5\pi (0.5 \pm 0.02) \sin 25000\pi t$

The resultant wave $y = 2 \times 10^{-6} \cos 5\pi x \sin 25000\pi t$ can be represented as

$y = 10^{-6}[\sin(5\pi x + 25000\pi t) - \sin(5\pi x - 25000\pi t)]$

$= [10^{-6} \sin(5\pi x + 25000\pi t) + 10^{-6} \sin(25000\pi t - 5\pi x)]$

The required constitutent waves are

$Y_1 = 10^{-6} \sin[25000\pi t + 5\pi x]$ and $Y_2 = 10^{-6} \sin[25000\pi t - 5\pi x]$

9.42. Stationary waves are produced in a 120 cm long string. The amplitude for points separated by distance of 15 cm is 3.5 mm. Find (i) the overtone for these oscillations (ii) the maximum displacement amplitude.

Solution:

(i) In view of figure 9.5 we have

$2x_1 = 15$

$x_1 = 7.5\ \text{cm}$

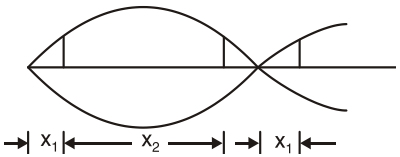


Fig. 9.5

Further $x_2 - x_1 = 15\ \text{cm}$

$\therefore x_2 = (15 + x_1)\ \text{cm}$

$y = 2a \sin \frac{2\pi}{\lambda} \cdot x_1 = 2a \sin \frac{2\pi}{\lambda} (15 + x_1)$

$$\text{or} \quad \frac{2\pi}{\lambda} \cdot x_1 = \pi - \frac{2\pi}{\lambda}(15 + x_1)$$

$$x_1 = \frac{\lambda}{2} - 15 - x_1$$

$$2x_1 + 15 = \frac{\lambda}{2}$$

$$\lambda = 60\,\text{cm}$$

The string will have four loops hence it will oscillate with third overtone.

(ii) $$3.5 = 2a \sin \frac{2\pi}{60} \times 7.5$$

$$= 2a \sin \frac{\pi}{4}$$

$$= \frac{2a}{\sqrt{2}}$$

$$\therefore 2a = 3.5\sqrt{2} = 4.95\,\text{mm}$$

9.43. A uniform rope of mass 0.1 kg and length 2.45 m hangs from the ceiling (a) Find the speed of transverse wave in the rope at a point 0.5 m distance from the lower end. (b) Calculate the time taken by a transverse wave to travel the full length of the rope. (Roorkee 1991)

Solution:

(a) Tension at a point h metre high from the lower end of the rope = mgh where m is mass per unit length.

Speed of the transverse wave $V = \sqrt{\frac{T}{m}} = \sqrt{\frac{mhg}{m}} = \sqrt{gh}$

On putting the values we have

$V = \sqrt{9.8 \times 0.5} = 2.21\,\text{m/s}$

(b) At the lower end of the rope tension is zero hence the speed of the transverse wave is zero

The speed of transverse wave at the upper end = $\sqrt{gl}$

Acceleration = $\frac{\sqrt{gl}}{t}$

$$l = \frac{1}{2} \cdot \frac{\sqrt{gl}}{t} \cdot t^2$$

$$\therefore t = 2\sqrt{\frac{l}{g}} = 2 \times \sqrt{\frac{2.45}{9.8}} = 1\,\text{sec.}$$

9.44. A point source S producing spherical wave is placed on a straight line between two points P and Q as shown in the Fig. 9.6. The oscillation amplitudes of the particles of the medium at these points are a_1 and a_2 respectively. Assuming the radius vector for the two points as $\vec{r_1}$ and $\vec{r_2}$ find the radius vector of the source.

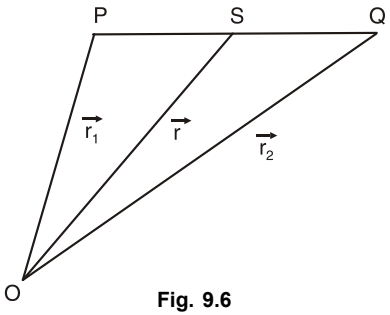


Fig. 9.6

Solution: Let $PS = d_1$ and $SQ = d_2$

$$\frac{I_1}{I_2} = \frac{\text{Intensity at } P}{\text{Intensity at } Q} = \frac{d_2^2}{d_1^2}$$

$$\frac{a_1^2}{a_2^2} = \frac{d_2^2}{d_1^2}$$

or $a_1 d_1 = a_2 d_2 = k$ (constant)

hence $d_1 = \dfrac{k}{a_1}$ and $d_2 = \dfrac{k}{a_2}$

Let i be the unit vector along PQ

$$\overrightarrow{PS} = i\frac{k}{d_1} \text{ and } \overrightarrow{SQ} = i\frac{k}{d_2}$$

Now $\overrightarrow{OP} + \overrightarrow{PS} = \overrightarrow{OS}$

$$r_1 + i\frac{k}{a_1} = r$$

or $a_1 r_1 + ik = a_1 r$

Similarly $a_2 r_2 - ik = a_2 r$

On adding the two equation we have

$$(a_1 + a_2)\, r = a_1 r_1 + a_2 r_2$$

$$r = \frac{a_1 r_1 + a_2 r_2}{a_1 + a_2}$$

9.45. Two coherent sources of sound of intensity 50 watt/m^2 and frequency 500 Hz are at distances $x_1 = 0.14$ m and $x_2 = 0.25$ m from a point. Assuming the velocity of sound in the medium as 330 m/s find the resultant intensity at that point.

Solution:

Path diff $= x_2 - x_1$

Phase diff $= \dfrac{2\pi}{\lambda} \cdot (x_2 - x_1)$

Resultant intensity $I = I_0 + I_0 + 2\sqrt{I_0^2} \cos \dfrac{2\pi}{\lambda} \cdot (x_2 - x_1)$

$$= 2I_0 + 2I_0 \cos\frac{2\pi \cdot n}{v}(x_2 - x_1)$$

$$= 4I_0 \cos^2\frac{\pi n}{v}(x_2 - x_1)$$

On substituting the values we have

$$I = 4 \times 50 \times \cos^2 \cdot \frac{\pi \times 500}{330} \times 0.11$$

$$= 200 \cdot \cos^2 \cdot \frac{\pi \cdot 500}{3000}$$

$$= 200\cos^2\frac{\pi}{6}$$

$$= 200 \times \frac{3}{4} = 150 \text{ watt/m}^2$$

9.46. A plane elastic wave $\xi = a \cos(\omega t - kx)$ propogates in a stationary frame of reference S. Find the equation of the wave in the frame of reference S' moving with a velocity V along x axis relative to S.

Solution: The coordinate of frame of reference S' is given by

$$x' = x - Vt$$

$$\therefore x = (x' + Vt)$$

$$\xi = a\cos[\omega t - Kx' - KVt]$$

$$= a\cos\left[\omega t\left(1 - \frac{V}{v}\right) - Kx'\right] \text{ where } v = \frac{\omega}{K}$$

9.47. Two plane waves propogate in a homogenous elastic medium. Of the two, one wave along x axis and the other along y axis: $\xi_1 = a\cos(\omega t - kx)$, $\xi_2 = a\cos(\omega t - ky)$. Find the resultant wave motion in the xy plane when both the waves (a) are transverse and their oscillation directions coincide (b) are longitudinal.

Solution:

(a) $\xi = \xi_1 + \xi_2$

$$= a\left[\cos(\omega t - kx) + \cos(\omega t - ky)\right]$$

$$= 2a\cos\frac{k(y-x)}{2} \cdot \cos\left(\omega t - \frac{ky + kx}{2}\right)$$

This is the equation of progressive wave of amplitude $2a\cos\frac{k(y-x)}{2}$ and travelling along a line equally inclined to x and y.

When $\frac{k}{2}(y-x) = \pm(2s+1)\frac{\pi}{2}$; $\xi = 0$

$\therefore y = x \pm (2s+1)\frac{\lambda}{2}$ (see Fig. 9.7a)

When $\frac{k}{2}(y-x) = \pm s\pi$,

amplitude is maximum

$y = x \pm s\lambda$

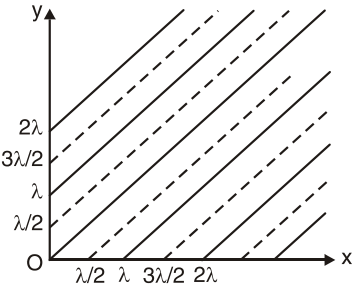


Fig. 9.7a

(b) $\vec{\xi} = \xi_1 i + \xi_2 j$

$\omega t - kx = \cos^{-1}\frac{\xi_1}{a}$

$\omega t - ky = \cos^{-1}\frac{\xi_2}{a}$

$k(y-x) = \cos^{-1}\frac{\xi_1}{a} - \cos^{-1}\frac{\xi_2}{a} = \alpha - \beta$

$\cos k(y-x) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos\phi$

$\phi = k(y-x)$

$$\frac{\xi_1}{a}\cdot\frac{\xi_2}{a} + \sqrt{1-\frac{\xi_1^2}{a^2}}\cdot\sqrt{1-\frac{\xi_2^2}{a^2}} = \cos\phi$$

$$\left(1-\frac{\xi_1^2}{a^2}\right)\left(1-\frac{\xi_2^2}{a^2}\right) = \left(\cos\phi - \frac{\xi_1\xi_2}{a^2}\right)^2$$

$\xi_1^2 + \xi_2^2 - 2\xi_1\xi_2\cos\phi = a^2\sin^2\phi$ where $\phi = k(y-x)$

This is equation of an ellipse

when $\phi = k(y-x) = \pm 2s\pi$, $s = 0, 1, 2, 3$

$y = x \pm s\lambda$

$\xi_1^2 + \xi_2^2 - 2\xi_1\xi_2 = 0$

$\xi_1 = \xi_2$

This represents a *st* line along a line $y = x \pm s\lambda$

when $\phi = k(y-x) = \pm(2s+1)\frac{\pi}{2}$

$y = x \pm (2s+1)\frac{\lambda}{4}$

Now $\xi_1^2 + \xi_2^2 = a^2$

This is equation of a circle on the line

$y = x \pm (2s+1)\frac{\lambda}{4}$

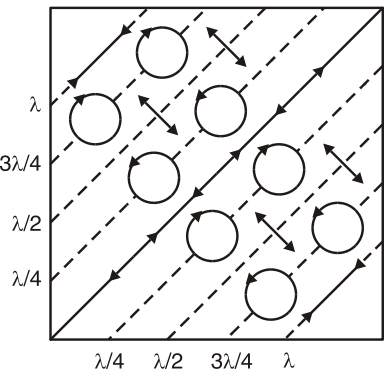


Fig. 9.7b

when $\phi = k(y-x) = \pm(2s+1)\pi$

$$y = x \pm \left(s + \frac{1}{2}\right)\lambda \text{ (see Fig. 9.7b)}$$

Now $(\xi_1 + \xi_2)^2 = 0;\ \xi_1 = -\xi_2$

9.48. Find the nature of motion of the particles in an elastic medium in which two plane transverse waves propogates along x and y axes respectively. The oscillations for both the waves is along z axis. The wavelength and amplitude of the two waves is same and equal to λ and A. The initial phase difference between them is θ.

Solution:

$z = A\sin(\omega t - kx - \theta)$ and $z = A\sin(\omega t - ky)$

On comparing the two

$$y = x + \frac{\theta\cdot\lambda}{2\pi}$$

The maximum amplitude is equal to $2A$ and is represented by heavy line (see Fig. 9.8)

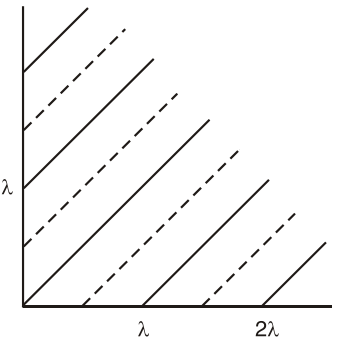


Fig. 9.8

For $\theta = 0, y = x + \frac{\theta\lambda}{2\pi} \pm n\lambda;$

$n = 0, 1, 2,$ etc.

The minimum amplitude is zero and are represented by

$$y = x + \frac{\theta\lambda}{2\pi} \pm \left(n + \frac{1}{2}\right)\lambda\ ;\ n = 0,\ 1,\ 2$$

and are shown by dotted lines.

9.49. Two thin wires of length l_1 and l_2 and having mass per unit length as m_1 and m_2 are joined together to form a single wire. The composite wire is stretched between the two supports A and B as shown in Fig. 9.9. Stationary waves are formed such that l_1 length contains four loops and l_2 length contains 2 loops. Taking $\frac{m_2}{m_1}$ as $\frac{4}{3}$. Find the ratio of l_1 and l_2.

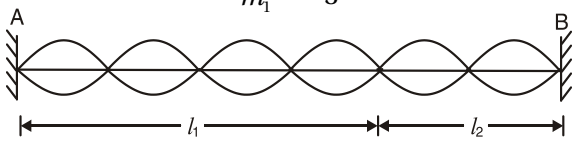


Fig. 9.9

Solution:

$$n\cdot\frac{l_1}{2} = \sqrt{\frac{T}{m_1}} \text{ and } n\cdot l_2 = \sqrt{\frac{T}{m_2}}$$

On dividng the two

$$\frac{l_1}{2l_2} = \sqrt{\frac{m_2}{m_1}}$$

$$\therefore \frac{l_1}{l_2} = 2\sqrt{\frac{4}{3}} = \frac{4}{\sqrt{3}}$$

9.50. An aluminium wire of length l_1 and a lead wire of length l_2 are joined together to form a single wire of length 2 m. A copper wire of length $l_3 = l_1 + l_2$ and the composite wire are arranged as shown in the figure 9.10. For what value of $\frac{l_1}{l_2}$ sound waves will take the same time to travel a length l_3 through the two path? $Y_{Al} = 7 \times 10^{10}$ N/m^2, $Y_{\text{lead}} = 1.6 \times 10^{10}$ N/m^2, $Y_{Cu} = 13 \times 10^{10}$ N/m^2, $\rho_{Al} = 2.7 \times 10^3$ kg/m^3, $\rho_{\text{lead}} = 11.3 \times 10^3$ kg/m^3 and $\rho_{Cu} = 8.9 \times 10^3$ kg/m^3.

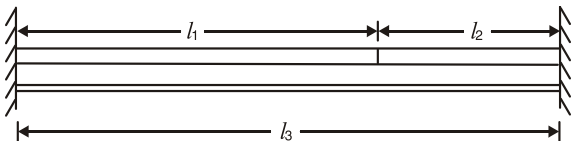


Fig. 9.10

Solution:

$$V_{Al} = \sqrt{\frac{70}{2.7}} \times 10^3 = 5.09175 \times 10^3 \text{ m/s}$$

$$V_{\text{lead}} = \sqrt{\frac{16}{11.3}} \times 10^3 = 1.18993 \times 10^3 \text{ m/s}$$

$$V_{Cu} = \sqrt{\frac{130}{8.9}} \times 10^3 = 3.82188 \times 10^3 \text{ m/s}$$

Now $$\frac{2 - l_2}{5.09175} + \frac{l_2}{1.18993} = \frac{2}{3.82188}$$

$$2.37986 - 1.18993\, l_2 + 5.09175\, l_2 = 3.17060$$

$$3.90182\, l_2 = 0.79074$$

$$\therefore l_2 = 0.2026$$

$$l_1 = 1.7974$$

$$\therefore \frac{l_1}{l_2} = 8.87$$

9.51. Stationary longitudinal waves are produced in a rod of length l. Determine the complete harmonic series when (i) the rod is free at both ends and not supported at any point of its length and (ii) the rod is clamped at both the ends.

Solution:

Let $y_1 = a \sin \frac{2\pi}{\lambda} (vt - x)$ (see Fig. 9.11)

be the displacement at point P.

The displacement produced at B is given by

$$y = a \sin \frac{2\pi}{\lambda} (vt - l)$$

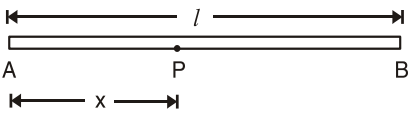


Fig. 9.11

Let B be taken as origin for the reflected wave. Assuming $BP = x'$ towards A. The equation for the displacement of P due to reflected wave

$$y_2 = a \sin \frac{2\pi}{\lambda} (vt - l - x')$$

$$x + x' = l$$

$$\therefore y_2 = a \sin \frac{2\pi}{\lambda} (vt + x - 2l)$$

Resultant displacement

$$y = y_1 + y_2$$

$$= a \sin \frac{2\pi}{\lambda} (vt - x) + a \sin \frac{2\pi}{\lambda} (vt + x - 2l)$$

$$= 2a \sin \frac{2\pi}{\lambda} (vt - l) \cos \frac{2\pi}{\lambda} (l - x)$$

(i) At each end we have antinode. The displacement is maximum at $x = 0$ and $x = l$.

$\left(\frac{dy}{dx}\right)$ at $x = 0$ and $x = l$ is zero

$$\frac{dy}{dx} = 2a \cdot \frac{2\pi}{\lambda} \sin(l - x) \sin \frac{2\pi}{\lambda} (vt - l)$$

at $x = 0$ we have

$$\frac{4a\pi}{\lambda} \cdot \sin \frac{2\pi l}{\lambda} \cdot \sin \frac{2\pi}{\lambda} (vt - l) = 0$$

This is possible when

$$\frac{2\pi l}{\lambda} = r\pi \text{ where } r = 1, 2, 3, \ldots$$

$$n \cdot \frac{2\pi l}{v} = r\pi; v = \text{velocity and } n = \text{frequency}$$

$$n_1 = \frac{v}{2l}, n_2 = \frac{2v}{2l}, n_3 = \frac{3v}{2l} \ldots \text{represents the complete harmonics.}$$

(ii) The equation for the direct and reflected wave is given by

$$y_1 = a \sin \frac{2\pi}{\lambda}(vt - x)$$

$$y_2 = -a \sin \frac{2\pi}{\lambda}(vt + x - 2l)$$

The negative sign is due to phase reversal of π on reflection, resultant displacement at P is given by

$$y = y_1 + y_2$$

$$= a \sin \frac{2\pi}{\lambda}(vt - x) - a \sin \frac{2\pi}{\lambda}(vt + x - 2l)$$

$$= 2a \cos \frac{2\pi}{\lambda}(vt - l) \sin \frac{2\pi}{\lambda}(l - x)$$

$y = 0$ when $x = 0$

$$0 = 2a \cos \frac{2\pi}{\lambda}(vt - l) \sin \frac{2\pi l}{\lambda}$$

This is possible when

$$\frac{2\pi l}{\lambda} = r\pi \text{ where } r = 1, 2, 3 \text{ etc.}$$

$$\frac{2\pi l \cdot n}{v} = r\pi$$

or $$n = \frac{rv}{2l}$$

$$n_1 = \frac{v}{2l},\ n_2 = \frac{2v}{2l},\ n_3 = \frac{3v}{2l}$$ etc. represents the complete harmonics.

9.52. Two radio stations broadcast their programmes at the same amplitude A and at slightly different frequencies ω_1 and ω_2 respectively. Where $\omega_2 - \omega_1 = 10^3$ Hz. A detector receives the signal from the two stations simultaneously. It can detect signals of intensity $> 2A^2$. (a) Find the time interval between successive maxima of the intensity of the signal received by the detector. (b) Find the time for which the detector remains idle in each cycle of the intensity of the signal. (IIT 1993)

Solution:

$$y_1 = A \sin 2\pi\omega_1 t \text{ and } y_2 = A \sin 2\pi\omega_2 t$$

$$y = \boldsymbol{y_1} + \boldsymbol{y_2} = 2A \cos 2\pi \left(\frac{\omega_2 - \omega_1}{2}\right) t \sin 2\pi \left(\frac{\omega_1 + \omega_2}{2}\right) t$$

$$y = A' \sin 2\pi \left(\frac{\omega_1 + \omega_2}{2}\right) t\ ;\ A' = 2A \cos 2\pi \left(\frac{\omega_2 - \omega_1}{2}\right) t$$

$I = A'^2 = 4A^2 \cos^2 \pi (\omega_2 - \omega_1)\, t$

I will be maximum when $\cos \pi (\omega_2 - \omega_1)\, t = \pm 1$

or $\pi (\omega_2 - \omega_1)\, t = 0, \pi, 2\pi$ etc.

or $$t = 0, \frac{1}{\omega_2 - \omega_1}, \frac{2}{\omega_2 - \omega_1}$$

Δt = time between two consecutive maxima = $\dfrac{1}{\omega_2 - \omega_1} = 10^{-3}$ sec

$I = 2A^2 = 4A^2 \cos^2 \pi (\omega_2 - \omega_1)\, t$ with the value of $\cos \pi (\omega_2 - \omega_1)\, t = \pm \dfrac{1}{\sqrt{2}}$

or $\pi (\omega_2 - \omega_1)\, t = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$

or $$t = \frac{1}{4(\omega_2 - \omega_1)}, \frac{3}{4(\omega_2 - \omega_1)} \text{ etc.}$$

$\Delta t'$ = time interval between two successive minima = $\dfrac{1}{2(\omega_2 - \omega_1)}$
$= 5 \times 10^{-4}$ sec

9.53. A tuning fork has a particular frequency at room temperature of 25°C. It is now put in boiling water (100°C). After taking it out of water its frequency is lowered by 1%. Calculate the temperature coefficient of the elastic constant. $\alpha_{\text{steel}} = 12 \times 10^{-6}/°\text{C}$

Solution: The frequency of a tuning fork varies $a\sqrt{Yl}$ where Y is Young's modulus and l is length of the prong.

$$\frac{\text{Freq at 100°C}}{\text{Freq at 25°C}} = \sqrt{\frac{Y_{100} \times l_{100}}{Y_{25} \times l_{25}}} = \sqrt{\frac{Y_{25}(1+75\beta)\, l_{25}(1+75\alpha)}{Y_{25} \times l_{25}}}$$

$$\frac{99}{100} = \sqrt{(1+75\beta)(1+75\alpha)}$$

$$1 + 75\beta = \left(\frac{99}{100}\right)^2 \times \frac{1}{1 + 75 \times 12 \times 10^{-6}}$$

$$= \left(\frac{99}{100}\right)^2 \times \frac{1}{1.0009}$$

$$= 0.9792$$

$$75\beta = -0.0208$$

$$\beta = -2.77 \times 10^{-4}$$

9.54. Four violin strings all of the same length and material, but of diameters in the ratio 4 : 3 : 2 : 1 are to be stretched so that each gives a note

whose frequency is 3/2 times that of the preceding string. If the stretching force of the first string is 3 kg wt, calculate the tension in the other strings.

Solution:

$$\frac{n_2}{n_1}=\frac{n_3}{n_2}=\frac{n_4}{n_3}=\frac{3}{2}$$

$n=\frac{1}{2l}\sqrt{\frac{4T}{\pi d^2\rho}}$ is an expression for the frequency.

For all the four strings we have

$nd = k\cdot\sqrt{T}$

In view of this

$n_1d_1 = k\sqrt{T_1}$, $n_2d_2 = k\sqrt{T_2}$, $n_3d_3 = k\sqrt{T_3}$ and $n_4d_4 = k\sqrt{T_4}$

$$\frac{n_2d_2}{n_1d_1} = \frac{\sqrt{T_2}}{\sqrt{T_1}}$$

$$T_2 = \left(\frac{3}{2}\times\frac{3}{4}\right)^2\cdot T_1$$

$$= \frac{81}{64}\times 3$$

$$= 3.797 \text{ Kg wt}$$

$$T_3 = \left(\frac{n_3d_3}{n_1d_1}\right)^2\cdot T_1$$

$$= \left(\frac{9}{4}\times\frac{1}{2}\right)^2\times 3$$

$$= \frac{81\times 3}{64}$$

$$= 3.797 \text{ kg wt}$$

$$T_4 = \left(\frac{n_4d_4}{n_1d_1}\right)^2\cdot T_1$$

$$= \left(\frac{27}{8}\times\frac{1}{4}\right)^2\times 3$$

$$= 2.136 \text{ Kg wt}$$

9.55. In the Melde's experiment it has been found that the string vibrates in 5 loops when 10 gm were placed on the pan. What mass must be placed in the pan to make the string to vibrate in 7 loops.

Solution: Let l be the length of the thread then

$$l = 5\times\frac{\lambda}{2}$$

$$\therefore \lambda = \frac{2l}{5}$$

$$v = \sqrt{\frac{T}{m}} = n\lambda$$

$$\therefore n = \frac{5}{2l}\sqrt{\frac{T}{m}}$$

$$= \frac{5}{2l}\sqrt{\frac{10g}{m}} = \frac{7}{2l}\sqrt{\frac{M\cdot g}{m}}$$

$$5\sqrt{10} = 7\sqrt{M}$$

$$\therefore M = \frac{25\times 10}{49} = 5.1 \text{ gm}$$

9.56. A metal wire of mass 9.8 gm/m is stretched with a tension of 10 kg wt between two rigid supports one meter apart. The wire passes at its middle point between the poles of a permanent magnet and it vibrates in resonance when carrying an alternating current of frequency n. Calculate the frequency of the alternating source.

Solution:

$$L = 1\text{m} = \frac{\lambda}{2}$$

$$\therefore \lambda = 2 \text{ m}$$

$$v = \sqrt{\frac{T}{m}} = \sqrt{\frac{10\times 9.8}{9.8\times 10^{-3}}} \text{ 100 m/sec}$$

$$n = \frac{v}{\lambda} = \frac{100}{2} = 50 \text{ c/s}$$

9.57. How many times the fundamental frequency of a stretched wire changes when its length is shortened by 35% and the tension is increased by 70%.

Solution:

$$n = \frac{1}{2l}\sqrt{\frac{T}{m}}$$

$$n' = \frac{1}{2(l-\Delta l)}\sqrt{\frac{T+\Delta T}{m}}$$

$$= \frac{1}{2l}\sqrt{\frac{T}{m}} \cdot \frac{\sqrt{1+\Delta T/T}}{\left(1-\Delta l/l\right)}$$

$$n' = \frac{n\sqrt{1+\Delta T/T}}{\left(1-\Delta l/l\right)}$$

$$\frac{n'}{n} = \frac{\sqrt{1+\Delta T/T}}{\left(1-\Delta l/l\right)}$$

On putting the values we have

$$\frac{n'}{n} = \frac{\sqrt{1+0.7}}{1-0.35} = \frac{\sqrt{1.7}}{0.65} = \frac{1.30}{0.65} = 2$$

9.58. *AB* is a cylinder of length 1.0 m, fitted with a thin flexible diaphragm *C* at the middle and two other thin flexible diaphragms *A* and *B* at the ends (Fig. 9.12). The portions *AC* and *BC* contains hydrogen and oxygen respectively. The diaphragms *A* and *B* are set into vibrations of the same frequency. What is the minimum frequency of these vibrations for which the diaphragm *C* is a node? Under the conditions of the experiment the velocity of sound in hydrogen is 1,100 m/s and in that oxygen 300 m/s.

(IIT 1978)

Solution: The two parts *AC* and *BC* form an organ pipe each closed at one end. For both the organ pipes, diaphragm *C* forms the closed end.

Let n_1 and n_2 be the fundamental frequencies in H_2 and O_2 respectively. Then

$$n_1 = \frac{V_H}{4L} = \frac{1100}{4\times 0.5} = 550 \text{ Hz}$$

and $$n_2 = \frac{V_O}{4L} = \frac{300}{4\times 0.5} = 150 \text{ Hz}$$

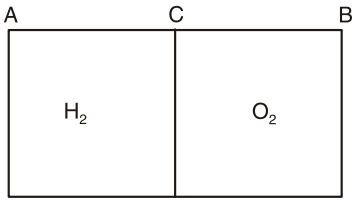


Fig. 9.12

Now $$\frac{n_1}{n_2} = \frac{550}{150} = \frac{11}{3}$$

i.e., $$3n_1 = 11n_2$$

Thus the common minimum frequency $= 3n_1 = 3 \times 550 = 1{,}650$ Hz

$= 11\, n_2 = 11 \times 150 = 1{,}650$ Hz

9.59. A string 25 cm long and having mass of 2.5 g is under tension. A pipe closed at one end is 40 cm long. When the string is set vibrating in its first

overtone and the air in the pipe in its fundamental frequency, 8 beats per second are heard. It is observed that decreasing the tension in the string decreases the beats frequency. If the speed of sound in air is 320 m/sec, find the tension in the string. (IIT 1982)

Solution: Let n_s bet the first overtone of the string and n_p the fundamental frequency of the organ pipe closed at one end.

Now, either

$$n_s - n_p = 8$$

or $$n_p - n_s = 8$$

But in this case the number of beats decreases on decreasing the tension of the string. Thus

$$n_s - n_p = 8$$

Now $$n_p = \frac{V}{4L} = \frac{320}{4 \times 0.4} = 200 \text{ Hz}$$

$$\therefore n_s = 208 \text{ Hz}$$

$$n_s = 208 = \frac{1}{l}\sqrt{\frac{T}{m}} \quad (n_s = 2 \times \text{fundamental frequency})$$

$$m = \frac{2.5 \times 10^{-3}}{25 \times 10^{-2}} = 10^{-2} \text{ kg/m}$$

$$T = 208^2 \times l^2 \times m$$

$$= 208^2 \times (0.25)^2 \times 10^{-2}$$

$$= 27.05 \text{ N}$$

9.60. A system consisting of a pipe with a piston and a membrane at one of its ends resonates at two positions of the piston separated by a distance $l = 16.5$ cm at a frequency of 1,000 Hz. Find the velocity of sound.

Solution: In this case the displacement of the piston is equal to $\lambda/2$.

Therefore $\frac{\lambda}{2} = 16.5 \text{ cm or } \lambda = 33 \text{ cm} = 0.33 \text{ m}$

Now $V = n\lambda = 1{,}000 \times 0.33 = 330 \text{ m/s}$

9.61. At room temperature one performs an experiment with resonance tube. The first and the second resonance are observed at lengths $l_1 = 20.1$ and $l_2 = 62.4$ cm, of air column, respectively. Find the length of air column at the third resonance.

Solution: For the first and second resonance at the two lengths l_1 and l_2, we have

$l_1 + e = \frac{\lambda}{4}$; here e is the end correction

and $l_2 + e = \frac{3\lambda}{4}$

Now $l_2 + e = 3\,(l_1 + e)$

or $e = \frac{l_2 - 3l_1}{2}$

On substituting the values of l_1 and l_2, we have

$$e = \frac{62.4 - 60.3}{2} = \frac{2.1}{2} = 1.05 \text{ cm}$$

Therefore $\frac{\lambda}{4} = 20.1 + 1.05 = 21.15$ cm

For the third resonance, we have

$$l_3 + e = \frac{5\lambda}{4} = 5 \times 21.15 = 105.75$$

$$\therefore\ l_3 = 105.75 - 1.05 = 104.7 \text{ cm}$$

9.62. When a tuning fork and an air column having temperature equal to 51°C are sounded together, four beats are observed. On decreasing the temperature of the air column the number of beats decreases. When the temperature of the air column is 16°C only one beat is observed. Find the frequency of the tuning fork.

Solution: At 51°C, we have

$n_{51} - n = 4$

where n_{51} and n represents frequency of the air column and tuning fork respectively. Similarly at 16°C, we have

$n_{16} - n = 1$

Therefore $\frac{n_{51}}{n_{16}} = \frac{n+4}{n+1}$

Now $\frac{n_{51}}{n_{16}} = \frac{V_{51}}{V_{16}} = \frac{\sqrt{273+51}}{\sqrt{273+16}} = \sqrt{\frac{324}{289}}$; V_{51} and V_{16} represent the velocity of sound at 51 and 16°C respectively.

Therefore $\sqrt{\frac{324}{289}} = \frac{n+4}{n+1}$

or $\frac{n+4}{n+1} = 1.059$

or $n + 4 = 1.059\,n + 1.059$

or $0.059\,n = 2.941$

or $n = \frac{2.941}{0.059} = 49.83$ Hz

9.63. How many times more intense is a 90 *db* sound than a 40 *db* sound.

Solution:

$$L_2 - L_1 = 10 \log \frac{I_2}{I_1}$$

$$90 - 40 = 10 \log \frac{I_2}{I_1}$$

$$50 = \log \frac{I_2}{I_1}$$

$$\therefore \frac{I_2}{I_1} = 10^5$$

9.64. The power of sound from the speaker of a radio is 20 mW. By turning the knob of the volume control the power of sound is increased to 400 mW. What is the power increase in *db* as compared to original power.

Solution:

$$L_2 - L_1 = \Delta L = 10 \log \frac{I_2}{I_1} = 10 \log \frac{P_2}{P_1} = 10 \log \frac{400}{20}$$

$$= 10 \log 20$$

$$= 10\ [1.3010] = 13.0 \text{ db}$$

9.65. A dog barking delivers about 1 mW of power. If this power is uniformly distributed over a hemispherical area, what is the sound level at a distance of 5 m? What would the sound level be if instead of 1 dog, 5 dogs start barking at the same time each delivers 1 mW of power?

Solution:

$$I = \frac{P}{\text{area}} = \frac{10^{-3}}{2\pi \times 5^2} = 6.37 \frac{\mu w}{m^2}$$

$$L = 10 \log \frac{I}{I_0} = 10 \log \frac{6.37 \times 10^{-6}}{10^{-12}}$$

$$= 10\ [\log (6.37) + 6 \log 10]$$

$$= 10\ [0.8 + 6]$$

$$= 68\ db$$

$$L_2 - L_1 = 10 \log \frac{I_2}{I_1} = 10 \log \frac{5 I_1}{I_1}$$

$$L_2 - L_1 = 10 \log 5$$

$$L_2 = 68 + 10 \times 0.7 = 75\ db$$

9.66. Find the level of the sound waves propagating in air with a velocity of $V = 330$ m/s. Atmospheric pressure 10^5 N/m^2 and density of air 1.29 kg/m^3.

Solution:

$$I = \frac{P^2}{2\rho V} = \frac{10^{10}}{2 \times 1.29 \times 330} = 1.17 \times 10^7 \text{ W/m}^2$$

$$L = 10 \log \frac{I}{I_0} = 10 \log \frac{10^7}{10^{-2}} = 190\ db$$

9.67. A spherical sound wave propagates in air from an isotropic source. At a distance of r_0 = 100 m from the source the sound level is L_0 = 100 db. Assuming that the sound is not absorbed by the air find the loudness level L at a distance of (a) 2 km (b) 4 km.

Solution:

$$SL = 10 \log \frac{I}{I_0} = 10 \log \frac{P}{4\pi r_0^2 I_0}$$

$$\frac{P}{4\pi r_0^2 I_0} = \frac{P}{4\pi I_0 \times 10^4} = 10^{10}$$

$$\therefore \frac{P}{4\pi I_0} = 10^{14}$$

(a) $$L_1 = 10 \log \frac{P}{4\pi I_0 \times 4 \times 10^6}$$

$$= 10 \log \frac{10^{14}}{4 \times 10^6} = 10 \log 0.25 \times 10^8$$

$$= 10\,[1.39 + 6] = 74 \text{ db}$$

(b) $$L_2 = 10 \log \frac{10^{14}}{16 \times 10^6} = 10 \log \left[6.25 \times 10^6\right]$$

$$= 10\,[6 + 0.7959]$$

$$= 67.96$$

$$= 68 \text{ db}$$

9.68. A window whose area is 3 m^2 opens on a street. The sound level at the window due to the street noise is 60 db. Calculate the acoustic power which enters the window. If a glass sheet is fitted at the window then calculate the energy collected by the sheet in a period of 12 hrs.

Solution:

$$SL = 10 \log \frac{I}{I_0} = 60 \text{ given}$$

$$\therefore \frac{I}{I_0} = 10^6$$

$$I = 10^6 \times 10^{-12} = 10^{-6} \text{W/m}^2$$

$$\text{Power} = I \cdot A = 10^{-6} \times 3 = 3\ \mu\text{w}$$

$$\text{Energy } E = P \times \text{time} = 3 \times 10^{-6} \times 12 \times 60 \times 60$$

$$= 0.1296\ \text{J}$$

9.69. A plane sound wave propagates along the x axis in air. The sound absorption coefficient $\alpha = 2.07 \times 10^{-3}/\text{m}$. In the plane $x = 0$ the loudness level is $L_0 = 100$ db. Calculate the loudness at $x = 2$ km.

Solution:

$$I = I_0' e^{-2.07 \times 10^{-3} \times 2 \times 10^3} = \frac{I_0'}{e^{4.14}}$$

$$= \frac{I_0'}{2.718^{4.14}} = 0.0159\, I_0'$$

$$\text{Further } \log \frac{I_0'}{I_0} = 10^{10} \text{ given}$$

$\therefore$ Required sound level $L = 10 \log (0.0159 \times 10^{10})$

$= 10\ [2.2014 + 6]$

$= 82.014$ db

9.70. An isotropic source produces a spherical sound wave with a frequency of 3 kHz in air. At a distance of $r_1 = 100$ m from the source the sound loudness level is $L_1 = 60$ db. Determine the loudness level L_2 at a distance of $r_2 = 200$ m and also the distance r at which the sound stops being heard assuming the sound absorption coefficient in air to be $\alpha = 2.42 \times 10^{-3}/\text{m}$.

Solution:

$$I = \frac{I_0' e^{-2.42 \times 10^{-3} \times 100}}{4\pi \times 100^2}$$

$$60 = 10 \log \left(\frac{I_0'}{e^{0.242} \times 4\pi I_0 \times 10^4} \right)$$

$$\frac{I_0'}{2.718^{0.242} \times 4\pi I_0 \times 10^4} = 10^6$$

$$\therefore \frac{I_0'}{I_0} = 4\pi \times 10^{10} \times 2.718^{0.242}$$

$$= \frac{4\pi \times 10^{10}}{0.7849}\ ;\ 2.718^{0.242} = \frac{1}{0.7849}$$

At a distance of 200 m

$$SL = 10 \log \left[\frac{I_0' e^{-2.42 \times 10^{-3} \times 200}}{4\pi \times 4 \times 10^4 \cdot I_0} \right]$$

$$= 10 \log \left[\frac{I_0'}{I_0} \cdot \frac{1}{2.718^{0.484}} \cdot \frac{1}{4\pi \times 4 \times 10^4} \right]$$

$$= 10 \log \left[\frac{4\pi \times 10^4 \times 10^6 \times 0.6161}{4\pi \times 10^4 \times 4 \times 0.7849} \right]; \quad \frac{1}{2.718^{0.484}} = 0.6161$$

$$= 53 \text{ db}$$

For determining r

$$SL = 0 = 10 \log \left[\frac{I_0' \cdot e^{-2.42 \times 10^{-3} \times r}}{4\pi r^2 \times I_0} \right]$$

$$= \left[\frac{4\pi \times 10^{10}}{4\pi r^2 \, e^{2.42 \times 10^{-3} r} \times 0.7849} \right] = 1$$

$$10^{10} = r^2 \, e^{2.42 \times 10^{-3} r} \times 0.7849$$

The above equation holds good with the value of $r = 3$ km

$$r^2 e^{2.42 \times 10^{-3} r} \times 0.7849 = 9 \times 10^6 \, e^{2.42 \times 3} \times 0.7849$$

$$= 9 \times 10^6 \times 0.7849 \times 2.718^{7.26}$$

$$\text{log value} = 6.9542 - 0.1052 + 3.1526$$

$$= 10.0016$$

$$\therefore r = 3 \text{ km}$$

9.71. Two sounds in a gas differ in their loudness level by $L_{12} = 20$ db. Find the ratio of the amplitudes of the pressure oscillations for these sounds.

Solution:

$$\frac{(\Delta pm)_1}{(\Delta pm)_2} = \frac{\rho v A_1 \omega}{\rho v A_2 \omega} = \frac{A_1}{A_2} = \sqrt{\frac{I_1}{I_2}}$$

Further $$L_1 = 10 \log \left(\frac{I_1}{I_0} \right)$$

$$L_2 = 10 \log \left(\frac{I_2}{I_0} \right)$$

Let $$\frac{I_1}{I_0} = 10^{a_1} \text{ and } \frac{I_2}{I_0} = 10^{a_2} \text{ then}$$

$$L_1 = 10a_1 \text{ and } L_2 = 10a_2$$

$$10 (a_1 - a_2) = L_1 - L_2 = L_{12} = 20 \text{ db}$$

$$\therefore a_1 - a_2 = 2$$

Now $\frac{A_1}{A_2} = \sqrt{\frac{I_1}{I_2}} = 10^{\frac{a_1 - a_2}{2}}$

$= 10$

9.72. The intensity of a sound wave 20 m away from the source is 3 nw/m^2. Find the intensity of the wave 32 m away from the source if the half thickness for sound of this frequency is 120 m.

Solution:

$\frac{I_1}{I_2} = \left(\frac{r_2}{r_1}\right)^2 \cdot 2^{\frac{r_2 - r_1}{L}}$ where I_1 and I_2 are the wave intensities at distances r_1 and r_2 and L is half thickness.

On putting the values we have

$$\frac{3 \times 10^{-9}}{I_2} = \left(\frac{32}{20}\right)^2 \cdot 2^{\frac{32-20}{120}}$$

or $$\frac{3 \times 10^{-9}}{I_2} = 1.6^2 \cdot 2^{1/10}$$

$$I_2 = \frac{3 \times 10^{-9}}{1.6^2 \times 1.072}$$

$$= 1.093 \times 10^{-9}\ \text{W/m}^2$$

9.73. The loudness level of a 1000 Hz pure tone is 60 phons. How many such tones must be sounded together in order to produce a loudness level of twice that produced by one tone.

Solution:

$$L = 60 = 10 \log \frac{I}{I_0}$$

$$120 = 10 \log \frac{nI}{I_0}$$

$$12 = \log n + \log \frac{I}{I_0}$$

$$= \log n + 6$$

$$\log n = 6 \text{ or } n = 10^6$$

9.74. A point isotropic source of sound is located at the axis of a ring of radius R at a distance l. Find the mean energy flow across the area enclosed by the ring if at the centre of the ring the intensity of sound is I_0.

Solution:

$I_0 = \frac{P}{l^2}$; P = Power of the source

Intensity at $O' = \dfrac{P\cos\theta}{l^2 + x^2} = \dfrac{I_0 l^3}{\left(l^2 + x^2\right)^{3/2}}$; see Fig. 9.13

Flux through the area enclosed between radii x & $x + dx$ $= \dfrac{2\pi x dx \cdot I_0 l^3}{\left(x^2 + l^2\right)^{3/2}}$

$$\text{Net flux} = \int_O^R 2\pi I_0 l^3 \cdot \frac{x dx}{\left(x^2 + l^2\right)^{3/2}}$$

$$= 2\pi I_0 l^2 \left[1 - \frac{1}{\sqrt{1 + R^2/l^2}}\right]$$

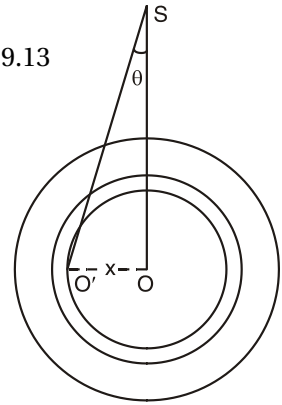


Fig. 9.13

9.75. The first overtone of an open organ pipe beats with the first overtone of a closed organ pipe with a beat frequency of 2.2 Hz. The fundamental frequency of the closed pipe is 110 Hz. Find the lengths of the pipes. (IIT 1997)

Solution:

For a closed organ pipe $n_C = (2r - 1)\ \dfrac{V}{4L_C}$

where L_C = length of the closed pipe, $r = 1, 2, \ldots$

For open organ pipe $n_0 = \dfrac{rV}{2L_0}$

where L_0 = length of the open pipe, $r = 1, 2, \ldots$

$$n_C = \frac{V}{4L_C} = 110$$

$$\therefore L_C = \frac{330}{4 \times 110} = 0.75 \text{ m}$$

$$\frac{3V}{4L_C} - \frac{2V}{2L_0} = 2.2$$

$$\frac{330}{L_0} = 330 \mp 2.2; \left(\frac{3V}{3L_C} = 3n_C = 330\right)$$

$$L_0 = \frac{330}{330 \mp 2.2} = 1.0067 \text{ m or } 0.9937 \text{ m}$$

9.76. A source of sound of frequency 256 Hz is moving rapidly towards a wall with a velocity of 5 m/s. How many beats per second will be heard if sound travels at a speed of 330 m/s? (IIT 1981)

Solution: Here we have two cases, depending upon the positions of the source and the observer: (i) When we have the arrangement "source-observer-wall", no beat will be observed because the apparent frequency before and after reflection is the same and is equal to $\frac{nv}{v - v_s}$. (ii) When arrangement is "observer-source-wall", the apparent frequency before reflection is

$$n_1 = \frac{nv}{v + v_s}$$

and $n_2 = \frac{nv}{v - v_s}$ after reflection.

Thus, number of beats $= n_2 - n_1 = \frac{2nvv_s}{v^2 - v_s^2} = \frac{2nvv_s}{(v + v_s)(v - v_s)}$

$$= \frac{2 \times 256 \times 330 \times 5}{335 \times 325} = 7.7$$

Alternative Solution: In this case two sounds will be heard, one directly from the source and the other reflected from the wall. The reflected sound can be imagined as coming from an acoustic image of the source.

Now $n' = \frac{n(v + v_0)}{v - v_s}$

Here $v_0 = v_s = 5$ m/s

$$\therefore n' = \frac{n \times 335}{325} = n\left(1 + \frac{10}{325}\right)$$

$$\therefore \text{Number of beats } n' - n = \frac{n \times 10}{325} = \frac{256 \times 10}{325} = 7.9$$

9.77. A source of sound with a frequency $n = 2000$ Hz moves along a line at right angles to the wall with a velocity $v_s = 0.33$ m/s. Two stationary detectors D_1 and D_2 are placed on the path of the source in a manner shown in Fig. 9.14. Calculate the number of beats. The velocity of sound is equal to $v = 330$ m/s.

Solution: Before reflection for detector D_1 the apparent frequency is given by

$$n_1 = \frac{nv}{v + v_s}$$

similarly for detector D_2 the apparent frequency is given by

$$n_2 = \frac{nv}{v - v_s}$$

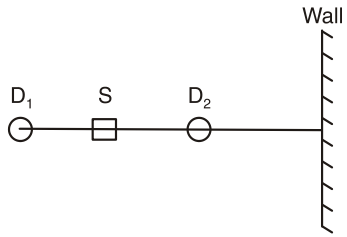


Fig. 9.14

After reflection the apparent frequency recorded by detector D_2 is

$$n_2' = \frac{nv}{v - v_s}$$

and the apparent frequency recorded by detector D_1 is

$$n_1' = \frac{nv}{v - v_s}$$

Thus D_2 will not record any beat since $n_2' = n_2$ whereas D_1 will record $n_1' - n_1$ beats.

Thus

$$\text{Number of beats} = n_1' - n_1 = nv\left(\frac{1}{v - v_s} - \frac{1}{v + v_s}\right) = \frac{2nv \cdot v_s}{v^2 - v_s^2} \cong \frac{2nv_s}{v}$$

$$= \frac{2 \times 2000 \times 0.33}{330} = 4 \text{ Hz}$$

9.78. There are two sources of sound with frequency n = 330 Hz, one moving towards a stationary detector and the other away from it with the same velocity v_s. The detector records four beats. Find the velocity of the source. Velocity of sound v = 330 m/s.

Solution: The source which is approaching to the stationary detector will have an apparent frequency given by

$$n' = \frac{nv}{v - v_s}$$

The source which is receding will have an apparent frequency given by

$$n'' = \frac{nv}{v + v_s}$$

$$\text{Number of beats } \Delta n = n' - n'' = nv\left[\frac{1}{v - v_s} - \frac{1}{v + v_s}\right] \cong \frac{2nv_s}{v}$$

$$\therefore v_s = \frac{\Delta n \cdot v}{2n}$$

On substituting the values of various terms, we have

$$v_s = \frac{4 \times 330}{2 \times 330} = 2 \text{ m/s}$$

9.79. A stationary source of sonic oscillation sends forth monochromatic sound. A wall approaches it with a velocity of u = 0.17 m/s. By what percentage does the wavelength of sound change after reflection from the wall? Velocity of sound v = 340 m/s.

Solution: Here we have two Doppler shifts. In the first phase the wall acts like a moving observer. The apparent frequency n' detected by it is given by

$$n' = \left(\frac{v+u}{v}\right) n$$

where n is the frequency of the source.

In the second phase the wall acts like a moving source and reflects sound. The apparent frequency is given by

$$n'' = \frac{n'v}{v-u} = \frac{n(v+u)}{v-u}$$

Now $$\frac{v}{\lambda'} = \frac{v}{\lambda}\frac{(v+u)}{(v-u)}$$

$$\therefore \frac{\lambda'}{\lambda} = \frac{v-u}{v+u}$$

or $$\frac{\lambda-\lambda'}{\lambda} = \frac{v+u-v+u}{v+u} = \frac{2u}{v+u}$$

$$\frac{\Delta\lambda}{\lambda} = \frac{2u}{v+u} = \frac{2\times 0.17\times 100}{340.17} \cong 0.1\%$$

9.80. A receiver and a source of sound with frequency $n = 1650$ Hz lies on a straight line normal to a wall. The wall is moving away from the stationary source with a velocity $u = 0.5$ m/s. Calculate the number of beats recorded by the stationary receiver. The velocity of sound is equal to $v = 330$ m/s.

Solution: This is a case of two Doppler shifts. First, the wall acts like a moving observer and detects an apparent frequency given by

$n' = \left(\frac{v-u}{v}\right) n$; where v is the velocity of sound.

Secondly the wall acts like a moving source and reflects sound. The apparent frequency is given by

$$n'' = \frac{n'v}{v+u} = \left(\frac{v-u}{v+u}\right) n$$

Number of beats $= n - n'' = n - \left(\frac{v-u}{v+u}\right) n = \frac{2nu}{v+u}$

On substituting the values of n, u and v we have

Number of beats $= \frac{2\times 1650\times 0.5}{330.5} = 5$ Hz

9.81. A source of sound of frequency 1000 Hz mounted on the rim of a circular table of radius $r = 33$ cm and a receiver are at rest and are located along the x axis (Fig. 9.15). With what angular velocity ω should be the table rotate about a vertical axis passing through its centre for the receiver to record a frequency difference $\Delta n = 100$ Hz. Velocity of sound $v = 330$ m/s.

Solution: Let S and R be the positions of the source and receiver respectively. The velocity of the source is $r\,\omega$. When the source follows the path ABC, it comes towards the receiver (Fig. 9.15). The apparent frequency recorded by the receiver is given by

$$n_1 = \frac{nv}{v - r\omega}$$

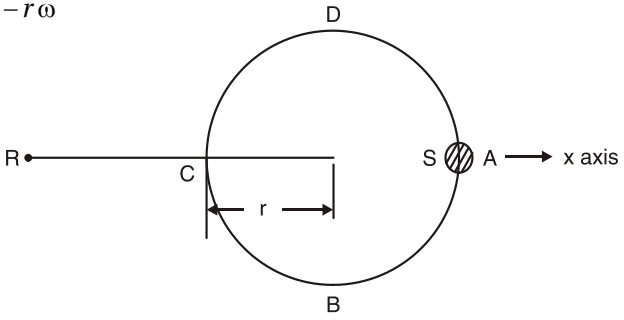


Fig. 9.15

When the source follows path CDA, it moves away from the receiver. The apparent frequency recorded by the receiver under this condition is given by

$$n_2 = \frac{nv}{v + r\omega}$$

$$\text{Now} \quad n_1 - n_2 = \Delta n = nv\left[\frac{1}{v - r\omega} - \frac{1}{v + r\omega}\right] = \frac{2nvr\omega}{v^2 - r^2\omega^2}$$

$$\text{or} \quad \Delta n \cdot v^2 - \Delta n r^2\omega^2 = 2nvr\omega$$

$$\text{or} \quad \Delta n r^2\omega^2 + 2nvr\omega - \Delta n \cdot v^2 = 0$$

$$\text{or} \quad \omega = -\frac{2nvr \pm \sqrt{4n^2v^2r^2 + 4\Delta n^2 r^2 v^2}}{2\Delta n r^2}$$

$$= -\frac{nv}{\Delta n r} + \sqrt{\frac{n^2v^2}{\Delta n^2 \cdot r^2} + \frac{v^2}{r^2}}$$

$$= \frac{nv}{\Delta n r}\left[\sqrt{1 + \left(\frac{\Delta n}{n}\right)^2} - 1\right]$$

On substituting the values of the various terms, we have

$$\omega = \frac{1000 \times 330}{100 \times 0.33}\left[\sqrt{1+\left(\frac{100}{1000}\right)^2} - 1\right]$$

$$= 10000\left[\sqrt{\frac{101}{100}} - 1\right]$$

$$= 10000\,[1.005 - 1] = 10000 \times 0.005 = 50 \text{ rad/s}$$

9.82. An astronaut aims to determine his velocity as he approaches the moon. To do this, he uses a source of electromagnetic waves to propagate waves of frequency 6000 MHz. On comparing it with its echo he records a difference of frequency of Δn = 100 kHz. What is the velocity of rocket relative to the moon?

Solution: Here both source and the observer are having the same velocity v. The apparent frequency is given by

$$n' = n\frac{c+v}{c-v}$$

where c is the velocity of light.

$$\frac{n'}{n} = \frac{c+v}{c-v}$$

$$\frac{\Delta n}{n} = \frac{n'-n}{n} = \frac{2v}{c-v} = \frac{2v}{c}; c - v \cong c$$

$$\therefore v = \frac{\Delta n}{2n} \cdot c$$

On substituting the values of the various terms, we have

$$v = \frac{100 \times 10^3 \times 3 \times 10^8}{2 \times 6000 \times 10^6} = \frac{10}{4} \times 10^3 = 2.5 \text{ km/s}$$

9.83. A source of sonic oscillation having frequency n = 3.2 kHz travels along the straight path SA with a velocity of 198 m/s (Fig. 9.16). A stationary observer O is at a distance of l = 250 m from this path. Caculate the apparent frequency recorded by the observer when the source is closest to it. Velocity of sound v = 330 m/s.

Solution: The source is moving along the path SA with velocity v_s. The relative speed of the source towards the observer O is $v_s \cos\theta$. The apparent frequency heard by O is given by

$$n' = \frac{nv}{v - v_s \cos\theta}$$

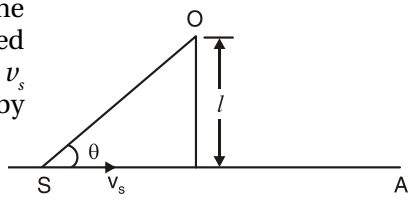


Fig. 9.16

or $$n' = \frac{n}{1-(v_s/v)\cos\theta};\ \eta = \frac{v_s}{v} = \frac{198}{330} = 0.6$$

$$= \frac{n}{1-\eta\cos\theta}$$

In view of the velocities of the source and sound, we have

$$\frac{v_s}{v + v_s\cos\theta} = \cos\theta$$

or $$\frac{\eta}{1+\eta\cos\theta} = \cos\theta$$

or $$\eta\cos^2\theta + \cos\theta - \eta = 0$$

or $$\cos\theta = \frac{-1 \pm \sqrt{1+4\eta^2}}{2\eta}$$

$$= \frac{1}{2\eta}\left(1 + 2\eta^2 - 1\right) = \eta$$

On substituting the value of $\cos\theta$ in the expressions for apparent frequency, we get

$$n' = \frac{n}{1-\eta^2}$$

On substituting the values of n and η, we have

$$n' = \frac{3.2\times10^3}{1-0.36} = \frac{3.2}{0.64}\text{ kHz} = 5\text{ kHz}$$

9.84. Two cars A and B are moving towards each other with a speed of 30 m/s. A person sitting in car A fires shots after every 3 seconds and person sitting in car B observes them. What will be the time difference recorded by him between two consecutive shots ? Velocity of sound $v = 330$ m/s.

Solution: The apparent frequency as heard by the person sitting in car B is given by

$$n' = n\cdot\frac{v+30}{v-30} = n\frac{330+30}{330-30} = \frac{6n}{5}$$

Let t and t' be the time intervals between two shots as recorded by the persons in car A and B respectively.

Then

$$\frac{t'}{t} = \frac{n}{n'} \text{ or } t' = \frac{t\cdot n}{n'} = 3\times\frac{5}{6} = 2.5\text{ s}$$

9.85. When a train is approaching the observer, the frequency of the whistle is 100 cps while when it has passed the observer, it is 50 cps. Calculate the frequency when the observer moves with the train.

(Roorkee 1997)

Solution:

$$n' = \frac{nV}{V - V_s}$$

$$100 = \frac{nV}{V - V_s} \text{ or } V - V_s = \frac{nV}{100}$$

Further $$n'' = \frac{nV}{V + V_s}$$

$$50 = \frac{nV}{V + V_s} \text{ or } V + V_s = \frac{nV}{50}$$

$$2V = \frac{3nV}{100}$$

$$n = \frac{200}{3} = 66.67 \text{ Hz}$$

9.86. A bat flies perpendicular to a wall with a speed of $v = 6$ m/s emitting ultrasonic waves with the frequency $\nu = 4.5 \times 10^4$ Hz. Calculate the frequencies that can be heard by the bat. The velocity of sound in air $V =$ 340 m/s.

Solution: One frequency that is heard by the bat will be 4.5×10^4

The other frequency will be

$$n' = n\frac{(V + v)}{(V - v)}$$

$$n' = 4.5 \times 10^4 \frac{(340 + 6)}{340 - 6}$$

$$= \frac{4.5 \times 346}{334} \times 10^4$$

$$= 4.66 \times 10^4 \text{ Hz}$$

9.87. An observer standing at the bed of a railroad observes a whistle of the train. When the train is approaching the observer, the frequency of the whistle as observed by him is n_1, while when it has passed the observer the frequency appears to be n_2. Determine (i) the speed of the train (ii) the frequency of the whistle. The speed of sound waves is V.

Solution:

(i) $$n_1 = \frac{nV}{V - V_s};\ n = \text{frequency of the whistle}$$

V_s = speed of the train

$$n_2 = \frac{nV}{V + V_s}$$

On dividing the two $\frac{n_1}{n_2} = \frac{V + V_s}{V - V_s}$

$$n_1 V - n_1 V_s = n_2 V + n_2 V_s$$

or $$V_s = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) V$$

(ii) $$n = \frac{n_1}{V}(V - V_s)$$

$$= \frac{n_1}{V}\left(V - \frac{n_1 - n_2}{n_1 + n_2} V\right)$$

$$= \frac{2 n_1 n_2}{n_1 + n_2}$$

9.88. Two trains run at the same speed of v = 25 m/sec along a straight track in the same direction keeping distance of x = 2 km between them (see Fig. 9.17). At an instant when they are symmetrically located with respect to the point O lying at a distance of y = 1 km from the track both trains blow a whistle of frequency 400 Hz. Taking the velocity of sound as 340 m/sec. Calculate the beat frequency recorded by the observer placed at O.

Solution: The frequency heard at O due to the engine 1 is given by

$$n_1 = \frac{nV}{V - v/\sqrt{2}}; \; V = \text{velocity of sound}$$

$$= \frac{400 \times 340}{340 - 25/\sqrt{2}}$$

$$= \frac{400 \times 340}{340 - 17.68}$$

$$= 421.9 \text{ Hz}$$

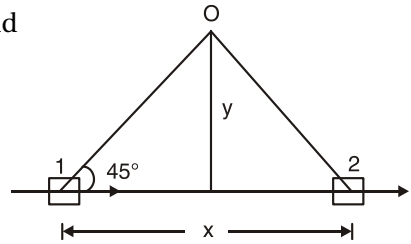


Fig. 9.17

The frequency heard at O due to the engine 2 is given by

$$n_2 = \frac{nV}{V + v/\sqrt{2}} = \frac{400 \times 340}{340 + 17.68}$$

$$= 380.2 \text{ Hz}$$

Beat freq. = 422 – 380 = 42

9.89. Two automobiles travel one after the other along a straight section of a road at the same speed of 25 m/sec in the same direction (see Fig. 9.18). When a third vehicle travelling at the speed of 20 m/sec in the opposite direction appeard at a distance, the driver of the second vehicle gave a long blast of his horn at a frequency 500 Hz. Sound of what frequency

will be heard by the drivers of the first and third vehicles? Velocity of sound = 340 m/sec.

Solution: For the first vehicle there will no Doppler effect and therefore its driver will hear a frequency of 500 Hz. However the driver of the third vehicle will hear the frequency n_1 given by

$$n_1 = 500 \times \frac{340 + 20}{340 - 25}$$
$$= 571.4 \text{ Hz}$$

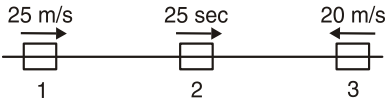


Fig. 9.18

9.90. A beam of He^+ ion moving with kinetic energy E emits a spectral line of wavelength λ. Find the Doppler shift $\Delta\lambda$ of the line when observed at an angle θ.

Solution:

$$\frac{1}{2} m_{He} v^2 = E$$

$$\therefore v = \sqrt{\frac{2E}{m_{He}}}$$

$$\nu' = \frac{\nu c}{c - v\cos\theta}$$

or
$$\frac{c - v\cos\theta}{\nu} = \frac{c}{\nu'}$$

$$\lambda\left(1 - \frac{v\cos\theta}{c}\right) = \lambda'$$

$$\Delta\lambda = \lambda' - \lambda = -\frac{\lambda v\cos\theta}{c}$$

$$= \frac{-\lambda}{c} \cdot \sqrt{\frac{2E}{m_{He}}} \cdot \cos\theta$$

$$= -\lambda \sqrt{\frac{2E}{m_{He} \cdot c^2}} \cdot \cos\theta$$

9.91. When a spectral line of wavelength λ is observed in the directions to the opposite edges of the solar disc along its equator a difference in wavelength equal to $\delta\lambda$ is noticed. Find the period of revolution of Sun about its own axis.

Solution: The two frequencies that will be observed will be given by

$$\nu' = \frac{\nu c}{C - R\omega} \text{ and } \nu'' = \frac{\nu c}{C + R\omega}; R = \text{radius of Sun}$$
$$\omega = \text{angular velocity of Sun}$$

$$\frac{C - R\omega}{\nu} = \frac{C}{\nu'} \text{ and } \frac{C + R\omega}{\nu} = \frac{C}{\nu''}$$

$$\lambda\left(1 - \frac{R\omega}{C}\right) = \lambda' \text{and } \lambda\left(1 + \frac{R\omega}{C}\right) = \lambda''$$

$$\Delta\lambda_1 = \lambda - \lambda' = \frac{R\omega\lambda}{C}$$

$$\Delta\lambda_2 = \lambda'' - \lambda = \frac{R\omega\lambda}{C}$$

$$\delta\lambda = \Delta\lambda_1 + \Delta\lambda_2 = \frac{2R\omega\lambda}{C} = \frac{4\pi R\lambda}{CT}$$

$$\therefore \quad T = \frac{4\pi\lambda R}{C \cdot \delta\lambda}$$

9.92. The beauty of the Doppler's effect is to discover distant double stars which could not be resolved by a telescope. Let us consider two stars of equal masses revolving about their centre of mass. The spectral lines emitted by the two stars periodically becomes doublets. Find the distance between the stars by assuming that every time the maximum splitting of the spectral lines $(\Delta\lambda/\lambda)_m$ occurs after a time τ.

Solution:

$$\Delta\lambda = \frac{\lambda R}{C} \cdot \frac{2\pi}{t}\text{; R is radius of path}$$

$$\left(\frac{\Delta\lambda}{\lambda}\right)_m = \frac{d\pi}{c\tau}\text{; } d = 2R = \text{distance between the stars}$$

$$\therefore d = \frac{C\tau}{\pi}\left(\frac{\Delta\lambda}{\lambda}\right)_m$$

9.93. A radar operates at wavelength λ. Calculate the velocity of an approaching aircraft if the beat frequency between the transmitted signal the signal reflected from the aircraft is $\Delta\nu$.

Solution:

$$\nu' = \nu\frac{(c + v)}{c - v}\text{; } v \text{ is the velocity}$$

$$\Delta\nu = \nu' - \nu = \frac{2\nu v}{c - v} = \frac{2\nu v}{c}$$

$$\therefore v = \frac{1}{2}\frac{c}{\nu} \cdot \Delta\nu = \frac{1}{2}\lambda \cdot \Delta\nu$$

9.94. A spectral line of hydrogen atom of wavelength λ is emitted from the beam of hydrogen atoms having kinetic energy *E*. Find the transverse Doppler shift of that line.

Solution: The most general representation of the Doppler's effect is given by

$$n' = n\frac{\sqrt{1-v^2/c^2}}{1-\frac{v}{c}\cos\theta}\ ;\ c \text{ is velocity of light}$$

For transverse Doppler's effect, $\theta = \frac{\pi}{2}$

$$\therefore n' = n\sqrt{1-v^2/c^2}$$

$$1-\frac{n'}{n} = \frac{\Delta n}{n} = \frac{\Delta\lambda}{\lambda} = \frac{1}{2}\frac{v^2}{c^2}$$

$$\therefore \Delta\lambda = \frac{1}{2}\lambda\cdot\frac{v^2}{c^2}$$

$\frac{1}{2}m_0v^2 = E$; m_0 = mass of hydrogen atom

$$\therefore v^2 = \frac{2E}{m_0}$$

$$\Delta\lambda = \frac{\lambda E}{m_0c^2}$$

9.95. An astronaut is approaching the moon. He sends out a radio signal of frequency 5×10^9 Hz and finds out that the frequency shift in echo received is 10^3 Hz. Find his speed of approach.

Solution:

$$n' = n\left[\frac{c+v}{c-v}\right] = n\left[1+\frac{2v}{c}\right]$$

$$\frac{n'-n}{n} = \frac{\Delta n}{n} = \frac{2v}{c}$$

$$\therefore v = \frac{c\cdot\Delta n}{2n}$$

On putting the values we have

$$v = \frac{3\times10^8\times10^3}{2\times5\times10^9} = 30 \text{ m/sec}$$

9.96. A source S and a receiver R are arranged in a room with a false ceiling, at a distance $l = 6$ m apart. The waves received directly from the source are in phase with those reflected from the false ceiling at a height $h_1 = 7$ m above the ground. The roof of the room is 3 m above the ceiling. On removing the ceiling the waves are reflected from the roof but no signal

is registered by the receiver. Find the wavelength of the wave assuming that incident and reflected rays make the same angle.

Solution: After reflection from the false ceiling, the total path length of the reflected wave

$$= 2\sqrt{h_1^2 + \frac{l^2}{4}} = \sqrt{4h_1^2 + l^2}$$

Phase for reflected wave $\phi_1' = \frac{2\pi}{\lambda}$ path

$$= \frac{2\pi}{\lambda} \cdot \sqrt{4h_1^2 + l^2}$$

Apart from this a phase difference of π will also be introduced due to reflection. Hence the net phase difference is given by

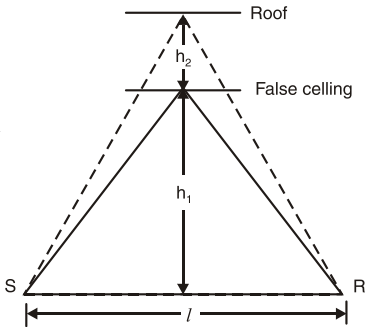


Fig. 9.19

$$\phi_1 = \frac{2\pi}{\lambda} \cdot \sqrt{4h_1^2 + l^2} + \pi$$

After reflection from the roof,

Total path length of reflected wave $= 2\sqrt{(h_1 + h_2)^2 + \frac{l^2}{4}}$

$$= \sqrt{4(h_1 + h_2)^2 + l^2}$$

The net phase for this reflected wave is given by

$$\phi_2 = \frac{2\pi}{\lambda}\sqrt{4(h_1 + h_2)^2 + l^2} + \pi$$

Now no signal will be obtained by the receiver R when there is destructive interference. This is possible when $\phi_2 - \phi_1 = \pi$

or $$\frac{2\pi}{\lambda}\left[\sqrt{4(h_1 + h_2)^2 + l^2} - \sqrt{4h_1^2 + l^2}\right] = \pi$$

or $$\lambda = 2\left[\sqrt{4(h_1 + h_2)^2 + l^2} - \sqrt{4h_1^2 + l^2}\right]$$

On substituting the values of h_1, h_2 and l we have

$$\lambda = 2\left[\sqrt{4 \times (3+7)^2 + 6^2} - \sqrt{4 \times 7^2 + 6^2}\right]$$

$$= 2\left(\sqrt{436} - \sqrt{232}\right)$$

$$= 2\,(20.88 - 15.24)$$

$$= 11.28 \text{ m}$$

9.97. Two sources S_1 and S_2, each producing sound of frequency 660 Hz, have an acoustic output of $4\pi \times 10^{-3}$ W. The two sources are 3 and 2 m from

a point *O*, respectively. Calculate the resultant intensity at *O*. Velocity of sound = 330 m/s.

Solution:

Path difference $\Delta x = S_1O - S_2O = 3.0 - 2.0 = 1.0$ m

Phase difference $= \dfrac{2\pi}{\lambda} \cdot \Delta x$

$= \dfrac{2\pi}{v} \cdot n \times 1 = \dfrac{2\pi \times 660}{330} = 4\pi$

Resultant intensity at $O = I = I_1 + I_2 + 2\sqrt{I_1 \cdot I_2}$

$= \dfrac{4\pi \times 10^{-3}}{4\pi \times 9} + \dfrac{4\pi \times 10^{-3}}{4\pi \times 4} + 2\sqrt{\dfrac{4\pi \times 10^{-3}}{4\pi \times 9} \times \dfrac{4\pi \times 10^{-3}}{4\pi \times 4}}$

$= \left(\dfrac{1}{9} + \dfrac{1}{4} + \dfrac{1}{3}\right)10^{-3} = \dfrac{25}{36} \times 10^{-3}$ W/m^2 □ 0.7×10^{-3} W/m^2